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UNIQUE CONTINUATION OF WEAKLY CONFORMAL MAPPINGS BETWEEN RIEMANNIAN MANIFOLDS

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ABSTRACT. In this note we show that weakly conformal mappings between two Riemannian manifolds satisfy the strong unique continuation property in a sense that if the conformal factor of a weakly conformal mapping vanishes to infinite order at a point, then it is a constant map.

1. Introduction. Let (M, g) and (N, h) be Riemannian manifolds. We assume in this paper that M and N are connected. A smooth map $\phi : M \to N$ is said to be weakly conformal if $\phi^* h = \lambda g$ with $\lambda \geq 0$ and conformal if $\lambda > 0$ on M, where the scalar function λ is a so-called conformal factor of ϕ. It is well-known that solutions of elliptic partial differential equations satisfy the strong unique continuation property in a sense that if they vanish to infinite order at a point then they are identically zero. In this note we show that weakly conformal mappings between Riemannian manifolds also satisfy the strong unique continuation property. More precisely,

Theorem. *Let $\phi : M \to N$ be a weakly conformal mapping between two Riemannian manifolds of equal dimension. Then ϕ satisfies the strong unique continuation property; that is, given a point p in M, if the conformal factor λ of ϕ vanishes to infinite order at p, then $\phi \equiv \phi(p)$ on M.*

Corollary. *If two weakly conformal maps agree in a neighborhood of a point, then they are equal.*

Corollary. *Let $\phi : (M, g) \to (N, h)$ be a nonconstant weakly conformal mapping between two Riemannian manifolds of equal dimension. Then $\phi^*(h)$, the pullback of metric h on N, induces a metric on M ex-*

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cept at a discrete set of points on M and degenerates to at most finite order at those points.

Remark 1. In the CR geometry setting, one may ask the same question as follows: Given two strongly pseudoconvex CR manifolds (M, θ) and (N, ω) with the contact forms θ and ω, respectively, if $f : M \to N$ is a CR mapping, i.e., $f^*(\omega) = \lambda\theta$ on M, does f have unique continuation? This question would be very simple to answer if the dim M is greater than 5 because in this case one has the embedding theorem of complex structures. On the other hand, there have been results concerning unique continuation on holomorphic mappings or CR functions by Alinhac, Baouendi and Rothschild [**1**], Bell and Lempert [**2**], Rosay [**9**] (also see [**4**, **5**, **6**, **7**, and **8**].

Remark 2. We also obtain a result on weakly conformal harmonic maps. In [**3**], Sampson proved that if $\phi : M \to N$ is a weakly conformal harmonic map where dim M is greater than two and M is compact, then ϕ is homothetic, that is, its conformal factor is constant. We can prove a similar result by assuming that M is complete and has nonnegative Ricci curvature.

Proposition. *Suppose that* $\dim M > 2$. *If $\phi : M \to N$ is a weakly conformal harmonic map, then its conformal factor λ is harmonic. Consequently, if M is compact or M is complete and has nonnegative Ricci curvature, then ϕ is homothetic.*

2. Proof of theorem. Our observation is that if $\phi : M \to N$ is a weakly conformal mapping between Riemannian manifolds (M, g) and (N, h), then its conformal factor λ satisfies the estimate

$$|\Delta\lambda| \leq C|\lambda|$$

on M provided $\dim M = m > 2$. In the case of dimension two, we shall show that weakly conformal maps are complex harmonic functions.

To show this, let us first set up some basic notations and notions that are needed below. We write the metric g on M as follows

$$g = \sum (\theta^i)^2, \tag{1}$$

where θ^i is a base of the orthonormal 1-form in the tangent space TM. Similarly, we may write

$$h = \sum (\omega^i)^2, \tag{2}$$

where ω^i is a base of the orthonormal 1-form in the tangent space TN. If $\phi : M \to N$ is smooth, so we can define $a_{\alpha i}$ by the expression

