ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 3, Summer 1995

ON THE TRUNCATION OF FUNCTIONS IN LORENTZ AND MARCINKIEWICZ SPACES

J. APPELL AND E.M. SEMENOV

ABSTRACT. Given a measurable function x on $[0,1]$, we study the family $Q(x)$ of all quasi-concave functions ψ such that $||x_h||_{M(\psi)} = o(||x_h||_{\Lambda(\psi)})$ as $h \to \infty$, where x_h denotes the truncation of x at height h. We show, in particular, that $Q(x)$ is nonempty if and only if $x \in L_1 \backslash L_\infty$.

Recall that a Banach space E of measurable functions on $[0,1]$ is called *symmetric space* or *rearrangement invariant* (r.i.) *space* if the following holds:

(a) from $|x(t)| \le |y(t)|$ and $y \in E$ it follows that $x \in E$ and $||x||_E \le ||y||_E$;

(b) if x is equi-measurable to $y \in E$, then $x \in E$ and $||x||_E = ||y||_E$.

Denote by χ_e the characteristic function of a measurable set $e \subseteq [0,1]$. By (b), the norm $||\chi_e||_E$ depends only then on the measure μe of e. Consequently, the function $\varphi_E : [0,1] \to [0,\infty)$ given by $\varphi_E(\mu e) = ||\chi_e||_E$ (the so-called *fundamental function* of E) is well-defined.

Examples of r.i. spaces are the classical Lebesgue, Orlicz, Lorentz and Marcinkiewicz spaces. Denote by Ω the set of all quasi-concave functions $\psi : [0,1] \to [0,\infty)$, i.e., $\psi(0) = 0$, and both functions $t \mapsto \psi(t)$ and $t \mapsto t/\psi(t)$ are increasing. Given $\psi \in \Omega$, let

$$(1) \qquad ||x||_{\Lambda(\psi)} = \int_0^1 x^*(t)\, d\psi(t)$$

and

$$(2) \qquad ||x||_{M(\psi)} = \sup_{0<\tau\le 1} \frac{\psi(\tau)}{\tau} \int_0^\tau x^*(t)\, dt$$

where $x^*(t)$ denotes the decreasing rearrangement of $|x(t)|$. The space $\Lambda(\psi)$ defined by the norm (1) is usually called *Lorentz space*, the space

Received by the editors on December 22, 1992.
1991 *Mathematics Subject Classification.* 46E30, 47A57.

$M(\psi)$ defined by the norm (2) *Marcinkiewicz space* (see, e.g., [**3, 5, 7**]). Even in the very special case $\psi(t) = t^\alpha$, $0 < \alpha \leq 1$, these spaces are extremely important in interpolation theory [**3, 4, 9**–**13**]. Recall that the fundamental function of an r.i. space is always quasi-concave (see [**7**, Chapter II, Theorem 4.7]). If E is an r.i. space whose fundamental function φ_E is concave, then

$$\Lambda(\varphi_E) \subseteq E \subseteq M(\varphi_E), \tag{3}$$

and the corresponding imbedding operators have norm 1. On the space L_1 we define an ordering $\preceq$ by requiring that $x \preceq y$ if and only if

$$\int_0^\tau x^*(t)\,dt \leq \int_0^\tau y^*(t)\,dt$$

for all $\tau \in [0,1]$. If an r.i. space E is separable, or isomorphic to a separable space, then $x \preceq y$ implies that $||x||_E \leq ||y||_E$. In particular, this holds for any Lorentz space. For more information on the preceding notions and results, we refer to the monographs [**3, 7, 8**].

In case $E = L_1$ we have $\varphi_E(t) = t$ and $\Lambda(\varphi_E) = M(\varphi_E) = L_1$. Similarly, in case $E = L_\infty$ we have $\varphi_E(t) = \operatorname{sign} t$ and $\Lambda(\varphi_E) = M(\varphi_E) = L_\infty$. These two cases are quite exceptional; in fact, the inclusion $\Lambda(\varphi_E) \subset M(\varphi_E)$ is always strict for $E \neq L_1, L_\infty$.

