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DUALITY FOR LOCALLY COMPACT LIPSCHITZ SPACES

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ABSTRACT. It is known that $\mathrm{lip}\,(X)^{**}$ is isometrically isomorphic to $\mathrm{Lip}\,(X)$ for a large class of compact metric spaces X. A generalization for certain locally compact spaces has been asserted but is false. We give a correct generalization and determine, for a large class of locally compact X, in exactly which cases it holds.

0. Introduction. Let X be a metric space. Then $\mathrm{Lip}\,(X)$ is the Banach space consisting of all bounded scalar-valued Lipschitz functions on X with the norm

$$||f||_L = \max\left(||f||_\infty, \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|f(x)-f(y)|}{\rho(x,y)}\right),$$

and $\mathrm{lip}\,(X)$ is the closed subspace of $\mathrm{Lip}\,(X)$ consisting of those functions with the property that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\rho(x,y) \le \delta \quad \text{implies} \quad \frac{|f(x)-f(y)|}{\rho(x,y)} \le \varepsilon.$$

Several researchers have investigate the equation $\mathrm{lip}\,(X)^{**} \cong \mathrm{Lip}\,(X)$, where $\cong$ stands for isometric isomorphism. This does not always hold; for example, if the unit interval $[0,1]$ is endowed with the usual metric, then any function in $\mathrm{lip}\,[0,1]$ has zero derivative at every point, hence is constant. However, $\mathrm{Lip}\,[0,1]$ is infinite-dimensional (in fact [**10**], it is isomorphic to $L^\infty[0,1]$), so $(\mathrm{lip}\,[0,1])^{**} \cong \mathrm{Lip}\,[0,1]$ fails abysmally.

The equation $\mathrm{lip}\,(X)^{**} \cong \mathrm{Lip}\,(X)$ was first proved by de Leeuw [**3**] in the case that X is the unit circle with the metric ρ^α where ρ is normalized arc length and $\alpha \in (0,1)$. Jenkins [**6**] then showed that if

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the scalar field is real the result holds for any compact metric space with a "Hölder" metric, i.e., a metric of the form ρ^α where $\alpha \in (0,1)$ and ρ is also a metric. Jenkins also proved the result in the complex case under a certain restriction on X.

Using a more subtle argument, Johnson [**7**] removed this extra condition, thus establishing the result for any compact Hölder metric space with real or complex scalars. This result was rediscovered by Bade, Curtis and Dales in [**2**], where it was given a somewhat different proof, and the special case where $X \subset \mathbf{R}^n$ and the scalars are real was rediscovered by Wulbert [**15**]. Finally, assuming real scalars, Hanin [**5**] gave a condition on $\mathrm{lip}\,(X)$ which is equivalent to $\mathrm{lip}\,(X)^{**} \cong \mathrm{Lip}\,(X)$ and used it to prove this equation for a certain generalization of compact Hölder metric spaces.

Jenkins and Johnson also claimed the result $(\mathrm{lip}\,(X) \cap C_0(X))^{**} \cong \mathrm{Lip}\,(X)$ in case X is a "finitely compact" Hölder metric space, meaning the metric is of the form ρ^α and every closed ball is compact. This result is false but the flaw is not that difficult to correct (see Section 2). Much later, Jonsson [**8**], apparently aware only of de Leeuw's result, proved that $\mathrm{Lip}\,(X)$ is the double dual of a certain subspace of $\mathrm{lip}\,(X)$ in the case that X is a closed subset of $\mathbf{R}^n$ with a Hölder metric; the definition of this subspace is heavily dependent on the fact that $X \subset \mathbf{R}^n$.

In correcting Jenkins' and Johnson's result, we find that the assumption of finite compactness is not necessary and can be replaced by the weaker assumption that X is "rigidly locally compact," meaning that for every $x \in X$ and $k < 1$ the closed ball of radius k about x is compact. The following well-known example is therefore also an instance of our result. Give $\mathbf{N}$ the metric $\rho(x,y) = 2$ for all $x \neq y$; then $\mathrm{Lip}\,(\mathbf{N}) \cong l^\infty$ and $\mathrm{lip}\,(\mathbf{N}) \cap C_0(\mathbf{N}) = c_0$, and we have $c_0^{**} \cong l^\infty$. Here $\mathbf{N}$ is neither finitely compact nor Hölder, but it is rigidly locally compact.

Rather than deal with Lipschitz spaces as defined above, we prefer to treat the spaces Lip_0 (see Section 1). These have appeared before (e.g., in [**11**] and [**15**], among others), but apparently without the realization that they represent a generalization of the Lipschitz spaces. (This result is quite trivial but perhaps deserves special emphasis nonetheless; see the end of Section 1.) We also define a subspace $\mathrm{lip}_0(X)$ of $\mathrm{Lip}_0(X)$ and under the assumption that X is rigidly locally compact, for a more general definition of this term, we give exact conditions under which $\mathrm{lip}_0(X)^{**}$ is naturally isometrically isomorphic to $\mathrm{Lip}_0(X)$. Our condition simplifies and generalizes Hanin's condition and is suitable for complex as well as real scalars. The spaces $\mathrm{lip}(X)$ and $\mathrm{Lip}(X)$ are special cases of the spaces $\mathrm{lip}_0(X)$ and $\mathrm{Lip}_0(X)$, so that our results generalize all of the results described above. Our results also imply a complex version of Hanin's generalization of Johnson's result.

(Incidentally, the space $\mathrm{lip}_0(X)$ in general depends on a choice of "base point" for X, but the space $\mathrm{Lip}_0(X)$ does not. Thus, our results yield a large class of examples of Banach spaces which are not naturally isometric but whose duals are naturally isometric. The well-known fact that $c_0^* \cong c^* \cong l^1$ is one such example.)

