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NONLINEAR OSCILLATION OF FIRST ORDER DELAY DIFFERENTIAL EQUATIONS

NORIO YOSHIDA

ABSTRACT. First order delay differential equations with forcing term and the related differential equations are studied and sufficient conditions are derived for all solutions to be oscillatory.

1. Introduction. Oscillation properties of first order functional differential equations has been investigated by many authors. We refer the reader to Bainov and Mishev [**1**], Györi and Ladas [**3**], Ladde, Lakshmikantham and Zhang [**4**] and the references cited therein. In particular, the oscillation of functional differential equations with forcing term was studied by Onose [**5, 6**] and Tomaras [**7**]. However, it seems that very little is known about the sufficient conditions which imply that all solutions of certain nonlinear functional differential equation with forcing term are oscillatory.

The objective of this paper is to establish oscillation criteria for the delay differential equation

$$y'(t) + p(t)f(y(\sigma(t))) = q(t), \qquad t > t_0, \tag{1}$$

and the related differential equation

$$y'(t)+a(t)y(t)+\sum_{i=1}^{k} b_i(t)y(\rho_i(t))+p(t)f(y(\sigma(t))) = q(t), \qquad t>t_0, \tag{2}$$

where t_0 is a positive number. In what follows, by a *solution* of (1) or (2), we mean a function $y(t) \in C([t_{-1}, \infty); \mathbf{R}^1) \cap C^1([t_0, \infty); \mathbf{R}^1)$ which satisfies (1) or (2) for all $t > t_0$, where

$$t_{-1} = \begin{cases} \inf_{t\geq t_0} \sigma(t) & \text{in the case of (1)} \\ \min\{\inf_{t\geq t_0} \sigma(t), \min_{1\leq i\leq k} \inf_{t\geq t_0} \rho_i(t)\} & \text{in the case of (2).} \end{cases}$$

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A solution of (1) or (2) is said to be *oscillatory* if it has arbitrarily large zeros. In Section 2 we consider equation (1), and in Section 3 the oscillation results of Section 2 are extended to equation (2).

2. Delay differential equations. In this section we derive sufficient conditions for every solution $y(t)$ of (1) to be oscillatory. The following hypotheses are assumed to hold:

(H1) $p(t) \in C([t_0, \infty); [0, \infty))$ and $q(t) \in C([t_0, \infty); \mathbf{R}^1)$;

(H2) $f(s) \in C(\mathbf{R}^1; \mathbf{R}^1)$, $f(s) \geq 0$ for $s \geq 0$, $f(-s) = -f(s)$ for $s > 0$, and $f(s)$ is nondecreasing for $s \geq 0$;

(H3) $\sigma(t) \in C([t_0, \infty); \mathbf{R}^1)$, $\lim_{t\to\infty} \sigma(t) = \infty$, $\sigma(t) \leq t$ for $t \geq t_0$, and $\sigma(t)$ is nondecreasing for $t \geq t_0$.

Theorem 1. *Assume that* (H1)–(H3) *hold. Assume, moreover, that the following hypothesis holds*:

(H4) *there exists a positive constant* β *such that* $f(s) \geq \beta s$ *in* $(0, \infty)$.

If there is a sequence $\{t_n\} \subset (t_0, \infty)$ *such that*

$$\begin{aligned}
&\lim_{n\to\infty} t_n = \infty, \qquad \sigma(\sigma(t_n)) > t_0, \\
&q(t) = 0 \quad \text{in } [\sigma(\sigma(t_n)), \sigma(t_n)], \\
&\int_{\sigma(t_n)}^{t_n} q(s)\, ds = 0, \\
&\int_{\sigma(t_n)}^{t_n} p(s)\, ds \geq \frac{1}{\beta},
\end{aligned}$$