Given a function $\psi \in \Omega$, by $\tilde{\psi}$ we denote the *concave majorant* of ψ. The functions ψ and $\tilde{\psi}$ are equivalent in the sense that

$$\psi(t) \leq \tilde{\psi}(t) \leq 2\psi(t), \qquad 0 \leq t \leq 1$$

(see [**7**, Chapter II, Corollary to Theorem 1.1]). Furthermore, by $\hat{\psi}$ we denote the *conjugate function* of ψ defined by

$$\hat{\psi}(t) = \frac{t}{\psi(t)}. \tag{4}$$

Lemma 1. *Suppose that $\hat{\psi}$ is concave and*

$$\lim_{t\to 0} \psi(t) = \lim_{t\to 0} \hat{\psi}(t) = 0. \tag{5}$$

Then

$$\int_0^1 \tilde{\psi}'(t)\hat{\psi}'(t)\,dt = \infty.$$

Proof. Suppose that

$$\int_0^1 \tilde{\psi}'(t)\hat{\psi}'(t)\,dt = C < \infty;$$

by (1), this means that $\hat{\psi}' \in \Lambda(\tilde{\psi})$. For any $x \in M(\psi)$ with $||x||_{M(\psi)} \leq 1$ we have then, by (2) and (4),

$$\int_0^\tau x^*(t)\,dt \leq \frac{\tau}{\psi(\tau)} = \int_0^\tau \hat{\psi}'(t)\,dt, \qquad 0 < \tau \leq t,$$

i.e., $x \preceq \hat{\psi}'$. By what we have observed before, this implies that

$$||x||_{\Lambda(\tilde{\psi})} \leq ||\hat{\psi}'||_{\Lambda(\tilde{\psi})} = C.$$

We have shown that $M(\psi) \subseteq \Lambda(\tilde{\psi})$ and hence, by (3), that $M(\psi) = \Lambda(\tilde{\psi})$ with equivalent norms. But (5) implies that the space $\Lambda(\tilde{\psi})$ is separable (see [**7**, Chapter II, Lemma 5.1]), while $M(\psi)$ is not. $\square$

Given a measurable function $x : [0,1] \to \mathbf{R}$ consider the truncation

$$x_h(t) = \begin{cases} x(t) & \text{if } |x(t)| \leq h, \\ h \operatorname{sign} x(t) & \text{if } |x(t)| > h, \end{cases}$$

and let

$$Q(x) = \left\{ \psi : \psi \in \Omega, \lim_{h\to\infty} \frac{||x_h||_{\Lambda(\psi)}}{||x_h||_{M(\psi)}} = \infty \right\}.$$

For example, if we take

$$\psi_\alpha(t) = t^\alpha, \qquad x_\beta(t) = t^{-\beta}$$

for $0 < \alpha < 1$ and $-\infty < \beta < \infty$, a straightforward computation shows that $\psi_\alpha \in Q(x_\alpha)$, but $\psi_\alpha \notin Q(x_\beta)$ for any $\beta \neq \alpha$. In particular, $Q(x_\beta)$ is nonempty if $0 < \beta < 1$. This is not accidental, as the following theorem shows which generalizes and improves some results from [**1, 2**] and is the main result of the present paper.

Theorem. *Let $x : [0, 1] \to \mathbf{R}$ be a measurable function. Then $Q(x)$ is nonempty if and only if $x \in L_1 \backslash L_\infty$.*

For proving this theorem we need some auxiliary lemmas. Denote by $\mathcal{A}$ the set of all increasing positive sequences $y = (y_k)_k$ such that

$$\lim_{k\to\infty} y_k = \infty, \tag{6}$$

and by $\mathcal{T}$ the set of all positive sequences $\lambda = (\lambda_k)_k$ such that

$$\sum_{k=1}^{\infty} \lambda_k = \infty. \tag{7}$$

For such sequences, we have, for $j \le n$,

$$\sum_{k=1}^{n} \lambda_k y_k \ge \sum_{k=j}^{n} \lambda_k y_k \ge y_j \sum_{k=j}^{n} \lambda_k,$$

hence

$$\frac{\sum_{k=1}^{n} \lambda_k y_k}{\max_{1\le j\le n} y_j \sum_{k=j}^{n} \lambda_k} \ge 1. \tag{8}$$

Consider the functional $\Phi : \mathcal{T} \times \mathcal{A} \to [1, \infty)$ defined by

$$\Phi(\lambda, y) = \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \lambda_k y_k}{\max_{1\le j\le n} y_j \sum_{k=j}^{n} \lambda_k}.$$

Lemma 2. *For any $y \in \mathcal{A}$ and $\lambda \in \mathcal{T}$ one can find $z \in \mathcal{A}$ and $\mu \in \mathcal{T}$ such that $z_{k+1} \ge 2z_k$, $k = 1, 2, 3, \dots$, and $\Phi(\lambda, y) \le 2\Phi(\mu, z)$.*