1. $\mathrm{Lip}(X)$ **and** $\mathrm{Lip}_0(X)$**.** If X is a metric space and f is any map from X into the scalar field, define the *Lipschitz number* of f to be

$$L(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)}$$

and say that f is *Lipschitz* if $L(f) < \infty$. More generally, say that a map between metric spaces is Lipschitz and define a Lipschitz number accordingly if the same sup is finite, replacing $|f(x) - f(y)|$ by the distance between $f(x)$ and $f(y)$ in the range space. For any metric space X we let $\mathrm{Lip}(X)$ be the space of all bounded scalar-valued Lipschitz functions on X, with the norm $||f||_L = \max(||f||_\infty, L(f))$. If X is equipped with a base point e we let $\mathrm{Lip}_0(X)$ be the set of all scalar-valued Lipschitz functions on X which vanish at e; this is also a vector space and is given the norm $L(f)$. The spaces $\mathrm{Lip}(X)$ and $\mathrm{Lip}_0(X)$ are called *Lipschitz spaces*.

As we mentioned in the introduction, complex scalars present more difficulties than real scalars. Therefore, we shall assume complex scalars throughout; the same proofs work for real scalars, with occasional simplifications.

It is standard that $\mathrm{Lip}(X)$ and $\mathrm{Lip}_0(X)$ are Banach spaces. One also often defines the spaces $\mathrm{Lip}^\alpha(X)$ and $\mathrm{Lip}_0^\alpha(X)$ for $\alpha \in (0, 1)$ by replacing ρ with ρ^α in the preceding definitions. For example, a typical member of $\mathrm{Lip}_0^\alpha(X)$ is a scalar-valued function f which satisfies

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{(\rho(x, y))^\alpha} < \infty$$

and vanishes at the base point. We prefer to treat these spaces as special cases of the spaces $\mathrm{Lip}\,(X)$ and $\mathrm{Lip}_0(X)$ which arise when the metric of X has the special form ρ^α for some other metric ρ (that is, when some power $1/\alpha > 1$ of the metric is again a metric). We call such a metric a *Hölder metric* and the corresponding metric space a *Hölder metric space.*

Note that up to isometric isomorphism the space $\mathrm{Lip}_0(X)$ does not depend on the choice of base point of X: if e and e' are elements of X, then the map $f \mapsto f - f(e')$ takes $\mathrm{Lip}_0(X)$ with base point e isometrically onto $\mathrm{Lip}_0(X)$ with base point e'. Thus, $\mathrm{Lip}_0(X)$ can be identified with the space of all scalar-valued Lipschitz functions on X with the seminorm $L(\cdot)$, modulo the constant functions; it is sometimes defined in this way.

For any metric space X, let X' be the space remetrized by $\rho'(x, y) = \min(\rho(x, y), 2)$. Then it is easy to check that $\mathrm{Lip}\,(X) \cong \mathrm{Lip}\,(X')$. (This was proved by Vasavada [**13**] and independently rediscovered by Weaver [**14**].) Now if Y is any metric space with diameter less than or equal to 2, let $Y \cup \{e\}$ be the same space with an additional base point e, where we set $\rho(x, e) = 1$ for all $x \in Y$. Then $\mathrm{Lip}\,(Y) \cong \mathrm{Lip}_0(Y \cup \{e\})$ by the map which extends a function $f \in \mathrm{Lip}\,(Y)$ to $Y \cup \{e\}$ by setting $f(e) = 0$. Combining these two observations yields the following trivial but nonobvious result (if one can use such a phrase), which appears to be new:

Proposition 1.1. *Let X be a metric space, and let X' be as above. Then* $\mathrm{Lip}\,(X) \cong \mathrm{Lip}_0(X' \cup \{e\})$.

This result shows that Lip spaces are a special case of Lip_0 spaces. Note that if X is compact, locally compact, or equipped with a Hölder metric, then it is easy to check that $X' \cup \{e\}$ has the same properties.

2. $\mathrm{lip}\,(X)$ **and** $\mathrm{lip}_0(X)$**.** For any metric space X let $\hat{X}$ be the topological space $X^2 - \{(x, x) : x \in X\}$. Then we have a (nonsurjective) isometry $\Phi : \mathrm{Lip}_0(X) \to C_b(\hat{X})$ (the bounded continuous scalar-valued functions on $\hat{X}$) defined by

$$\Phi f(x, y) = \frac{f(x) - f(y)}{\rho(x, y)}.$$

Let $X_0 = X' \cup \{e\}$, as in the last section. If X is compact it is fairly easy to check that for $f \in \mathrm{Lip}\,(X) \cong \mathrm{Lip}_0(X_0)$, the condition $\Phi f \in C_0(\hat{X}_0)$ is equivalent to the requirement that for every $\varepsilon > 0$ there should exist $\delta > 0$ such that

$$\rho(x,y) \le \delta \quad \text{implies} \quad |\Phi f(x,y)| \le \varepsilon.$$

We abbreviate the latter condition as "$\Phi f(x,y) \to 0$ as $\rho(x,y) \to 0$," and for any metric space X we define the *little Lipschitz space* $\mathrm{lip}\,(X)$ to be the subspace of $\mathrm{Lip}\,(X)$ consisting of all $f \in \mathrm{Lip}\,(X)$ such that $\Phi f(x,y) \to 0$ as $\rho(x,y) \to 0$. It is standard that this subspace is complete.

The main tool used in studying $\mathrm{lip}\,(X)$ when X is compact is the fact that $\Phi(\mathrm{lip}\,(X)) \subset C_0(\hat{X}_0)$. Note that for locally compact X, $\Phi(\mathrm{lip}\,(X) \cap C_0(X))$ is generally *not* contained in $C_0(\hat{X}_0)$, e.g., consider a function on $X = \mathbf{R}$ which goes to zero at infinity while its derivative does not. This falsifies Jenkins' [**6**] and Johnson's [**7**] argument that $(\mathrm{lip}\,(X) \cap C_0(X))^{**} \cong \mathrm{Lip}\,(X)$ when X is a finitely compact Hölder metric space. (A *finitely compact* metric space is one in which every closed ball is compact.)