then every solution of (1) *is oscillatory.*

Proof. Suppose to the contrary that there is a solution $y(t)$ of (1) which has no zero in $[t_1, \infty)$ for some $t_1 > t_0$. First we assume that $y(t) > 0$ in $[t_1, \infty)$. Since $\lim_{t\to\infty} \sigma(t) = \infty$, there exists a number $t_2 \geq t_1$ such that $y(\sigma(t)) > 0$ in $[t_2, \infty)$. Then, equation (1) implies that

$$y'(t) \leq q(t), \qquad t \geq t_2. \tag{3}$$

If $s \in [\sigma(t), t]$, then we see that $\sigma(s) \leq \sigma(t)$. Integrating (3) over $[\sigma(s), \sigma(t)]$ yields

$$y(\sigma(t)) - y(\sigma(s)) \leq \int_{\sigma(s)}^{\sigma(t)} q(r)\, dr, \qquad s \in [\sigma(t), t],$$

which is equivalent to

$$y(\sigma(s)) \geq y(\sigma(t)) - \int_{\sigma(s)}^{\sigma(t)} q(r)\, dr, \qquad s \in [\sigma(t), t].$$

It follows from (H2) that $f(s)$ is nondecreasing in $\mathbf{R}^1$ and therefore

$$f(y(\sigma(s))) \geq f\left(y(\sigma(t)) - \int_{\sigma(s)}^{\sigma(t)} q(r)\, dr\right), \qquad s \in [\sigma(t), t]. \tag{4}$$

We integrate (1) over $[\sigma(t), t]$ and find

$$y(t) - y(\sigma(t)) + \int_{\sigma(t)}^{t} p(s) f(y(\sigma(s)))\, ds = \int_{\sigma(t)}^{t} q(s)\, ds, \qquad t \geq t_2. \tag{5}$$

Combining (4) with (5), we obtain

$$\begin{aligned} y(t) - y(\sigma(t)) + \int_{\sigma(t)}^{t} p(s) f\left(y(\sigma(t)) - \int_{\sigma(s)}^{\sigma(t)} q(r)\, dr\right) ds& \\ \leq \int_{\sigma(t)}^{t} q(s)\, ds, \qquad t \geq t_2,& \end{aligned}$$

or equivalently,

$$\begin{aligned} y(t) &+ y(\sigma(t))\left(\frac{f(y(\sigma(t)))}{y(\sigma(t))} \int_{\sigma(t)}^{t} p(s)\, ds - 1\right) \\ &+ \int_{\sigma(t)}^{t} p(s)\left[f\left(y(\sigma(t)) - \int_{\sigma(s)}^{\sigma(t)} q(r)\, dr\right) - f(y(\sigma(t)))\right] ds \\ &\leq \int_{\sigma(t)}^{t} q(s)\, ds, \qquad t \geq t_2. \end{aligned} \tag{6}$$

By the hypothesis (H4) there exists $T \geq t_2$ for which

$$\frac{f(y(\sigma(t)))}{y(\sigma(t))} \geq \beta \quad \text{for } t \geq T. \tag{7}$$

We see from (6) and (7) that

$$\begin{aligned} y(t) + \beta y(\sigma(t))\left(\int_{\sigma(t)}^{t} p(s)\,ds - \frac{1}{\beta}\right) \\ + \int_{\sigma(t)}^{t} p(s)\left[f\left(y(\sigma(t)) - \int_{\sigma(s)}^{\sigma(t)} q(r)\,dr\right) - f(y(\sigma(t)))\right] ds \\ \leq \int_{\sigma(t)}^{t} q(s)\,ds, \qquad t \geq T. \end{aligned} \tag{8}$$

There exists a number $n_0 \in \mathbf{N}$ such that $t_n > T$ for any $n \geq n_0$. Letting $t = t_n$ in (8), we find that the lefthand side of (8) is positive in view of the fact that

$$\int_{\sigma(s)}^{\sigma(t_n)} q(r)\,dr = 0 \quad \text{for } s \in [\sigma(t_n), t_n].$$

