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AN OSCILLATION CRITERION OF ALMOST-PERIODIC STURM-LIOUVILLE EQUATIONS

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ABSTRACT. The class $\Omega \subset L^1_{\text{loc}}(\mathbf{R})$ of Besicovitch almost-periodic functions is the closure of the set of all finite trigonometric polynomials with the Besicovitch seminorm. Consider the half-linear second order differential equation

$$(E) \qquad \frac{d}{dt}\phi(u'(t)) - \lambda c(t)\phi(u(t)) = 0,$$

where $\phi(s) = |s|^{p-2}s$ with $p > 1$ a fixed number and $c(t) \in \Omega$. We show that if $M\{c\} := \lim_{t\to\infty}(1/t)\int_0^t c(s+\alpha)\,ds = 0$ and $M\{|c|\} > 0$, then (E) is oscillatory at $+\infty$ and $-\infty$ for every $\lambda \in \mathbf{R} - \{0\}$.

1. Introduction. Let $\mathbf{R}$ denote the real line. The class $\Omega \subset L^1_{\text{loc}}(\mathbf{R})$ of Besicovitch almost-periodic functions is the closure of the set of all finite trigonometric polynomials with the Besicovitch seminorm $||\cdot||_B$:

$$||c||_B = \limsup_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} |c(s)|\,ds,$$

where $c \in \Omega$. The mean value, $M\{c\}$, of $c \in \Omega$, always exists, is finite and is uniform with respect to α for $\alpha \in \mathbf{R}$, where

$$M\{c\} = \lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} c(s+\alpha)\,ds,$$

for some $t_0 \geq 0$ (see [**1**] and [**3**] for details).

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Consider the following half-linear second order differential equation

$$\text{(E)} \qquad \frac{d}{dt}\phi(u'(t)) - \lambda c(t)\phi(u(t)) = 0,$$

where

(i) $\phi : \mathbf{R} \to \mathbf{R}$ is defined by $\phi(s) = |s|^{p-2}s$, with $p > 1$ a fixed real number,

(ii) $c(t) \in \Omega$.

If $p = 2$, then (E) becomes the second order linear differential equation

$$u''(t) - \lambda c(t)u(t) = 0.$$

If $u > 0$ and $u' > 0$ (or $u' < 0$), then (E) is reduced to the Euler-Lagrange differential equation

$$\frac{d}{dt}[u'(t)]^{p-1} - \lambda c(t)u^{p-1}(t) = 0,$$

or

$$\frac{d}{dt}[-u'(t)]^{p-1} + \lambda c(t)u^{p-1}(t) = 0.$$

Half-linear equation (E) was first considered by Bihari [**2**] in 1957 and then Elbert [**5**] in 1987. For other related papers, we refer the reader to Kaper, Knaap and Kwong [**8**], Lalli and Kusano [**9**], and Pino and Manasevich [**13**].

In [**5**], Elbert established the existence and uniqueness of solutions to the initial value problem for (E) on $[T, \infty)$, for some $T \geq 0$. Note that any constant multiple of a solution of (E) is also a solution.

We say that equation (E) is oscillatory at $+\infty$ and $-\infty$ if every solution of (E) has an infinity of zeros clustering only at $+\infty$ and $-\infty$, respectively.

In 1989, Dzurnak and Mingarelli [**4**] proved the following very interesting result by using Levin's comparison theorem [**10**].

Theorem A. *Let $c \in \Omega$ and $M\{|c|\} > 0$. If $p = 2$, then $M\{c\} = 0$ if and only if* (E) *is oscillatory at $+\infty$ and $-\infty$ for every $\lambda \in \mathbf{R} - \{0\}$.*

Recently, Wong and Yeh [**14**] used the nonlinear Levin comparison theorem [**15**] to extend the sufficient condition of Theorem A to the following second nonlinear differential equation

$$u''(t) - \lambda c(t) f(u(t)) = 0,$$

if f satisfies some suitable conditions.

The main purpose of this paper is to extend Theorem A to equation (E), which involves constructing three hard-to-come-by auxiliary functions (see (9), (16) and (17) below).

For other related results, we refer the reader to [**6, 11, 12**].