The correct definition of $\mathrm{lip}_0(X)$, for X a locally compact metric space with base point e, is: the set of all $f \in \mathrm{Lip}_0(X)$ such that $f(x)/\rho(x,e) \in C_0(X)$ and $\Phi f(x,y) \to 0$ as either $\rho(x,y) \to 0$ or $x, y \to \infty$, where "$\Phi f(x,y) \to 0$ as $x,y \to \infty$" means that for every $\varepsilon > 0$ there exists a compact subset $K \subset X$ such that $|\Phi f(x,y)| \le \varepsilon$ for $x, y \in X - K$. If X is finitely compact, then with this definition, $\Phi(\mathrm{lip}_0(X)) \subset C_0(\hat{X})$. We also call $\mathrm{lip}_0(X)$ a "little Lipschitz space."

With the preceding definition of $\mathrm{lip}_0(X)$, the argument of Jenkins and Johnson can easily be made into a correct proof that $\mathrm{lip}_0(X)^{**} \cong \mathrm{Lip}_0(X)$ when X is a finitely compact Hölder metric space. However, as we indicated in the introduction, this excludes the example $X = \mathbf{N} \cup \{e\}$ (when $\mathrm{lip}_0(X) \cong c_0$ and $\mathrm{Lip}_0(X) \cong l^\infty$). In this case $\Phi(\mathrm{lip}_0(X))$ is *not* contained in $C_0(\hat{X})$, which introduces serious difficulties. However, we are able to get around these difficulties; the main consequence of containment in $C_0(\hat{X})$ used in earlier papers, namely that every element of the dual of $\mathrm{lip}_0(X)$ is representable as a measure on $\hat{X}$, remains true provided only that X is rigidly locally compact (see Section 4).

Note that if X is compact then $\mathrm{lip}\,(X) \cong \mathrm{lip}_0(X \cup \{e\})$. In this case the conditions having to do with behavior near infinity are vacuous.

This shows that results about lip_0 spaces will specialize to results about lip spaces.

We omit the formal proof of the following proposition. Briefly, one proves that $\mathrm{lip}_0(X)$ is complete in the same way that one proves $\mathrm{lip}(X)$ is complete, and the rest of the proposition follows from the fact that

$$|\Phi h(x,y)| \le \max(|\Phi f(x,y)|, |\Phi g(x,y)|)$$

for $h = \mathrm{Re}\, f$, $\mathrm{Im}\, f$, $f \vee g$ or $f \wedge g$.

Proposition 2.1. *Let X be a locally compact metric space with base point. Then $\mathrm{lip}_0(X)$ is a closed subspace of $\mathrm{Lip}_0(X)$. The real and imaginary parts of a function in $\mathrm{lip}_0(X)$ are in $\mathrm{lip}_0(X)$, and the join and meet of real functions in $\mathrm{lip}_0(X)$ are in $\mathrm{lip}_0(X)$.*

One interesting facet of $\mathrm{lip}_0(X)$ is that it *does* depend on the choice of base point. Shifting to a new base point e' by the map $f \mapsto f - f(e')$ is not possible since this map does not preserve the property of vanishing at infinity (unless X is compact).

This ambiguity is especially interesting in light of the fact, to be proved in Section 5, that under certain conditions $\mathrm{lip}_0(X)^{**} \cong \mathrm{Lip}_0(X)$, for the latter space does not depend on a choice of base point. This gives rise to examples of nonisometric Banach spaces whose duals are isometric. The best-known example of this phenomenon, $c_0^* \cong c^* \cong l^1$, is a special case. For $\mathrm{lip}(\mathbf{N}) \cong \mathrm{lip}_0(\mathbf{N} \cup \{e\}) \cong c_0$, but if the base point of $\mathbf{N} \cup \{e\}$ is changed to, say, $1 \in \mathbf{N}$, then $\mathrm{lip}_0(\mathbf{N} \cup \{e\})$ is isometrically isomorphic to c by the map taking the function $f : \mathbf{N} \cup \{e\} \to \mathbf{C}$ to the sequence whose nth term is $f(n+1) - f(e)$.

3. The separation property. The condition relevant to satisfaction of $\mathrm{lip}_0(X)^{**} \cong \mathrm{Lip}_0(X)$ is contained in the following proposition.

Proposition 3.1. *Let X be a locally compact metric space with base point e. Then of the following conditions,* a) *and* b) *are equivalent, and both imply* c)*.*

a) *There exists $k_1 > 0$ such that for any $x, y \in X$, $\varepsilon > 0$ and*

$a \in [0, \rho(y,e)]$, some real-valued $g \in \mathrm{lip}_0(X)$ satisfies $L(g) \leq k_1$, $g(x) \leq \varepsilon + \max(0, a - \rho(x,y))$, and $|g(y) - a| \leq \varepsilon$;

b) *there exists $k_2 > 0$ such that for any $\varepsilon > 0$, $f \in \mathrm{Lip}_0(X)$, and finite subset $A \subset X$, some $g \in \mathrm{lip}_0(X)$ satisfies $L(g) \leq k_2 \cdot L(f)$ and $||(g-f)|_A||_\infty \leq \varepsilon$;*

c) *there exists $k_3 > 1$ such that the closed $\rho(y,e)/k_3$-ball about y is compact for every $y \in X$.*

Proof. a) $\Rightarrow$ b). Suppose a) holds; let $\varepsilon > 0$, $f \in \mathrm{Lip}(X)$ and $A \subset X$ be finite. We may assume f is real by separating it into real and imaginary parts and using Proposition 2.1. (Passing to the complex case introduces a factor of $\sqrt{2}$ into k_2 and ε.) By separating f into positive and negative parts, i.e., by approximating $f \vee 0$ and $-f \vee 0$ separately on A and then combining the result, we can also reduce to the case where f is positive. (This introduces a factor of 2 into k_2 and ε.) Dividing f by $L(f)$ reduces to the case that $L(f) = 1$. Finally, we may also assume that $e \in A$.