However, the righthand side of (8) is zero. This is a contradiction. In the case where $y(t) < 0$ in $[t_1, \infty)$ for some $t_1 > t_0$, $z(t) \equiv -y(t)$ satisfies

$$z'(t) + p(t)f(z(\sigma(t))) = -q(t), \qquad t > t_0.$$

Proceeding as in the case where $y(t) > 0$, we are led to a contradiction. This completes the proof. □

Corollary 1. *Assume that* (H1)–(H4) *hold. If there is a sequence* $\{t_n\} \subset (t_0, \infty)$ *such that*

$$\begin{aligned} &\lim_{n\to\infty} t_n = \infty, \qquad \sigma(\sigma(t_n)) > t_0, \\ &q(t) \leq 0 \quad \text{in } [\sigma(\sigma(t_n)), \sigma(t_n)], \\ &\int_{\sigma(t_n)}^{t_n} q(s)\,ds \leq 0, \\ &\int_{\sigma(t_n)}^{t_n} p(s)\,ds \geq \frac{1}{\beta}, \end{aligned}$$

then the differential inequality

$$y'(t) + p(t)f(y(\sigma(t))) \le q(t) \tag{9}$$

has no eventually positive solution.

The proof follows by using the same arguments as in Theorem 1 and will be omitted.

Theorem 2. *Assume that* (H1)–(H3) *hold. If*

$$\int_T^\infty p(s)f(Q_+(\sigma(s)))\,ds = \infty, \tag{10}$$

$$\int_T^\infty p(s)f(Q_-(\sigma(s)))\,ds = \infty \tag{11}$$

for all large T, then every solution of (1) *is oscillatory, where $Q(t)$ is a C^1-function such that $Q'(t) = q(t)$, $t > t_0$ and*

$$Q_\pm(t) = \max\{\pm Q(t), 0\}.$$

Proof. Suppose that $y(t)$ is a solution of (1) such that $y(t) > 0$ in $[t_1, \infty)$ for some $t_1 > t_0$. As in the proof of Theorem 1, the inequality (3) holds for some $t_2 \ge t_1$. It is clear that (3) reduces to

$$(y(t) - Q(t))' \le 0, \qquad t \ge t_2.$$

Then, either $y(t) - Q(t) \ge 0$ in $[t_2, \infty)$ or $y(t) - Q(t) < 0$ in $[t_3, \infty)$ for some $t_3 > t_2$. If $Q(t)$ is eventually nonnegative (respectively, eventually nonpositive), then $Q_-(t) \equiv 0$ (respectively, $Q_+(t) \equiv 0$) for all sufficiently large t, which implies that (11) (respectively, (10)) does not hold. Hence, $Q(t)$ must change sign as $t \to \infty$. Then it does not occur that $0 < y(t) < Q(t)$ in $[t_3, \infty)$. Consequently, $y(t) \ge Q(t)$ in $[t_2, \infty)$. Since $y(t) > 0$ in $[t_2, \infty)$, we find that $y(t) \ge Q_+(t)$ in $[t_2, \infty)$. Then there exists a number $T > t_2$ such that $y(\sigma(t)) \ge Q_+(\sigma(t))$ for $t \ge T$. Taking (H2) into account, we see from (1) that

$$y'(t) + p(t)f(Q_+(\sigma(t))) \le q(t), \qquad t \ge T.$$

Integrating the above inequality over $[T, t]$, we obtain

$$y(t)-y(T)+\int_T^t p(s)f(Q_+(\sigma(s)))\,ds \le \int_T^t q(s)\,ds, \qquad t \ge T,$$

or

$$y(t)-Q(t)+Q(T)-y(T) \le -\int_T^t p(s)f(Q_+(\sigma(s)))\,ds, \qquad t \ge T. \tag{12}$$

Since $y(t) - Q(t) \ge 0$ in $[T, \infty)$, the lefthand side of (12) is bounded from below. However, the righthand side of (12) is not bounded from below by the hypothesis (10). This is a contradiction. In the case where $y(t) < 0$ in $[t_1, \infty)$ for some $t_1 > t_0$, the same arguments as in the case where $y(t) > 0$ lead us to a contradiction. □

The following corollary is an immediate consequence of Theorem 2.