2. Main result. In order to discuss our main results, we need the following three lemmas, the first one is a half-linear extension of Levin's comparison theorem [**10**].

Lemma 2.1. *Let* $c_1, c_2 \in L^1_{\text{loc}}(\mathbf{R})$*, and let* $u(t)$ *and* $v(t)$ *be nontrivial solutions of*

$$(\mathrm{E}_1) \qquad u(t)\left\{\frac{d}{dt}\phi(u'(t)) - c_1(t)\phi(u(t))\right\} \le 0$$

and

$$(\mathrm{E}_2) \qquad \frac{d}{dt}\phi(v'(t)) - c_2(t)\phi(v(t)) = 0,$$

respectively, on a closed subinterval $[\alpha, \beta]$ *of* $[T, \infty)$ *satisfying either*

(i) $v(\alpha) \ge u(\alpha) > 0$*,* $u > 0$ *on* $[\alpha, \beta]$*,*

or

(ii) $v(\alpha) \le u(\alpha) < 0$*,* $u < 0$ *on* $[\alpha, \beta]$*.*

If

$$(\mathrm{H}_1) \qquad -\frac{\phi(u'(\alpha))}{\phi(u(\alpha))} - \int_\alpha^t c_1(s)\,ds > \left| -\frac{\phi(v'(\alpha))}{\phi(v(\alpha))} - \int_\alpha^t c_2(s)\,ds \right|$$

for all $t \in [\alpha, \beta]$*, then* $v(t)$ *does not vanish on* $[\alpha, \beta]$*, and we have* $u(t)u'(t) < 0$*,* $v(t) \ge u(t) > 0$ *if* (i) *holds,* $v(t) \le u(t) < 0$ *if* (ii) *holds, and*

$$(\mathrm{R}_1) \qquad -\frac{\phi(u'(t))}{\phi(u(t))} > \left|\frac{\phi(v'(t))}{\phi(v(t))}\right|$$

for all $t \in [\alpha, \beta]$. If the inequality sign "$>$" in (H_1) *is replaced by "$\geq$," then the inequality sign "$>$" in* (R_1) *should be replaced by "$\geq$."*

Proof. Since the proofs for (i) and (ii) are similar, we prove only case (i). Since $u > 0$ on $[\alpha, \beta]$, the continuous function

$$w(t) := -\frac{\phi(u'(t))}{\phi(u(t))} \tag{1}$$

on $[\alpha, \beta]$ satisfies

$$\begin{aligned} w(t) &\geq w(\alpha) - \int_\alpha^t c_1(s)\,ds + (p-1)\int_\alpha^t |w(s)|^q\,ds \\ &\geq w(\alpha) - \int_\alpha^t c_1(s)\,ds > 0, \end{aligned} \tag{2}$$

where $1/p + 1/q = 1$. Thus, $u'(t) < 0$ for all $t \in [\alpha, \beta]$. Since $v(\alpha) > 0$,

$$z(t) := -\frac{\phi(v'(t))}{\phi(v(t))} \tag{3}$$

is continuous on some interval $[\alpha, \gamma]$, where $\alpha < \gamma < \beta$. Clearly, $z(t)$ satisfies the integral equation

$$z(t) = z(\alpha) - \int_\alpha^t c_2(s)\,ds + (p-1)\int_\alpha^t |z(s)|^q\,ds \tag{4}$$

for all $t \in [\alpha, \gamma]$. From (4), (H_1) and (2), we obtain

$$z(t) \geq z(\alpha) - \int_\alpha^t c_2(s)\,ds > -w(\alpha) + \int_\alpha^t c_1(s)\,ds \geq -w(t).$$