Now for any distinct $x, y \in A$, we have $f(x) \geq \max(0, f(y) - \rho(x,y))$ since f is nonexpansive and positive. Thus, setting $a = f(y)$, by a) we can find a real-valued function $g_{xy} \in \mathrm{lip}_0(X)$ such that

$$L(g_{xy}) \leq k_1, \qquad g_{xy}(x) \leq f(x) + \varepsilon, \qquad |g_{xy}(y) - f(y)| \leq \varepsilon.$$

Then let

$$g = \bigvee_{x \in A} \bigwedge_{\substack{y \in A \\ y \neq x}} g_{xy};$$

this is in $\mathrm{lip}_0(X)$ and satisfies $||(g-f)|_A||_\infty \leq \varepsilon$ and $L(g) \leq k_1 = k_1 \cdot L(f)$. Thus b) holds with $k_2 = k_1$ in the real, positive case, which is enough.

b) $\Rightarrow$ a). Let x, y, ε and a be as in part a) and assume b) holds. By an analogue of the Tietze extension theorem due to McShane (see [**12**, Proposition 1.4]) we can extend the function $x \mapsto \max(0, a - \rho(x,y))$, $y \mapsto a$, $e \mapsto 0$ to a nonexpansive real-valued function $f \in \mathrm{Lip}(X)$. Then apply b) with $A = \{x, y\}$ and take the real part of the resulting function g; this has the properties required by a). Thus, a) holds with $k_1 = k_2$.

a) $\Rightarrow$ c). Suppose a) holds and c) fails. Then for each $n \in \mathbf{N}$ there exists $y_n \in X$ such that the closed $\rho(y_n, e)/n$-ball about y is not compact. Apply a) with $x = e$, $y = y_n$, $\varepsilon > 0$, and $a = \rho(y_n, e)$; we get $g \in \mathrm{lip}_0(X)$ such that $L(g) \leq k_1$ and $|g(y_n) - \rho(y_n, e)| \leq \varepsilon$.

Since the $\rho(y_n, e)/n$-ball about y_n is bounded but not compact and $g \in \mathrm{lip}_0(X)$, g must take arbitrarily small values on this ball; hence there exists $x_n \in X$ such that $\rho(x_n, y_n) \leq \rho(y_n, e)/n$ and $|g(x_n)| \leq \varepsilon$. It follows that

$$L(g) \geq \frac{|g(x_n) - g(y_n)|}{\rho(x_n, y_n)} \geq \frac{n(\rho(y_n, e) - 2\varepsilon)}{\rho(y_n, e)}.$$

Taking $\varepsilon \to 0$ implies that $L(g) \geq n$, hence $k_1 \geq n$. As n was arbitrary, this is a contradiction. □

We say that $\mathrm{lip}_0(X)$ has the *separation property* if it satisfies either condition a) or condition b) of Proposition 3.1; the terminology derives from part a), which asserts that the little lipschitz functions separate the points of X in a certain manner. In the compact case we have the following simplification.

Proposition 3.2. *Let X be a compact metric space and recall the notation of Proposition* 1.1. *Then* $\mathrm{lip}(X) \cong \mathrm{lip}_0(X' \cup \{e\})$ *has the separation property if and only if there exists $k_1 > 0$ such that for all $x, y \in X$, some $g \in \mathrm{lip}(X)$ satisfies $||g||_L \leq k_1$ and $|g(x) - g(y)| = \min(\rho(x, y), 2)$.*

We omit the proof of this proposition. It is a fairly easy consequence of the a) form of the separation property and the fact that $\mathrm{lip}(X)$ contains the constant functions.

When we show in Section 5 that $\mathrm{lip}_0(X)$ having the separation property implies $\mathrm{lip}_0(X)^{**} \cong \mathrm{Lip}_0(X)$, we will deduce several equivalent conditions. We now discuss some examples where $\mathrm{lip}_0(X)$ has the separation property.

If X is any "uniformly discrete" metric spce, meaning the distances $\rho(x, y)$ are bounded away from zero (for x and y distinct), and if X has finite diameter, then $\mathrm{lip}_0(X)$ has the separation property for arbitrary base point. For in this case $\mathrm{Lip}_0(X)$ is isomorphic to $l^\infty(X - \{e\})$

and $\mathrm{lip}_0(X)$ is isomorphic to $c_0(X - \{e\})$, and given any element of the former and a finite subset of $X - \{e\}$, multiplying the element by the characteristic function of the finite subset yields an element of the latter, which shows satisfaction of the b) form of the separation property. This example includes the case $X = \mathbf{N} \cup \{e\}$ discussed earlier.

Now let X be a locally compact Hölder metric space. We will show that if X satisfies condition c) of Proposition 3.1 for every $k_3 > 1$ then it has the separation property. This requirement plays a big role later in the paper, so we give it a name: we say that such an X is *rigidly locally compact*. Note that if X is a locally compact metric space and $X' \cup \{e\}$ is defined as in Section 1, so that $\mathrm{Lip}(X) \cong \mathrm{Lip}_0(X' \cup \{e\})$, then $X' \cup \{e\}$ is rigidly locally compact if and only if every closed ball of radius less than 1 is compact.

Proposition 3.3. *Let X be a rigidly locally compact Hölder metric space with base point e. Then X has the separation property.*

Proof. We will verify the a) form of the separation property for any $k_1 > 1$. Thus, choose $x, y \in X$, $\varepsilon > 0$ and $a \in [0, \rho^\alpha(y, e)]$. Then we claim that the function

$$g(z) = [(a - (1+\varepsilon)\rho^\alpha(z, y) \wedge (a - \varepsilon)] \vee 0$$

is the desired.