Corollary 2. *Assume that* (H1)–(H3) *hold. If* (10) *holds for all large* T, *then the differential inequality* (9) *has no eventually positive solution.*

Remark 1. The hypothesis (H4) was used in the paper of Cui [**2**].

Remark 2. In Theorem 2 the function $Q(t)$ must change sign as $t \to \infty$, and therefore $q(t)$ must also change sign as $t \to \infty$.

Remark 3. We cannot apply Theorem 2 to the case where $q(t) \equiv 0$ in $[T, \infty)$ for some $T > t_0$. However, Theorem 1 can be applied to this case.

Remark 4. In Corollaries 1 and 2, the condition "$f(-s) = -f(s)$ for $s > 0$" is unnecessary.

Example 1. We consider the equation

$$y'(t) + e^{\pi/2}y(t-\pi/2) = e^t \sin t, \qquad t > 0. \tag{13}$$

Here $p(t) = e^{\pi/2}$, $f(s) = s$, $\sigma(t) = t - \pi/2$, $t_0 = 0$ and $q(t) = e^t \sin t$. We easily see that $Q(t) = (1/\sqrt{2})e^t \sin(t - \pi/4)$ satisfies $Q'(t) = q(t)$. It is clear that $Q_+(t - \pi/2) = (1/\sqrt{2})[e^{t-\pi/2} \sin(t - 3\pi/4)]_+$. There exists a positive integer $m \in \mathbf{N}$ such that $T \le 3\pi/4 + 2m\pi$. Let $n(t) \in \mathbf{N}$ be the largest integer which satisfies $3\pi/4 + (2n(t) + 1)\pi \le t$. An easy calculation shows that

$$
\begin{aligned}
(14)\quad & \int_T^t e^{\pi/2} \frac{1}{\sqrt{2}} \left[e^{s-\pi/2} \sin\left(s - \frac{3}{4}\pi\right)\right]_+ ds \\
&\ge \int_{3\pi/4+2m\pi}^{3\pi/4+(2n(t)+1)\pi} \frac{1}{\sqrt{2}} \left[e^{s} \sin\left(s - \frac{3}{4}\pi\right)\right]_+ ds \\
&= \frac{1}{\sqrt{2}} \sum_{j=m}^{n(t)} \int_{3\pi/4+2j\pi}^{3\pi/4+(2j+1)\pi} e^{s} \sin\left(s - \frac{3}{4}\pi\right) ds \\
&= \frac{1}{2\sqrt{2}} e^{(3\pi/4)} (e^{\pi} + 1) \sum_{j=m}^{n(t)} e^{2j\pi}.
\end{aligned}
$$

Since $\lim_{t\to\infty} n(t) = \infty$, we see from (14) that

$$
\int_T^\infty e^{\pi/2} \frac{1}{\sqrt{2}} \left[e^{s-\pi/2} \sin\left(s - \frac{3}{4}\pi\right)\right]_+ ds = \infty.
$$

Analogously, we obtain

$$
\int_T^\infty e^{\pi/2} \frac{1}{\sqrt{2}} \left[e^{s-\pi/2} \sin\left(s - \frac{3}{4}\pi\right)\right]_- ds = \infty.
$$

Theorem 2 implies that every solution y of (13) is oscillatory. One such solution is $y = e^t \sin t$.

3. Related differential equations with delay. This section is devoted to establishing oscillation criteria for the related differential equation (2) using the oscillation results obtained in Section 2.