Hence $w(t) > -z(t)$ on $[\alpha, \gamma]$. Next, we claim that $w(t) > z(t)$ on $[\alpha, \gamma]$. Suppose to the contrary that there exists a point $t_0 \in [\alpha, \gamma]$ such that $w(t_0) \leq z(t_0)$. From (H_1), we have $w(\alpha) > |z(\alpha)|$. As $w(t)$ and $z(t)$ are continuous on $[\alpha, \gamma]$, there exists $t_1 \in [\alpha, t_0]$ such that $z(t_1) = w(t_1)$ and $z(t) < w(t)$ on $t \in [\alpha, t_1)$. Since we have established $w(t) > -z(t)$

on $[\alpha, \gamma]$, we obtain that $|z(t)| < w(t)$ for all $t \in [\alpha, t_1)$. Thus, it follows from (H_1), (2) and (4) that

$$\begin{aligned} z(t_1) &= z(\alpha) - \int_\alpha^{t_1} c_2(s)\,ds + (p-1)\int_\alpha^{t_1} |z(s)|^q\,ds \\ &< w(\alpha) - \int_\alpha^{t_1} c_1(s)\,ds + (p-1)\int_\alpha^{t_1} |w(s)|^q\,ds \\ &\le w(t_1), \end{aligned}$$

which is a contradiction. Thus

$$|z(t)| < w(t) \tag{5}$$

for $t \in [\alpha, \gamma]$.

Next we show that $v(t)$ cannot vanish on $[\alpha, \beta]$. Suppose that the first point to the right of α at which $v(t)$ vanishes is $t = \delta \le \beta$, that is, $v(t) > 0$ on $[\alpha, \delta)$ and $v(\delta) = 0$. We claim that $v'(\delta) \ne 0$. Suppose that $v'(\delta) = 0$. Then, for $t \in [\alpha, \delta]$,

$$\phi(v'(t)) = \int_t^\delta c_2(s)\phi(v(s))\,ds,$$

which implies

$$v'(t) = \phi^{-1}\left\{\int_t^\delta c_2(s)\phi(v(s))\,ds\right\},$$

where $\phi^{-1}(s) = |s|^{q-2}s$ is the inverse function of ϕ. Thus

$$v(t) - v(\delta) = -\int_t^\delta \phi^{-1}\left\{\int_x^\delta c_2(s)\phi(v(s))\,ds\right\}dx.$$

Hence,

$$\phi(v(t)) \le (\delta-\alpha)^{p-1}\int_t^\delta |c_2(s)|\phi(v(s))\,ds \quad \text{for } \alpha \le t \le \delta.$$

It follows from the Gronwall inequality that $v(t) \equiv 0$ for each $t \in [\alpha, \delta]$, which is impossible. Thus $v'(\delta) \ne 0$. This means that the solutions of

(E_2) have only simple zeros. However, since $|z(t)| < w(t)$ on $[\alpha, \delta)$ and $w(t)$ is bounded on $[\alpha, \beta]$, we get

$$\infty = \limsup_{t\to\delta^-} |z(t)| \le \lim_{t\to\delta^-} w(t) = w(\delta) < \infty,$$

which is absurd. This contradiction proves that $v(t)$ cannot vanish on $[\alpha, \beta]$. Thus, (5) holds on any interval $[\alpha, \gamma] \subset [\alpha, \beta]$ on which z is continuous. But this implies that z is continuous on the entire interval $[\alpha, \beta]$ since $w(t)$ is bounded on $[\alpha, \beta]$ and $v(t)$ cannot vanish on $[\alpha, \beta]$. Thus, (5) holds on the interval $[\alpha, \beta]$.

Clearly, it follows from (1), (3) and (5) that $v(t) \ge u(t)$ on $[\alpha, \beta]$. Hence our proof is complete. □

Lemma 2.2. *If $\gamma > 1$, $a > 0$ and $b > 0$, then $(a + b)^\gamma > a^\gamma + b^\gamma$.*

Lemma 2.3. *Suppose that*

(C_1) $c : [t_0, \infty) \to \mathbf{R}$ *is locally Lebesgue integrable and has a mean value $M\{c\}$, where $t_0 \ge 0$,*

(C_2) $$M\{c\} = 0$$

If $u(t) \ne 0$ is a solution of

(E_3) $$\frac{d}{dt}\phi(u'(t)) - c(t)\phi(u(t)) = 0$$

on $[t_0, \infty)$, then

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \left|\frac{u'(s)}{u(s)}\right|^p ds = 0.$$

Proof. Define

$$x(t) = -\frac{\phi(u'(t))}{\phi(u(t))} \quad \text{for all } t \in [t_0, \infty).$$