First, we have

$$g(x) \leq \max(0, a - (1+\varepsilon)\rho^\alpha(x, y)) \leq \max(0, a - \rho^\alpha(x, y))$$

and $g(y) = \max(0, a - \varepsilon)$ hence $|g(y) - a| \leq \varepsilon$. Also, $L(g) \leq 1 + \varepsilon$. So we have only to show that $g \in \mathrm{lip}_0(X)$.

Since g is supported on the $\rho^\alpha(y, e)/(1+\varepsilon)$-ball about y, which is compact, the only thing to check is that $\Phi f(x', y') \to 0$ as $\rho^\alpha(x', y') \to 0$. By compactness it is enough to check that for every z in the support of g and for every $\varepsilon' > 0$ there exists $\delta' > 0$ such that $|\Phi g(x, z)| \leq \varepsilon'$ for all x within δ' of z. This is certainly true for $z = y$ since g is constant in a neighborhood of y. On the other hand, if $b = \rho(z, y) > 0$ then for any $x \in X$, letting $\delta = \rho(x, z)$ we have $\rho(x, y) \geq b - \delta$, hence

$$|\Phi g(x, z)| \leq \frac{|\rho^\alpha(x, y) - \rho^\alpha(z, y)|}{\rho^\alpha(x, z)} \leq \frac{b^\alpha - (b - \delta)^\alpha}{\delta^\alpha}.$$

This goes to zero as $\delta \to 0$, since differentiability at b of the function $t \mapsto t^\alpha$ on $\mathbf{R}$ implies that the numerator is $O(\delta)$. So for some δ' we do have $|\Phi g| \leq \varepsilon'$ on the δ'-ball about z, as desired. □

We mention that the above proof works equally well for a certain generalization of Hölder metric spaces due to Hanin [**5**]; these are the spaces whose metric is of the form $\omega(\rho)$ where ρ is a metric and $\omega : \mathbf{R}^+ \to \mathbf{R}^+$ satisfies the conditions $\omega(0) = 0$, ω is continuous and nondecreasing, $\lim_{t\to 0} \omega(t)/t = \infty$, and ω has finite oscillation at each $t > 0$. (This definition is actually more general than Hanin's.)

If one has satisfaction of the b) form of the separation property for every $k_2 > 1$, the proof that $\mathrm{lip}_0(X)^{**} \cong \mathrm{Lip}_0(X)$ becomes much easier. This is why the case of complex scalars is genuinely harder than the case of real scalars; in the complex case one can usually only directly verify the separation property for $k_2 > \sqrt{2}$. Oddly, having the separation property for any k_2 actually implies having it for every $k_2 > 1$, as we will see in Corollary 5.5. However, in most cases there seems to be no direct verification of this fact. We mention this in order to make the point that our extension of Hanin's result to include complex scalars is a real generalization.

4. The dual of $\mathrm{lip}_0(X)$**.** The purpose of this section is to show that if X is rigidly locally compact then every element of the dual of $\mathrm{lip}_0(X)$ can be represented by a Borel measure on $\hat{X}$ of the same norm; in other words, for every $F \in \mathrm{lip}_0(X)^*$ there exists $\mu \in M(\hat{X})$, the space of finite Borel measurds on $\hat{X}$, such that $|\mu|(\hat{X}) = ||F||$ and

$$F(f) = \int (\Phi f)\, d\mu$$

for all $f \in \mathrm{lip}_0(X)$. We also say "$\Phi^*\mu$ agrees with F on $\mathrm{lip}_0(X)$." In the finitely compact case this result is trivial since then Φ takes $\mathrm{lip}_0(X)$ isometrically into $C_0(\hat{X})$ (and $M(\hat{X}) \cong C_0(\hat{X})^*$).

We use the notation βY to denote the Stone-Čech compactification of a completely regular topological space Y, and if $f : Y \to \mathbf{C}$ is a bounded continuous map we write $\tilde{f}$ for its unique continuous extension to βY.

Lemma 4.1. *Let Y be a completely regular topological space, and let E be a closed subspace of $C_b(Y)$. Suppose there exist Borel maps*

$\phi_1 : \beta Y \to Y$ and $\phi_2 : \beta Y \to [-1,1]$ such that $\tilde{f} = \phi_2 \cdot (f \circ \phi_1)$ for all $f \in E$. Then for every $F \in E^*$ there exists $\mu \in M(Y)$ such that $|\mu|(Y) = ||F||$ and

$$F(f) = \int f\, d\mu$$

for all $f \in E$.

Proof. Let $F \in E^*$ and extend it to a measure $\nu \in M(\beta Y) \cong C_b(Y)^*$ such that $|\nu|(\beta Y) = ||F||$. Then define $\mu \in M(Y)$ by setting

$$\mu(A) = \int_{\beta Y} \phi_2 \cdot (\chi_A \circ \phi_1)\, d\nu.$$

Since the integrand is always less than or equal to 1 in absolute value, we get $|\mu|(Y) \le |\nu|(\beta Y) = ||F||$. Also, for any $f \in E$, we have

$$\begin{aligned} \int_Y f\, d\mu &= \int_{\beta Y} \phi_2 \cdot (f \circ \phi_1)\, d\nu \\ &= \int_{\beta Y} \tilde{f}\, d\nu \\ &= F(f). \end{aligned}$$

(The first equality holds for all simple $f \in L^1(Y,\mu)$, hence for all $f \in L^1(Y,\mu)$, hence for all $f \in C_b(Y)$.) This also implies $|\mu|(Y) \ge ||F||$, which completes the proof. □

Theorem 4.2. *Let X be a rigidly locally compact metric space with base point e, and let $F \in \mathrm{lip}_0(X)^*$. Then there exists $\mu \in M(\hat{X})$ such that $|\mu|(\hat{X}) = ||F||$ and*

$$F(f) = \int (\Phi f)\, d\mu$$

for all $f \in \mathrm{lip}_0(X)$.