Theorem 3. *Assume* (H1) *and* (H3) *hold, and that the following hypotheses hold:*

(H5) $a(t) \in C([t_0,\infty);\mathbf{R}^1)$, $b_i(t) \in C([t_0,\infty);[0,\infty))$, $\rho_i(t) \in C([t_0,\infty);\mathbf{R}^1)$, *$\rho_i(t) \le t$ for $t \ge t_0$, and $\lim_{t\to\infty} \rho_i(t) = \infty (i = 1,2,\dots,k)$;*

(H6) *$f(s) \in C(\mathbf{R}^1;\mathbf{R}^1)$, $f(-s) = -f(s)$ for $s > 0$ and $f(s_1 s_2) \ge f_1(s_1) f_2(s_2)$ for $s_1 > 0$, $s_2 \ge 0$, where $f_1(s_1) \ge 0$ for $s_1 > 0$, $f_2(s_2) \ge 0$ for $s_2 \ge 0$ and $f_2(s_2)$ is nondecreasing for $s_2 \ge 0$.*

Every solution $y(t)$ of (2) is oscillatory if the differential inequalities

$$Y'(t) + P(t) f_1\left(\exp\left(-\int_{t_0}^{\sigma(t)} A(s)\,ds \right)\right) f_2(Y(\sigma(t))) \le \exp\left(\int_{t_0}^{t} A(s)\,ds \right) B(t), \tag{15}$$

$$Y'(t) + P(t) f_1\left(\exp\left(-\int_{t_0}^{\sigma(t)} A(s)\,ds \right)\right) f_2(Y(\sigma(t))) \le -\exp\left(\int_{t_0}^{t} A(s)\,ds \right) B(t), \tag{16}$$

have no eventually positive solutions, where

$$A(t) = a(t) + \sum_{i=1}^{k} b_i(t),$$
$$B(t) = q(t) + \sum_{i=1}^{k} b_i(t) \int_{\rho_i(t)}^{t} q(s)\,ds,$$
$$P(t) = p(t) \exp\left(\int_{t_0}^{t} A(s)\,ds \right).$$

Proof. Let $y(t)$ be a solution of (2) with the property that $y(t) > 0$ in $[t_1,\infty)$ for some $t_1 > t_0$. Then the inequality (3) holds. Integrating (3) over $[\rho_i(t), t]$, we obtain

$$y(t) - y(\rho_i(t)) \le \int_{\rho_i(t)}^{t} q(s)\,ds, \qquad t \ge t_2.$$

Combining the above inequality with (2) yields

$$y'(t) + a(t)y(t) + \sum_{i=1}^{k} b_i(t)\bigg(y(t) - \int_{\rho_i(t)}^{t} q(s)\,ds\bigg) + p(t)f(y(\sigma(t))) \le q(t), \qquad t \ge t_2,$$

which is equivalent to

$$y'(t) + A(t)y(t) + p(t)f(y(\sigma(t))) \le B(t), \qquad t \ge t_2. \tag{17}$$

Multiplying (17) by $\exp(\int_{t_0}^{t} A(s)\,ds)$, we have

$$\bigg(\exp\bigg(\int_{t_0}^{t} A(s)\,ds\bigg)y(t)\bigg)' + P(t)f\bigg(\exp\bigg(-\int_{t_0}^{\sigma(t)} A(s)\,ds\bigg) \exp\bigg(\int_{t_0}^{\sigma(t)} A(s)\,ds\bigg)y(\sigma(t))\bigg) \le \exp\bigg(\int_{t_0}^{t} A(s)\,ds\bigg)B(t), \qquad t \ge t_2. \tag{18}$$

Taking account of (H6), we see from (18) that

$$Y'(t) + P(t)f_1\bigg(\exp\bigg(-\int_{t_0}^{\sigma(t)} A(s)\,ds\bigg)\bigg)f_2(Y(\sigma(t))) \le \exp\bigg(\int_{t_0}^{t} A(s)\,ds\bigg)B(t), \qquad t \ge t_2,$$

where $Y(t) \equiv \exp(\int_{t_0}^{t} A(s)\,ds)y(t) > 0$ in $[t_2, \infty)$. This contradicts the hypothesis. The case where $y(t) < 0$ in $[t_1, \infty)$ can be handled similarly, and we are led to a contradiction. The proof is complete. □

Combining Theorem 3 with Corollaries 1 and 2, we can obtain the following results.