Proof. We construct a sequence $(n_i)_i$ of natural numbers by induction as follows. Let $n_1 = 1$. If $n_1, n_2, \dots, n_i$ are constructed, we put $n_{i+1} = \min\{n : y_n \ge 2y_{n_i}\}$. Now, defining z and μ by

$$z_k = y_{n_k}, \qquad \mu_k = \lambda_{n_k} + \lambda_{n_k+1} + \cdots + \lambda_{n_{k+1}-1},$$

we have

$$\begin{aligned}
\Phi(\lambda, y) &= \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \lambda_k y_k}{\max_{1\le j\le n} y_j \sum_{k=j}^{n} \lambda_k} \\
&\le \liminf_{m\to\infty} \frac{\sum_{k=1}^{n_m-1} \lambda_k y_k}{\max_{1\le i\le m-1} y_{n_i} \sum_{k=n_i}^{n_m-1} \lambda_k} \\
&\le \liminf_{m\to\infty} \frac{\sum_{r=1}^{m-1} 2y_{n_r} \sum_{k=n_r}^{n_{r+1}-1} \lambda_k}{\max_{1\le i\le m-1} y_{n_i} \sum_{r=i}^{m-1} \sum_{k=n_r}^{n_{r+1}-1} \lambda_k} \\
&= 2\liminf_{m\to\infty} \frac{\sum_{r=1}^{m-1} \mu_r z_r}{\max_{1\le i\le m-1} z_i \sum_{r=i}^{m-1} \mu_r} \\
&= 2\Phi(\mu, z).
\end{aligned}$$

This proves the assertion. □

Lemma 3. *Let $y \in \mathcal{A}$ be given with*

$$C = \sup_{k\ge 1} y_k \sum_{n=k}^{\infty} \frac{1}{\sum_{j=1}^{n} y_j} < \infty. \tag{9}$$

Then

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \lambda_k y_k}{\lambda_n \sum_{k=1}^{n} y_k} \le C$$

for every $\lambda \in \mathcal{T}$.

Proof. If the assertion is false, we find $d > C$ and $p \in \mathbf{N}$ such that

$$\sum_{k=1}^{n} \lambda_k y_k \ge d\lambda_n \sum_{k=1}^{n} y_k$$

for all $n \ge p$. For $q \ge p$, we have then

$$\sum_{n=p}^{q} \frac{1}{\sum_{j=1}^{n} y_j} \sum_{k=1}^{n} \lambda_k y_k \ge d \sum_{n=p}^{q} \lambda_n.$$

Interchanging the order of summation on the lefthand side, we obtain

$$\sum_{k=1}^{q} \lambda_k y_k \sum_{n=k}^{q} \frac{1}{\sum_{j=1}^{n} y_j} \ge d \sum_{n=p}^{q} \lambda_n,$$

which together with

$$y_k \sum_{n=k}^{q} \frac{1}{\sum_{j=1}^{n} y_j} \le y_k \sum_{n=k}^{\infty} \frac{1}{\sum_{j=1}^{n} y_j} \le C$$

implies that

$$C \sum_{k=1}^{q} \lambda_k \ge d \sum_{k=p}^{q} \lambda_k.$$

Letting q tend to infinity we get a contradiction, by (7) and by our choice of d. $\square$

Lemma 4. *For any $(\lambda, y) \in \mathcal{T} \times \mathcal{A}$, the estimate*

$$\Phi(\lambda, y) \le 8 \tag{10}$$

holds.

Proof. First let $y \in \mathcal{A}$ satisfy $y_{k+1} \ge 2y_k$, $k = 1, 2, 3, \dots$. We claim that y then satisfies the hypothesis (9) of Lemma 3. In fact, from $y_n \ge 2^{n-k} y_k$ for $n \ge k$, we get

$$y_k \sum_{n=k}^{\infty} \frac{1}{\sum_{j=1}^{n} y_j} \le y_k \sum_{n=k}^{\infty} \frac{1}{y_n} \le y_k \sum_{n=k}^{\infty} \frac{1}{2^{n-k} y_k} = 2,$$

which is (9) with $C = 2$. By Lemma 3, for every $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbf{N}$ such that

$$\Phi(\lambda, y) \le (2+\varepsilon) \frac{\lambda_n \sum_{k=1}^{N} y_k}{\max_{1\le j\le N} y_j \sum_{k=j}^{N} \lambda_k}. \tag{11}$$