It follows from (E_3) that $x(t)$ is a solution of

$$x'(t) - (p-1)|x(t)|^q + c(t) = 0 \quad \text{on } [t_0, \infty), \tag{6}$$

where $1/q + 1/p = 1$. Since $|x(t)|^q = |u'(t)/u(t)|^p \geq 0$ on $[t_0, \infty)$, it suffices to show that

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} |x(s)|^q \, ds = 0.$$

Assume to the contrary that

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} |x(s)|^q \, ds > 0. \tag{7}$$

Integrating (6) from t_0 to t and dividing it by t, we have

$$\frac{x(t)}{t} = \frac{x(t_0)}{t} + \frac{1}{t} \int_{t_0}^{t} c(s) \, ds + \frac{p-1}{t} \int_{t_0}^{t} |x(s)|^q \, ds, \tag{8}$$

for all $t > t_0$. It follows from (7), (8) and (C_2) that there exist a positive constant m and an increasing sequence $\{t_n\}_{n=1}^{\infty}$ of (t_0, ∞) with $\lim_{n\to\infty} t_n = \infty$ such that

$$\frac{x(t_n)}{t_n} > (p-1)m^p \quad \text{for all } n \text{ large enough.} \tag{9}$$

It follows from (C_2) that there exists t^* large enough such that

$$\left| \int_{t_0}^{t} c(s) \, ds \right| < (p-1)(m/2)^p t \quad \text{for all } t \geq t^*. \tag{10}$$

Using (10), we have

$$\begin{aligned} \int_{t_n}^{t} c(s) \, ds &= \int_{t_0}^{t} c(s) \, ds - \int_{t_0}^{t_n} c(s) \, ds \\ &< (p-1)(m/2)^p (t + t_n) \end{aligned} \tag{11}$$

for all $t \geq t_n \geq t^*$. It follows from (9) and (11) that

(12)

$$\begin{aligned} x(t_n) - \int_{t_n}^{t} c(s) \, ds &> (p-1)m^p t_n - (p-1)(m/2)^p (t + t_n) \\ &\geq (p-1)m^p t_n - (p-1)(m/2)^p [(2^p - 1)t_n + t_n] \\ &= 0 \end{aligned}$$

for all $t \in [t_n, (2^p-1)t_n] \subset [t^*, \infty)$. From the existence of solutions to the initial value problem, the differential equation

$$\frac{d}{dt}\phi(u_n'(t)) - (p-1)(m/2)^p \phi(u_n(t)) = 0 \tag{13}$$

has a solution $u_n(t)$ on $[t_n, (2^p-1)t_n]$ satisfying $u_n(t_n) = u(t_n)$ and

$$-\frac{\phi(u_n'(t_n))}{\phi(u_n(t_n))} = x(t_n) - 2(p-1)(m/2)^p t_n.$$

It follows from (11) and (12) that

$$\begin{aligned}
-\frac{\phi(u'(t_n))}{\phi(u(t_n))} - \int_{t_n}^{t} c(s)\,ds &= x(t_n) - \int_{t_n}^{t} c(s)\,ds \\
&> x(t_n) - (p-1)(m/2)^p (t+t_n) \\
&= \{x(t_n) - 2(p-1)(m/2)^p t_n\} \\
&\quad - (p-1)(m/2)^p (t-t_n) \\
&= -\frac{\phi(u_n'(t_n))}{\phi(u_n(t_n))} - \int_{t_n}^{t} (p-1)(m/2)^p\,ds \\
&\geq 0
\end{aligned}$$

on $[t_n, (2^p-1)t_n] \subset (t^*, \infty)$. Using Lemma 2.1, we have

$$-\frac{\phi(u'(t_n))}{\phi(u(t_n))} > \left| -\frac{\phi(u_n'(t))}{\phi(u_n(t))} \right| \quad \text{on } [t_n, (2^p-1)t_n] \subset [t^*, \infty). \tag{14}$$

Now, define

$$x_n(t) = -\frac{\phi(u_n'(t))}{\phi(u_n(t))} \quad \text{on } [t_n, (2^p-1)t_n] \subset [t^*, \infty).$$