Corollary 3. *Assume that* (H1), (H3), (H5), (H6) *hold, and that* $f_2(s)$ *satisfies* (H4). *Every solution* $y(t)$ *of* (2) *is oscillatory if there is* *a sequence* $\{t_n\} \subset (t_0, \infty)$ *with the properties that*:

$$\lim_{n\to\infty} t_n = \infty, \qquad \sigma(\sigma(t_n)) > t_0,$$
$$\exp\Big(\int_{t_0}^{t} A(s)\,ds\Big)B(t) = 0 \quad \text{in } [\sigma(\sigma(t_n)), \sigma(t_n)],$$
$$\int_{\sigma(t_n)}^{t_n} \exp\Big(\int_{t_0}^{r} A(s)\,ds\Big)B(r)\,dr = 0,$$
$$\int_{\sigma(t_n)}^{t_n} P(r) f_1\Big(\exp\Big(-\int_{t_0}^{\sigma(r)} A(s)\,ds\Big)\Big)\,dr \geq \frac{1}{\beta}.$$

Corollary 4. *Assume that* (H1), (H3), (H5), (H6) *hold. Every solution* $y(t)$ *of* (2) *is oscillatory if*

$$\int_T^{\infty} P(r) f_1\Big(\exp\Big(-\int_{t_0}^{\sigma(r)} A(s)\,ds\Big)\Big) f_2(\tilde{Q}_{\pm}(\sigma(r)))\,dr = \infty$$

for all large T, *where* $\tilde{Q}(t)$ *is a* C^1*-function for which*

$$\tilde{Q}'(t) = \exp\Big(\int_{t_0}^{t} A(s)\,ds\Big)B(t).$$

By the same arguments as were used in Theorem 3, we can prove the following.

Corollary 5. *Assume that* (H1), (H3), (H5) *and* (H6) *hold. The differential inequality*

$$y'(t) + a(t)y(t) + \sum_{i=1}^{k} b_i(t) y(\rho_i(t)) + p(t) f(y(\sigma(t))) \leq q(t), \qquad t > t_0$$

has no eventually positive solution if (15) *has no eventually positive solution.*

Remark 5. Corollary 5 holds true without the condition "$f(-s) = -f(s)$ for $s > 0$."

Example 2. We consider the equation

$$(19)\quad y'(t)+y(t)+\left(e^{\pi/2}y\left(t-\frac{\pi}{2}\right)+e^{\pi}y(t-\pi)\right)+e^{2\pi}y(t-2\pi) = 2e^{t}\cos t,\qquad t>0.$$

Here $t_0=0$, $a(t)=1$, $k=2$, $b_1(t)=e^{\pi/2}$, $\rho_1(t)=t-\pi/2$, $b_2(t)=e^{\pi}$, $\rho_2(t)=t-\pi$, $p(t)=e^{2\pi}$, $f(s)=s$, $\sigma(t)=t-2\pi$ and $q(t)=2e^t\cos t$. It is easy to see that

$$\begin{aligned} P(t)&=\exp(2\pi+(1+e^{\pi/2}+e^{\pi})t),\\ A(t)&=1+e^{\pi/2}+e^{\pi},\\ B(t)&=2e^t\cos t+e^{\pi/2}\int_{t-\pi/2}^{t}2e^s\cos s\,ds\\ &\quad+e^{\pi}\int_{t-\pi}^{t}2e^s\cos s\,ds\\ &=k_1e^t\cos t+k_2e^t\sin t,\end{aligned}$$