Proof. We apply Lemma 4.1 with $Y = X^2$ and $E = \Phi(\mathrm{lip}_0(X))$. For the purpose of this proof we consider $\Phi(\mathrm{lip}_0(X))$ as lying in $C_b(X^2)$, by extending the functions on $\hat{X}$ by zero on the diagonal. After applying

Lemma 4.1, the restriction of the resulting measure μ to $\hat{X}$ has the desired properties.

Thus, we need to exhibit Borel maps $\phi_1 : \beta(X^2) \to X^2$ and $\phi_2 : \beta(X^2) \to [-1,1]$ with the property that

$$(*) \qquad \widetilde{\Phi f} = \phi_2 \cdot (\Phi f \circ \phi_1)$$

for all $f \in \mathrm{lip}_0(X)$.

Let $\psi_1 : \beta(X^2) \to (\beta X)^2$ be the natural continuous map (defined via the universal property of the Stone-Čech compactification) and define $\psi_2 : \beta X \to X$ by $\psi_2(x) = x$ if $x \in X$ and $\psi_2(x) = e$ if $x \in \beta X - X$. Then we set $\phi_1 = (\psi_2 \times \psi_2) \circ \psi_1$. Since X is locally compact, it is open in βX, hence ψ_2 is Borel; and ψ_1 is continuous, so ϕ_1 is also Borel.

Let $\tilde{g} : \beta(X^2) \to [0,1]$ be the unique continuous extension of the function $g(x,y) = \min(1, \rho(x,e)/\rho(x,y))$. Then we define ϕ_2 by

$$\phi_2(\xi) = \begin{cases} 1 & \text{if } \psi_1(\xi) \in X^2 \\ 0 & \text{if } \psi_1(\xi) \in (\beta X - X)^2 \\ \tilde{g}(\xi) & \text{if } \psi_1(\xi) \in X \times (\beta X - X) \\ -\tilde{g}(\xi) & \text{if } \psi_1(\xi) \in (\beta X - X) \times X. \end{cases}$$

This is Borel because it is defined by patching together Borel maps on Borel sets.

To verify $(*)$, let $f \in \mathrm{lip}_0(X)$ and $\xi \in \beta(X^2)$. Suppose first that $\xi = (x,y) \in X^2$. Then $\phi_1(\xi) = \xi$ and $\phi_2(\xi) = 1$, so

$$\widetilde{\Phi f}(\xi) = \phi_2(\xi)\Phi f(\phi_1(\xi))$$

as desired.

If $\psi_1(\xi) \in (\beta X - X)^2$, let (x_λ, y_λ) be a net in X^2 converging to ξ; then $x_\lambda, y_\lambda \to \infty$, hence $\Phi f(x_\lambda, y_\lambda) \to 0$. Thus,

$$\widetilde{\Phi f}(\xi) = 0 = \phi_2(\xi),$$

which implies $(*)$ in this case.

If $\psi_1(\xi) \in X \times (\beta X - X)$, let $x \in X$ and let (x_λ, y_λ) be a net in X which converges to ξ. Then $x_\lambda \to x$ and $f(y_\lambda)/\rho(e, y_\lambda) \to 0$ (since $y_\lambda \to \infty$), hence $f(y_\lambda)/\rho(x_\lambda, y_\lambda) \to 0$, hence

$$\widetilde{\Phi f}(\xi) = \lim_\lambda (f(x_\lambda)/\rho(x_\lambda, y_\lambda)) = \Phi f(x,e) \cdot \tilde{g}(\xi)$$

(note that $\lim(\rho(x_\lambda, y_\lambda)) = \lim(\rho(x, y_\lambda)) \geq \rho(x, e)$ since X is rigidly locally compact). Thus $(*)$ holds in this case as well. The final case that $\psi_1(\xi) \in (\beta X - X) \times X$ is handled similarly. □

5. The equation $\mathrm{lip}_0(X)^{**} \cong \mathrm{Lip}_0(X)$**.** For any metric space with base point, it is possible to construct a Banach space whose dual space is isometrically isomorphic to $\mathrm{Lip}_0(X)$. Such a space was given in a neglected paper by Arens and Eells [**1**]; they did not prove isometric isomorphism but this is easy enough to check. By the natural embedding of a Banach space in its double dual, the space of Arens and Eells is identified with the linear span of the point evaluations on $\mathrm{Lip}_0(X)$. (For $x \in X$, the *point evaluation* at x, written χ_x, is defined by $\chi_x(f) = f(x)$.) Alternatively, Johnson's result [**7**, Theorem 4.1] that the point evaluations span a predual of $\mathrm{Lip}(X)$ can easily be extended to cover $\mathrm{Lip}_0(X)$. We therefore take as known that the space $\mathrm{Lip}_0(X)_*$ defined as the closed span in $\mathrm{Lip}_0(X)^*$ of the point evaluations, naturally satisfies $(\mathrm{Lip}_0(X)_*)^* \cong \mathrm{Lip}_0(X)$.

There is a natural nonexpansive map from $\mathrm{Lip}_0(X)_*$ into $\mathrm{lip}_0(X)^*$, which takes an element of $\mathrm{Lip}_0(X)_* \subset \mathrm{Lip}_0(X)^*$ to its restriction to $\mathrm{lip}_0(X)$. We now want to show that if X is rigidly locally compact this map has dense range, and if X also has the separation property then it is a surjective isometry. Our method of proof follows that used by Bade, Curtis and Dales [**2**] in the compact Hölder case; Johnson's proof in [**7**] relies crucially on separability of $\mathrm{lip}_0(X)^*$ and therefore is not suitable in the locally compact case (when X may have arbitrary cardinality).