$$\phi^*(\omega_\alpha) = \sum a_{\alpha i}\theta_i. \tag{3}$$

From (1), (2) and (3), we have

$$\phi^*(h) = \sum a_{\alpha i}a_{\alpha j}\omega_i\omega_j. \tag{4}$$

Let

$$\Lambda = \sum_{\alpha,i,j} a_{\alpha i}a_{\alpha j},$$

where the sum is taken over all α, i, j. The structure equations of M are

$$d\theta_i = \theta_i \wedge \theta_{ji}, \qquad \theta_{ij} + \theta_{ji} = 0$$
$$d\theta_{ij} = \sum \theta_{ik} \wedge \theta_{kj} + \Omega_{ij}$$

and

$$\Omega_{ij} = -1/2 \sum R_{ijkl}\theta_k \wedge \theta_l$$

are the structure 2-forms of M, and

$$R_{ik} = \sum R_{ij}k_j$$

are the components of the Ricci tensor of M. Similarly, the structure equations of N are

$$d\omega_a = \omega_\beta \wedge \omega_{\beta\alpha}, \qquad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0$$
$$d\omega_{\alpha,\beta} = \sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + K_{\alpha\beta}$$

and

$$K_{\alpha\beta} = -1/2 \sum K_{\alpha\beta\gamma\delta}\omega_\gamma \wedge \omega_\delta$$

are the structure 2-forms of N. For $y \in N$, the sectional curvature of the two-dimensional plane of T_yN spanned by the vectors ξ and η is

$$K(\xi,\eta) = \frac{K_{\alpha\beta\gamma\delta}\xi_\alpha\xi_\beta\eta_g\eta_\delta}{\sum\xi_\alpha^2\sum\eta_\alpha^2 - (\sum\xi_\alpha\eta_a)^2}.$$

By exterior differentiating (3) and using the structure equations of M and N, we obtain

$$(da_{\alpha i} + a_{\alpha k}\theta_{ki} + a_{\beta i}\phi^*(\omega_{\beta\alpha}))\wedge\theta_i = 0.$$

Let

$$da_{\alpha i} + a_{\alpha k}\theta_{ki} + a_{\beta i}\phi^*(\omega_{\beta\alpha}) = a_{\alpha ij}\theta_j. \tag{5}$$

Then we call $a_{\alpha ij}$ the covariant derivative of $a_{\alpha i}$, and we have

$$a_{\alpha ij} = a_{\alpha ji}. \tag{6}$$

By exterior differentiating (5), and using the structure equations, it follows that

$$(da_{\alpha ij} + a_{\alpha kj}\theta_{ki} + a_{\beta ij}\phi^*(\omega_{\beta\alpha}))\wedge\theta_j = a_{\alpha j}\Omega_{ji} + a_{\beta i}K_{\beta\alpha}.$$

Letting

$$(da_{\alpha ij} + a_{\alpha kj}\theta_{ki} + a_{\beta ij}\phi^*(\omega_{\beta\alpha}))\wedge\theta_j = a_{\alpha ijl}\theta_l,$$

we have the following Ricci identities

$$a_{\alpha ijl} - a_{\alpha ilj} = -a_{\alpha k}R_{kilj} - K_{\beta\alpha\gamma\delta}a_{\alpha i}a_{\gamma l}a_{\delta j}. \tag{7}$$

It follows from the definition of Λ that

$$\Delta\Lambda = \sum\Lambda_{jj} = 2\sum a_{\alpha ij}^2 + 2\sum a_{\alpha i}\alpha_{\alpha ijj}.$$

But, using (6) and (7), one gets

$$\begin{aligned}\sum a_{\alpha i}a_{\alpha ijj} = \sum a_{\alpha i}a_{\alpha jji} + \sum a_{\alpha j}a_{\alpha i}R_{ji}\\ + \sum K_{\beta\alpha\gamma\delta}a_{\alpha i}\alpha_{\beta j}a_{\gamma l}a_{\delta j}.\end{aligned} \tag{8}$$

Denoting

$$R = \sum a_{\alpha j} a_{\alpha i} R_{ij}, \quad \text{and} \quad K = \sum K_{\beta\alpha-\gamma\delta} a_{\alpha i} \alpha_{\beta j} a_{\gamma l} a_{\delta j},$$

it follows from (8) that, for any smooth map,

$$\Delta\Lambda = 2\sum a_{\alpha ij}^2 + 2\sum a_{\alpha i} a_{\alpha jji} + 2R - 2K. \tag{9}$$