It is clear that $x_n(t)$ is a solution of the differential equation

$$x_n'(t) - (p-1)|x_n(t)|^q + (p-1)(m/2)^p = 0 \tag{15}$$

on $[t_n, (2^p-1)t_n] \subset [t^*, \infty)$ with

$$x_n(t_n) = x(t_n) - 2(p-1)(m/2)^p t_n.$$

Let

$$r_n = [x_n(t_n) - (m/2)^{p/q}]^{1-q} \tag{16}$$

and

$$y_n(t) = (m/2)^{p/q} + (t_n - t + r_n)^{1-p} \tag{17}$$

on $[t_n, t_n + r_n) \subset [t^*, \infty)$, where n is large enough such that $x_n(t_n) > (m/2)^{p/q}$. Then $y_n(t_n) = x_n(t_n)$, and it follows from Lemma 2.2 that

$$\begin{aligned}
y_n'(t) &= (p-1)(t_n - t + r_n)^{-p} \\
&= (p-1)[(t_n - t + r_n)^{-p} + (m/2)^p] - (p-1)(m/2)^p \\
&< (p-1)[(m/2)^{p/q} + (t_n - t + r_n)^{1-p}]^q - (p-1)(m/2)^p \\
&= (p-1)|y_n(t)|^q - (p-1)(m/2)^p \quad \text{on } [t_n, t_n + r_n) \subset [t^*, \infty).
\end{aligned}$$

Thus

$$\begin{aligned}
y_n'(t) - (p-1)|y_n(t)|^q + (p-1)(m/2)^p &< 0 \\
&= x_n'(t) - (p-1)|x_n(t)|^q + (p-1)(m/2)^p
\end{aligned}$$

for all $t \in [t_n, (2^p - 1)t_n] \cap [t_n, t_n + r_n) \subset [t^*, \infty)$. A simple comparison argument shows that

$$y_n(t) \le x_n(t) \quad \text{on } [t_n, (2^p - 1)t_n] \cap [t_n, t_n + r_n) \subset [t^*, \infty).$$

It follows from

$$x_n(t_n) = x(t_n) - 2(p-1)(m/2)^p t_n > (p-1)(1 - 2^{1-p}) m^p t_n$$

that $t_n + r_n \in [t_n, (2^p - 1)t_n]$ for n large enough. By the definition of $y_n(t)$, we see that

$$\lim_{t \to (t_n + r_n)^-} y_n(t) = \infty \quad \text{for } n \text{ large enough.}$$

Hence,

$$\lim_{t \to (t_n + r_n)^-} x_n(t) = \infty \quad \text{for } n \text{ large enough.} \tag{18}$$

Now we take k large enough such that

$$t_k + r_k \in [t_k, (2^p - 1)t_k].$$

Clearly, there exists a positive constant M such that

$$-\frac{\phi(u'(t_n))}{\phi(u(t_n))} \leq M < \infty \quad \text{on } [t_k, (2^p-1)t_k] \subset [t^*, \infty).$$

It follows from (14) and (18) that

$$\infty = \lim_{t \to (t_k + r_k)^-} x_n(t) \leq \lim_{t \to (t_k + r_k)^-} \left\{ -\frac{\phi(u'(t_n))}{\phi(u(t_n))} \right\} \leq M < \infty,$$

which is a contradiction. Thus, the proof is complete. □

Theorem 2.4. *If $c \in \Omega$ satisfies (C_2) and $M\{|c|\} > 0$, then* (E) *is oscillatory at $+\infty$ and $-\infty$ for every $\lambda \in \mathbf{R} - \{0\}$.*

Proof. Without loss of generality, we only show that (E_3) is oscillatory at $+\infty$. Assume to the contrary that (E_3) has a solution $u(t)$ which is nonoscillatory at $+\infty$. Thus, we can assume that there exists $t_0 > 0$ such that $u(t) > 0$ on $[t_0, \infty)$. Define

$$x(t) = -\frac{\phi(u'(t_n))}{\phi(u(t_n))} \quad \text{for all } t \in [t_0, \infty).$$

Then $x(t)$ is a solution of (6) on $[t_0, \infty)$. Hence, for any fixed $\delta > 0$, we have