In the last section we determined that if X is rigidly locally compact, then for any $F \in \mathrm{lip}_0(X)^*$, there exists $\mu \in M(\hat{X})$ such that $|\mu|(\hat{X}) = ||F||$ and $\Phi^*\mu$ agrees with F on $\mathrm{lip}_0(X)$. Thus, the following lemma implies that if X is rigidly locally compact, then the natural map from $\mathrm{Lip}_0(X)_*$ into $\mathrm{lip}_0(X)^*$ has dense range.

Lemma 5.1. *Let X be a locally compact metric space with base point, let $\mu \in M(\hat{X})$, and let $\varepsilon > 0$. Then there exists an element $\sigma \in \mathrm{Lip}_0(X)_*$ which is a finite linear combination of point evaluations, such that $||\sigma - \Phi^*\mu|| \leq \varepsilon$.*

Proof. Find a compact subset $K \subset \hat{X}$ such that $|\mu|(\hat{X} - K) \leq \varepsilon/2$. Since ρ is bounded away from zero on K, it follows that $\mu' = (\mu/\rho)|_K$ is a Borel measure of finite total variation.

Define $\nu \in M(X)$ by $\nu(E) = \mu'(E \times X) - \mu'(X \times E)$. Then as an element of $\mathrm{Lip}_0(X)^*$, $\nu = \Phi^*(\mu|_K)$. Since ν is compactly supported, we can apply a lemma originally due to Kantorovich (see [**9**, Lemma 4.4.4] or [**6**, Lemma 4.5] or [**2**, Lemma 3.2]) to conclude that there exists a linear combination σ of point evaluations on $\mathrm{Lip}_0(X)$ such that $||\sigma - \nu|| \leq \varepsilon/2$. Since $\nu = \Phi^*(\mu|_K)$ and

$$||\Phi^*\mu - \Phi^*(\mu|_K)|| = ||\Phi^*(\mu|_{\hat{X}-K})|| \leq |\mu|(\hat{X} - K) \leq \frac{\varepsilon}{2},$$

we get $||\sigma - \Phi^*\mu|| \leq \varepsilon$, as desired. □

Lemma 5.2. *Suppose X is a locally compact metric space with base point which has the separation property. Then for any $\mu \in M(\hat{X})$, if $\Phi^*\mu \in \mathrm{Lip}_0(X)^*$ vanishes on $\mathrm{lip}_0(X)$, it is zero.*

Proof. Let $\mu \in M(\hat{X})$, suppose $\Phi^*\mu$ vanishes on $\mathrm{lip}_0(X)$, and let $f \in \mathrm{Lip}_0(X)$; we must show that $\Phi^*\mu(f) = 0$. Let $\varepsilon > 0$ and use Lemma 5.1 to find $\sigma = \sum_1^n a_i \chi_{x_i}$ such that $||\sigma - \Phi^*\mu|| \leq \varepsilon$. By the b) form of the separation property, we can then find $g \in \mathrm{lip}_0(X)$ which satisfies $L(g) \leq k_2 \cdot L(f)$ and agrees with f to within $\varepsilon/||\sigma||_1$ on the support of σ, where $||\sigma||_1 = \sum_1^n |a_i|$. We get

$$\begin{aligned} |\Phi^*\mu(f)| &\leq |\Phi^*\mu(f) - \sigma(f)| + |\sigma(f) - \sigma(g)| \\ &\quad + |\sigma(g) - \Phi^*\mu(g)| + |\Phi^*\mu(g)| \\ &\leq \varepsilon \cdot L(f) + \varepsilon + \varepsilon k_2 \cdot L(f) + 0 \\ &= \varepsilon(1 + L(f) + k_2 \cdot L(f)). \end{aligned}$$

Taking $\varepsilon \to 0$, we get $\Phi^*\mu(f) = 0$. □

We can now prove the main theorem.

Theorem 5.3. *Suppose X is a rigidly locally compact metric space with base point which has the separation property. Then the natural map from $\mathrm{Lip}_0(X)_*$ to $\mathrm{lip}_0(X^*)$ is a surjective isometric isomorphism.*

Proof. The map is automatically nonexpansive, and it has dense range by Lemma 5.1 and Theorem 4.2. Thus, we need only show it is noncontractive. It is enough to check this on elements of $\mathrm{Lip}_0(X)_*$ of the form $\sum_1^n a_i \chi_{x_i}$, since such elements are norm-dense in $\mathrm{Lip}_0(X)_*$.

Thus, let $\sigma = \sum_1^n a_1 \chi_{x_i}$. Let F be the restriction of σ to $\mathrm{lip}_0(X)$, and by Theorem 4.2 find a measure $\mu \in M(\hat{X})$ such that $|\mu|(\hat{X}) = ||F||$ and $\Phi^*\mu$ restricted to $\mathrm{lip}_0(X)$ agrees with F.

Defining the discrete measure $\nu \in M(\hat{X})$ by $\nu(x_i, e) = a_i \cdot \rho(x_i, e)$, $1 \leq i \leq n$, and zero elsewhere, we get $\Phi^*\nu = \sigma$. Since $\Phi^*(\mu - \nu)$ vanishes on $\mathrm{lip}_0(X)$, Lemma 5.2 implies that it is zero. It follows that $\Phi^*\mu = \sigma$, hence $||\sigma|| \leq |\mu|(\hat{X}) = ||F||$, as we needed to show. □

The following corollary follows from Proposition 3.3 and the comments immediately preceding and following it. This result includes all of the results mentioned in the introduction. (The result for X is proved by replacing X with $X' \cup \{e\}$ as in Proposition 1.1 and applying the result for Y.)