Since

$$\sum_{k=1}^{N} y_k \le y_N \sum_{k=1}^{N} 2^{k-N} < 2y_N,$$

we conclude that

$$\frac{\lambda_N \sum_{k=1}^{N} y_k}{\max_{1\le j\le N} y_j \sum_{k=j}^{N} \lambda_k} \le \frac{2\lambda_N y_N}{y_N \lambda_N} = 2. \tag{12}$$

Combining (11) and (12) yields $\Phi(\lambda, y) \le 4$. For general y the proof is reduced to the above case by using Lemma 2. The assertion is proved. $\square$

We point out that the estimate (10) is nontrivial only for sequences $y = (y_k)_k$ satisfying (6). In fact, if $(y_k)_k$ is bounded, then $\Phi(\lambda, y) \equiv 1$ for all $\lambda \in \mathcal{T}$. To see this, fix $\varepsilon > 0$ and choose $j \in \mathbf{N}$ such that $||y||_\infty = \sup\{y_1, y_2, \dots\} \le (1+\varepsilon) y_j$; we then get

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^n \lambda_k y_k}{\max_{1\le j\le n} y_j \sum_{k=j}^n \lambda_k} \le \liminf_{n\to\infty} \frac{||y||_\infty \sum_{k=1}^n \lambda_k}{\frac{||y||_\infty}{1+\varepsilon} \sum_{k=j}^n \lambda_k} = 1+\varepsilon,$$

which together with the trivial estimate (8) proves the assertion.

Similarly, condition (7) is also important for the validity of the estimate (10). In fact, one can prove that, if $\lambda = (\lambda_k)_k$ is a positive sequence such that

$$\sum_{k=1}^\infty \lambda_k < \infty,$$

one can always find a sequence $y = (y_k)_k \in \mathcal{A}$ such that $\Phi(\lambda, y) = \infty$. To see this, it suffices to put

$$y_j = \frac{1}{\sum_{k=j}^\infty \lambda_k}.$$

In fact, from the convergence of the series $\sum_{k=1}^\infty \lambda_k$, it follows that the series

$$\sum_{k=1}^\infty \frac{\lambda_k}{\sum_{i=1}^\infty \lambda_i}$$

is divergent [**6**].

We turn now to the proof of the theorem. Let $x : [0,1] \to \mathbf{R}$ be a measurable function. Suppose first that $Q(x)$ is nonempty and fix $\psi \in Q(x)$. With the measurable function x, we associate the function

$$u(t) = \sum_{k=0}^\infty x^*(2^{-k}) \chi_{(2^{-k-1}, 2^{-k}]}(t).$$

Then

$$u(t) \le x^*(t) \le u(t/2)$$

and

$$\|u\|_E \le \|x\|_E \le 2\|u\|_E \tag{13}$$

for any r.i. space E, hence $Q(x) = Q(u)$. Fix $n \in \mathbf{N}$ and put $h = x^*(2^{-n})$. For any $\psi \in \Omega$, we have

$$\begin{aligned}
&\|u_h\|_{\Lambda(\psi)} \\
&\quad = \Bigg\| x^*(2^{-n})\chi_{(0,2^{-n}]} + \sum_{k=0}^{n-1} x^*(2^{-k})\chi_{(2^{-k-1},2^{-k}]} \Bigg\|_{\Lambda(\psi)} \\
&\quad \le 2 \Bigg\| \sum_{k=0}^{n} x^*(2^{-k})\chi_{(2^{-k-1},2^{-k}]} \Bigg\|_{\Lambda(\psi)} \\
&\quad = 2 \sum_{k=0}^{n} x^*(2^{-k})[\psi(2^{-k}) - \psi(2^{-k-1})] \\
&\quad \le 2 \sum_{k=0}^{n} x^*(2^{-k})\psi(2^{-k})
\end{aligned}$$

and

$$\begin{aligned}
\|u_h\|_{M(\psi)} &= \max_{0\le j\le n} \frac{\psi(2^{-j})}{2^{-j}} \int_0^{2^{-j}} u_h(t)\,dt \\
&\ge \max_{0\le j\le n} \psi(2^{-j})2^j \sum_{k=j}^{n} x^*(2^{-k})2^{-k-1}.
\end{aligned}$$

Putting

$$\lambda_k = x^*(2^{-k})2^{-k}, \qquad y_k = \psi(2^{-k})2^k$$

we get

$$\begin{aligned}
\liminf_{h\to\infty} \frac{\|u_h\|_{\Lambda_\psi}}{\|u_h\|_{M_\psi}} &\le \liminf_{n\to\infty} \frac{2\sum_{k=0}^{n} x^*(2^{-k})\psi(2^{-k})}{\max_{0\le j\le n} \psi(2^{-j})2^j \sum_{k=j}^{n} x^*(2^{-k})2^{-k-1}} \\
&= 4\Phi(\lambda, y).
\end{aligned} \tag{14}$$

The sequence $y = (y_k)_k$ is increasing and tends to infinity. Indeed, the boundedness of the sequence $\psi(2^{-k})2^k$ is equivalent to the fact that

$\psi(t) \sim ct$ for some $c > 0$. But in this case we have $\Lambda(\psi) = M(\psi) = L_1$, and there is nothing to prove.