where $k_1=4+e^{\pi/2}+e^{\pi}$ and $k_2=e^{\pi/2}+e^{\pi}$. Therefore,

$$\exp\left(\int_0^t A(s)\,ds\right)B(t)=k_1e^{Lt}\cos t+k_2e^{Lt}\sin t,$$

where $L=2+e^{\pi/2}+e^{\pi}$. We observe that the function

$$\begin{aligned}\tilde{Q}(t)&=\frac{k_1+k_2L}{L^2+1}e^{Lt}\sin t+\frac{k_1L-k_2}{L^2+1}e^{Lt}\cos t\\ &=\sqrt{k_3^2+k_4^2}\,e^{Lt}\sin(t+\alpha)\end{aligned}$$

satisfies $\tilde{Q}'(t)=k_1e^{Lt}\cos t+k_2e^{Lt}\sin t$, where $k_3=(k_1+k_2L)/(L^2+1)$, $k_4=(k_1L-k_2)/(L^2+1)>0$ and $\alpha=\tan^{-1}(k_4/k_3)$, $0<\alpha<\pi/2$. Since $f(s_1s_2)=s_1s_2$, we may choose $f_i(s_i)=s_i$, $i=1,2$. It is obvious that

$$\begin{aligned}\int_T^t &\exp(2\pi+(1+e^{\pi/2}+e^{\pi})r)\\ &\qquad\exp\left(-\int_0^{r-2\pi}A(s)\,ds\right)\tilde{Q}_+(r-2\pi)\,dr\\ &=\sqrt{k_3^2+k_4^2}\int_T^t[e^{Lr}\sin(r-(2\pi-\alpha))]_+\,dr.\end{aligned}$$

There is a positive integer $M \in \mathbf{N}$ such that $T \leq 2\pi - \alpha + 2M\pi$. Let $N(t) \in \mathbf{N}$ be the largest integer which satisfies $2\pi - \alpha + (2N(t)+1)\pi \leq t$. We easily obtain

$$\begin{aligned}\sqrt{k_3^2+k_4^2}&\int_T^t [e^{Lr}\sin(r-(2\pi-\alpha))]_+\,dr\\ &\geq \sqrt{k_3^2+k_4^2}\int_{2\pi-\alpha+2M\pi}^{2\pi-\alpha+(2N(t)+1)\pi}[e^{Lr}\sin(r-(2\pi-\alpha))]_+\,dr\\ &= \sqrt{k_3^2+k_4^2}\sum_{j=M}^{N(t)}\int_{2\pi-\alpha+2j\pi}^{2\pi-\alpha+(2j+1)\pi} e^{Lr}\sin(r-(2\pi-\alpha))\,dr\\ &= \frac{\sqrt{k_3^2+k_4^2}}{L^2+1}e^{L(2\pi-\alpha)}(e^{L\pi}+1)\sum_{j=M}^{N(t)}e^{2Lj\pi}.\end{aligned}$$

Since $\lim_{t\to\infty} N(t) = \infty$, we conclude that

$$\int_T^\infty \exp(2\pi + (1+e^{\pi/2}+e^\pi)r)\exp\left(-\int_0^{r-2\pi}A(s)\,ds\right)\tilde{Q}_+(r-2\pi)\,dr = \infty.$$

In the same way, we obtain

$$\int_T^\infty \exp(2\pi + (1+e^{\pi/2}+e^\pi)r)\exp\left(-\int_0^{r-2\pi}A(s)\,ds\right)\tilde{Q}_-(r-2\pi)\,dr = \infty.$$

It follows from Corollary 4 that every solution y of (19) is oscillatory. For example, $y = e^t\cos t$ is such a solution.

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Department of Mathematics, Faculty of Science, Toyama University, Toyama 930, Japan