Corollary 5.4. *let X be a Hölder metric space in which every closed ball of radius less than 1 is compact, let Y be a rigidly locally compact Hölder metric space, and let Z be a rigidly locally compact uniformly discrete space with finite diameter. Then* $\mathrm{lip}_0(X' \cup \{e\})^{**} \cong \mathrm{Lip}(X)$, $\mathrm{lip}_0(Y)^{**} \cong \mathrm{Lip}_0(Y)$, *and* $\mathrm{lip}_0(Z)^{**} \cong \mathrm{Lip}_0(Z)$. *If X is compact, then* $\mathrm{lip}(X)^{**} \cong \mathrm{Lip}(X)$.

The same results hold if the Hölder hypothesis is replaced by the weaker condition described following Proposition 3.3.

We conclude by giving some equivalent forms of the separation property and showing that it is not only sufficient to imply $\mathrm{lip}_0(X)^{**} \cong \mathrm{Lip}_0(X)$, but also necessary.

Corollary 5.5. *Let X be a rigidly locally compact metric space with base point e. Then the following are equivalent*:

a) $\mathrm{lip}_0(X)$ *has the separation property*;

b) $\mathrm{lip}_0(X)^{**} \cong \mathrm{Lip}_0(X)$ *naturally*;

c) $\mathrm{lip}_0(X)$ *has the separation property with* $k_1 = k_2 = 1$;

d) *for all* $k_2 > 1$, $f \in \mathrm{Lip}_0(X)$, *and* $A \subset X$ *finite there exists* $g \in \mathrm{lip}_0(X)$ *such that* $f|_A = g|_A$ *and* $L(g) \leq k_2 \cdot L(f)$.

Proof. a) $\Rightarrow$ b). This follows immediately from Theorem 5.3.

b) $\Rightarrow$ c). Suppose b) holds; then the unit ball of $\mathrm{lip}_0(X)$ is weak* dense in the unit ball of $\mathrm{Lip}_0(X)$, i.e., [**7**, Corollary 4.4], it is dense in the topology of pointwise convergence. This shows that $\mathrm{lip}_0(X)$ satisfies the b) form of the separation property with $k_2 = 1$; and from the proof of Proposition 3.1 b) $\Rightarrow$ a), we also have satisfaction of the a) form of the separation property with $k_1 = 1$.

c) $\Rightarrow$ d). Suppose c) holds and let $k_2 > 1$ and $A \subset X$ be finite. Without loss of generality, suppose $e \notin A$. Consider the map $T : \mathrm{lip}_0(X) \to \mathbf{C}^A$ which takes a little Lipschitz function on X to its restriction to A. Give $\mathbf{C}^A$ the norm

$$||f||_{\mathbf{C}^A} = \inf\{L(g) : g \in \mathrm{Lip}_0(X) \text{ and } g|_A = f\}.$$

Then by c), the map T satisfies the hypotheses of a simple result of Grabiner [**4**] (the "approximation lemma") for $k = 1$ and t arbitrarily close to 0. This result asserts that if T is a bounded linear map between Banach spaces and k and t are positive constants, $t < 1$, such that for any y in the range space there exists x in the domain space such that $||x|| \leq k||y||$ and $||Tx - y|| \leq t||y||$, then for any y in the range space there exists x in the domain space such that $||x|| \leq k||y||/(1-t)$ and $Tx = y$. So, choosing $t \leq 1 - 1/k_2$, we get that for every $f \in \mathrm{Lip}_0(X)$ there exists $g \in \mathrm{lip}_0(X)$ such that $Tg = f|_A$, i.e., $g|_A = f|_A$, and

$$L(g) \leq ||f|_A||_{\mathbf{C}^A}/(1-t) \leq k_2 \cdot L(f).$$

d) $\Rightarrow$ a). Trivial. □

Another condition equivalent to the separation property is that the unit ball of $\mathrm{lip}_0(X)$ should be lattice-dense in the unit ball of $\mathrm{Lip}_0(X)$. This follows from Theorem 2 of [**14**].

REFERENCES

1. R.F. Arens and J. Eells, Jr., *On embedding uniform and topological spaces*, Pacific J. Math. **6** (1956), 397–403.

2. W.G. Bade, P.C. Curtis and H.G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc. **55** (1987), 359–372.

3. K. de Leeuw, *Banach spaces of Lipschitz functions*, Studia Math. **21** (1961/62), 55–66.

4. S. Grabiner, *The Tietze extension theorem and the open mapping theorem*, Amer. Math. Monthly **93** (1986), 190–191.

5. L.G. Hanin, *Kantorovich-Rubenstein norm and its application in the theory of Lipschitz spaces*, Proc. Amer. Math. Soc. **115** (1992), 345–352.

6. T.M. Jenkins, *Banach spaces of Lipschitz functions on an abstract metric space*, Ph.D. thesis, Yale University, 1968.

7. J.A. Johnson, *Banach spaces of Lipschitz functions and vector-valued Lipschitz functions*, Trans. Amer. Math. Soc. **148** (1970), 147–169.

8. A. Jonsson, *The duals of Lipschitz spaces defined on closed sets*, Indiana Math. J. **39** (1990), 467–476.

9. L.V. Kantorovich and G.P. Akilov, *Functional analysis* (2nd edition), New York, 1982.

10. J. Musielak and Z. Semadeni, *Some classes of Banach spaces depending on a parameter*, Studia Math. **20** (1961), 271–284.

11. E. Mayer-Wolf, *Isometries between Banach spaces of Lipschitz functions*, Israel J. Math. **38** (1981), 58–74.

12. D.R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. **111** (1964), 240–272.

13. M.H. Vasavada, *Closed ideals and linear isometries of certain function spaces*, Ph.D. thesis, University of Wisconsin, 1969.

14. N. Weaver, *Lattices of Lipschitz functions*, Pacific J. Math. **164** (1994), 179–193.

15. D.E. Wulbert, *Representations of the spaces of Lipschitz functions*, J. Funct. Anal. **15** (1974), 45–55.

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