(19)
$$\frac{1}{\delta}\int_t^{t+\delta} c(s)\, ds = -\frac{x(t+\delta)}{\delta} + \frac{x(t)}{\delta} + \frac{p-1}{\delta}\int_t^{t+\delta} |x(s)|^q\, ds \quad \text{on } [t_0, \infty).$$

Applying the Besicovitch semi-norm $||\cdot||_{B'}$, essentially a restriction of $||\cdot||_B$ to the interval $[t_0, \infty)$, defined by

$$||f||_{B'} = \limsup_{t \to \infty} \frac{1}{t}\int_{t_0}^t |f(s)|\, ds,$$

to (19), we find

$$
(20)\qquad \begin{aligned} 0 \le \left\| \frac{1}{\delta}\int_t^{t+\delta} c(s)\,ds \right\|_{B'} &\le \left\| \frac{p-1}{\delta}\int_t^{t+\delta} |x(s)|^q\,ds \right\|_{B'} \\ &+ \left\| \frac{x(t+\delta)}{\delta} \right\|_{B'} + \left\| \frac{x(t)}{\delta} \right\|_{B'} \end{aligned}
$$

for all $\delta > 0$. It follows from Lemma 2.3 that $M\{|x|^q\} = 0$; thus, $||x||_{B'} = ||x(t+\delta)||_{B'} = 0$ for all $\delta > 0$. Using Fubini's theorem, we have

$$
(21)\qquad \begin{aligned} \frac{1}{t\delta}\int_{t_0}^{t}\int_s^{s+\delta} |x(r)|^q\,dr\,ds &= \frac{1}{t\delta}\int_{t_0}^{t}\int_0^{\delta} |x(r+s)|^q\,dr\,ds \\ &= \frac{1}{t\delta}\int_0^{\delta}\int_{t_0}^{t} |x(r+s)|^q\,ds\,dr \\ &\le \frac{1}{t\delta}\int_0^{\delta}\int_{t_0}^{t+\delta} |x(s)|^q\,ds\,dr \\ &= \frac{1}{t}\int_{t_0}^{t+\delta} |x(s)|^q\,ds \end{aligned}
$$

for any fixed $\delta > 0$. Using (21) and Lemma 2.3, we have

$$
(22)\qquad \left\| \frac{p-1}{\delta}\int_t^{t+\delta} |x(s)|^q\,ds \right\|_{B'} = 0 \quad \text{for any fixed } \delta > 0.
$$

Applying (22) and $||x||_{B'} = ||x(t+\delta)||_{B'} = 0$ to (21), we see that

$$
(23)\qquad \left\| \frac{1}{\delta}\int_t^{t+\delta} c(s)\,ds \right\|_{B'} = 0 \quad \text{for all } \delta > 0.
$$

Since c is Besicovitch almost periodic, it follows from Besicovitch [**1**, page 97] that

$$
\lim_{\delta\to 0} \left\| c(t) - \frac{1}{\delta}\int_t^{t+\delta} c(s)\,ds \right\|_{B'} = 0.
$$

This and (23) imply $M\{|c|\} = ||c||_{B'} = 0$, which is a contradiction. Thus, the proof is complete. □

Example. Consider the differential equation

$$\frac{d}{dt}\phi(u'(t)) - \lambda\cos(t)\phi(u(t)) = 0. \tag{24}$$

Then $c(t) = \cos(t)$. Thus,

$$M\{c\} = \lim_{t\to\infty}\frac{1}{t}\int_0^t c(s)\,ds = \lim_{t\to 0}\frac{\sin(t)}{t} = 0$$

and

$$\begin{aligned} M\{|c|\} &= \lim_{t\to\infty}\frac{1}{t}\int_0^t |c(s)|\,ds \\ &= \lim_{n\to\infty}\frac{1}{2(n+1)\pi}\int_0^{2(n+1)\pi} |\cos(s)|\,ds \\ &= \lim_{n\to\infty}\frac{2}{\pi}\int_0^{\pi/2}\cos(s)\,ds \\ &= \frac{2}{\pi} > 0. \end{aligned}$$

It follows from Theorem 2.4 that for each $\lambda \in \mathbf{R} - \{0\}$, (24) is oscillatory at $+\infty$ and $-\infty$.

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