On the other hand, if ϕ is weakly conformal, that is, $\phi^*(h) = \lambda g$, then

$$\sum a_{\alpha i} a_{\alpha j} = \delta_{ij} \lambda.$$

It follows, after differentiating twice, that for any i and j,

$$\sum a_{\alpha iji} a_{\alpha j} + a_{\alpha ij} a_{\alpha ji} + a_{\alpha ii} a_{\alpha jj} + a_{\alpha i} a_{\alpha jji} = \delta_{ij} \lambda_{ji}. \tag{10}$$

Then one gets, using (7), (8) and (11) and noting $\Lambda = m\lambda$,

$$\begin{aligned} \Delta\Lambda &= m\sum \lambda_{jj} = m\sum \delta_{ij}\lambda_{ji} \\ &= m\sum a_{\alpha ij}^2 + 2m\sum a_{\alpha i} a_{\alpha jji} + m\sum a_{\alpha ii} a_{\alpha jj}. \end{aligned} \tag{11}$$

We conclude that if ϕ is weakly conformal, then by subtracting (9) from (11), it satisfies the following equations

$$(m-2)\left\{ \sum a_{\alpha ij}^2 + \sum a_{\alpha ii} a_{\alpha jj} + (R-K) \right\} + (2m-2)\sum a_{\alpha i} a_{\alpha jji} = 0. \tag{12}$$

Since

$$\lambda = \sum_i a_{\alpha i}^2$$

for each α, using the Cauchy inequality we see that

$$|R| = \left| \sum a_{\alpha j} a_{\alpha i} R_{ij} \right| \leq C\left(\sum a_{\alpha j}^2 \right)^{1/2} \left(\sum a_{\alpha i}^2 \right)^{1/2} = mC\lambda$$

where C is a constant such that $|R_{ij}| \leq C$. Similarly, we have $|K| \leq C\lambda$. Now if $\dim M > 2$, from (12), using the Cauchy inequality again, we obtain

$$\begin{aligned} \sum a_{\alpha ii}^2 + \sum a_{\alpha ii} a_{\alpha jj} &= \frac{1}{m-2}(K-R) + \frac{2-2m}{m-2}\sum a_{\alpha i} a_{\alpha jji} \\ &\leq C(|R| + |K|) + C\sum a_{\alpha i}^2 \sum a_{\alpha jji}^2 \\ &\leq C\lambda. \end{aligned} \tag{13}$$

We point out that the lefthand side of the above inequality is nonnegative. Now combining (11) and (13), it follows that

$$|\Delta\Lambda| \leq C\lambda.$$

Since $\Lambda = m\lambda$ we have $|\Delta\lambda| \leq C/m\lambda$. So the unique continuation property of a weakly conformal mapping follows from the well-known result for elliptic partial differential inequality applied to the conformal factor λ to conclude that $\lambda \equiv 0$ and hence $\phi(x) \equiv \phi(p)$.

Now we discuss the situation for dimension two. We consider M and N locally as the complex plane C with metrics $e^u|dz|^2$ and $e^v|dw|^2$, respectively. So a weakly conformal map is a complex function satisfying

$$f^*(e^v|dw|^2) = \lambda e^u|dz|^2.$$

If we write $d = \partial + \bar{\partial}$, then

$$\begin{aligned} f^*(e^v|dw|^2) &= e^{v\circ f}(\partial f + \bar{\partial} f)(\overline{\partial f} + \partial \bar{f}) \\ &= \lambda e^u|dz|^2. \end{aligned}$$

This implies that

$$\frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z} = 0,$$

from which it can be seen that

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0.$$

This is to say f is harmonic, and therefore the unique continuation follows.

Proof of Proposition. If ϕ is harmonic, then

$$\sum a_{\alpha kk} = 0$$

for all α. Then from (12), we see

$$\sum a_{\alpha ij}^2 + R - K = 0$$

provided $\dim M > 2$. It follows that

$$\Delta\lambda = 0,$$

i.e., λ is a harmonic function on M. Therefore, we conclude that λ is constant on M by a theorem of Yau [**10**] which states that any positive harmonic function on a complete Riemannian manifold with nonnegative Ricci curvature is constant. □

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