Now the assumption $x \notin L_1$ implies (7), i.e., $\lambda = (\lambda_k)_k \in \mathcal{T}$. From (13), (14) and Lemma 4 we conclude that

$$\liminf_{h\to\infty} \frac{||x_h||_{\Lambda(\psi)}}{||x_h||_{M(\psi)}} \le 2 \liminf_{h\to\infty} \frac{||u_h||_{\Lambda(\psi)}}{||u_h||_{M(\psi)}} \le 8\Phi(\lambda, y) \le 64.$$

In this way we have shown that $Q(x) \neq \varnothing$ implies that $x \in L_1$; the fact that $Q(x) \neq \varnothing$ implies that $x \notin L_\infty$ is obvious.

Conversely, suppose now that $x \in L_1 \backslash L_\infty$. Putting

$$\psi(t) = \frac{t}{\int_0^t x^*(\tau)\, d\tau}, \tag{15}$$

it is not hard to see that the function

$$\hat{\psi}(t) = \int_0^t x^*(\tau)\, d\tau$$

is concave and (5) holds. By Lemma 1,

$$\int_0^1 \tilde{\psi}'(t) x^*(t)\, dt = \int_0^1 \tilde{\psi}'(t) \hat{\psi}'(t)\, dt = \infty,$$

which shows that $x \notin \Lambda(\tilde{\psi})$. On the other hand, it follows immediately from definition (15) that $x \in M(\psi)$, and hence $\psi \in Q(x)$. This finishes the proof of the theorem. $\square$

Acknowledgment. The authors express their gratitude to the referee for several valuable and interesting remarks.

REFERENCES

1. J. Appell and E.M Semenov, *On the equivalence of the Lorentz and Marcinkiewicz norm on subsets of measurable functions*, J. Funct. Anal. **104** (1992), 47–53.

2. ———, *The asymptotic behaviour of the norms of truncations of functions in near function spaces*, Doklady Akad. Nauk Rossii **328** (1993), 279–281, in Russian; *Russian Acad. Sci.*, Doklady Math. **47** (1993), 54–56, English translation.

3. C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, London, 1988.

4. J. Bergh and J. Löfström, *Interpolation spaces—an introduction*, Springer, Berlin, 1976.

5. Ju. A. Brudnyj, S.G. Krejn and E.M. Semenov, *Interpolation of linear operators*, Itogi Nauki Tekhniki **24** (1986), 3–163, in Russian; J. Soviet Math. **42** (1988), 2009–2113, English translation.

6. G.H. Hardy, *Divergent series*, Oxford University Press, Oxford, 1949.

7. S.G. Krein, Ju. I. Petunin and E.M. Semenov, *Interpolation of linear operators*, Nauka, Moscow, 1978, in Russian; Math. Mono., Amer. Math. Soc., Providence, RI, 1982, English translation.

8. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces* II, Springer, Berlin, 1979.

9. J. Marcinkiewicz, *Sur l'interpolation d'opérateurs*, C.R. Acad. Sci. Paris **208** (1939), 1272–1273.

10. M. Riesz, *Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires*, Acta Math. **49** (1926), 465–497.

11. E.M. Stein and G. Weiss, *An extension of a theorem of Marcinkiewicz and some of its applications*, J. Math. Mech. **8** (1959), 263–284.

12. G.O. Thorin, *An extension of a convexity theorem due to M. Riesz*, Comm. Sem. Math. Lund. **4** (1939), 1–5.

13. A. Zygmund, *On a theorem of Marcinkiewicz concerning interpolation*, J. Pure Appl. Math. **9** (1956), 223–248.

Universität Würzburg, Mathematisches Institut, Am Hubland, 97074 Würzburg, Germany

Voronezhskij Universitet, Matematicheskij Fakul'tet, Universitetskaja Pl. 1, 394693 Voronezh, Russia