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ROBINSON'S THEOREM ON ASYMMETRIC DIOPHANTINE APPROXIMATION

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ABSTRACT. Let x be an irrational number. In this note we give two functions $A(K)$ and $B(K)$ defined on positive integers, for which the asymmetric approximation inequality $-1/(A(K)q^2) < x-p/q < 1/(B(K)q^2)$ has infinitely many rational solutions p/q. This result improves Robinson's classical asymmetric inequality found in 1947.

1. Introduction. In 1891, Hurwitz [**3**] proved the fundamental theorem on Diophantine approximation, which asserts that for any irrational number x, there are infinitely many rational numbers p/q such that $|x-p/q| < 1/(\sqrt{5}q^2)$. This inequality involves absolute value and is called symmetric approximation.

In 1945, Segre [**10**] initiated the study of asymmetric approximation. He proved that for any irrational numbers x and a given positive real number τ independent of x, there are infinitely many rational numbers p/q such that $-1/(\sqrt{1+4\tau}q^2) < x-p/q < \tau/(\sqrt{1+4\tau}q^2)$. Segre's result has been extensively investigated. See Mahler [**7**], Le Veque [**6**], Kopetzky and Schnitzer [**4, 5**], Prasad and Prasad [**8**], Szüsz [**11**] and Tong [**12**–**16**].

Right after Segre's discovery, Robinson [**9**] pointed out another direction of asymmetric approximation in 1947. He proved the following theorem:

Theorem 1. *Let x be an irrational number. Then for any given positive real number ε, there are infinitely many rational numbers p/q such that*

$$(1) \qquad -1/((\sqrt{5}-\varepsilon)q^2) < x-p/q < 1/((1+\sqrt{5})q^2).$$

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Inequality (1) is interesting. By Hurwitz's theorem, we have $-1/(\sqrt{5}q^2) < x - p/q < 1/(\sqrt{5}q^2)$. Inequality (1) shows that the righthand side of this inequality can be essentially improved with a tiny loss on the estimation of the left. It is easily seen that Segre's result and Robinson's result are independent, no one includes the other.

Contrary to the numerous study of Segre's theorem, the investigation of Robinson's theorem was entirely quiet. Nothing can be found in the literature for nearly half a century, although there are some questions apparently unsolved. For instance, on the lefthand side of inequality (1), the expression involves ε, but on the righthand side, ε does not appear. It is very reasonable to guess that ε should show up in the righthand inequality and get the estimate improved.

In this paper, using continued fraction and Fibonacci sequence, we obtained a more precise estimation on the difference $x - p/q$ than inequality (1). The explicit forms of functions $A(K)$, $B(K)$ are given such that $-1/(A(K)q^2) < x - p/q < 1/(B(K)q^2)$ for any given positive integer K. As a corollary, one can involve ε on the righthand side of inequality (1) and improve it.

2. Preliminaries. We need some basic facts about continued fractions.

Let x be an irrational number with simple continued fraction expansion $x = [a_0; a_1, a_2, \dots, a_i, \dots]$. Let $p_i/q_i = [a_0; a_1, \dots, a_i]$ be the ith convergent of x. If $\alpha_i = [a_{i+1}; a_{i+2}, \dots]$, then

$$\begin{aligned} x - p_i/q_i &= [a_0, a_1, \dots, a_i + \alpha_i^{-1}] - p_i/q_i \\ &= (p_{i-1}q_i - p_i q_{i-1})/((\alpha_i + q_i - 1/q_i)q_i^2). \end{aligned}$$

Since $p_{i-1}q_i - p_i q_{i-1} = (-1)^i$ and $q_{i-1}/q_i = [0; a_i, a_{i-1}, \dots, a_1]$, writing

$$M_i = a_{i+1} + [0; a_{i+2}, a_{i+3}, \dots] + [0; a_i, a_{i-1}, \dots, a_1], \tag{2}$$

we have

$$x - p_i/q_i = (-1)^i/(M_i q_i^2). \tag{3}$$

We will use equality (3) frequently.

Let $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_{n+1} + u_n$ be the Fibonacci sequence. The following two lemmas are simple consequences of induction.

Lemma 1. *Let $c = [0; 1, 1, \dots, 1]$ with k consecutive 1s. Then*

$$c = u_k / u_{k+1}. \tag{4}$$

Lemma 2. *Let $d = [0; 1, 1, \dots, 1, 2]$ with k consecutive 1s followed by a 2. Then*

$$d = u_{k+2} / u_{k+3}. \tag{5}$$

Lemma 3 (Binet's formula [**2**]). *Let u_k be the kth Fibonacci number. Then*

$$u_k = (\alpha^k - \beta^k)/\sqrt{5}, \tag{6}$$

where $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$.

Lemma 4.

$$1 + 2u_{2k}/u_{2k+1} = \sqrt{5} - 2\sqrt{5}/[((3+\sqrt{5})/2)^{2k+1} + 1].$$

Proof.

$$\begin{aligned} 1 + 2u_{2k}/u_{2k+1} &= (u_{2k+2} + u_{2k})/u_{2k+1} \\ &= ((\alpha - \alpha^{-1})(\alpha^{2k+1} - \beta^{2k+1}) \\ &\qquad + (\alpha - \beta + \alpha^{-1} - \beta^{-1})\beta^{2k+1})/(\alpha^{2k+1} - \beta^{2k+1}) \\ &= \alpha + \alpha^{-1} + (\alpha + \alpha^{-1} - \beta - \beta^{-1})/((\alpha/\beta)^{2k+1} - 1) \\ &= \sqrt{5} - 2\sqrt{5}/[((3+\sqrt{5})/2)^{2k+1} + 1]. \end{aligned}$$

Lemma 5.

$$\begin{aligned} &(3+\sqrt{5})/2 + u_{4k+1}/u_{4k+2} \\ &\qquad = (1+\sqrt{5})/2 + u_{4k+3}/u_{4k+2} \\ &\qquad = 1 + \sqrt{5} + \sqrt{5}/[((3+\sqrt{5})/2)^{4k+2} - 1]. \end{aligned}$$

3. Main theorem. Now we turn to the main result.

Theorem 2. *Let x be an irrational number. Then for any given positive integer K, there are infinitely many convergents p_i/q_i of x such that*

$$-1/(A(K)q^2) < x - p_i/q_i < 1/(B(K)q^2), \tag{7}$$

where $A(K) = \sqrt{5} - 2\sqrt{5}/[((3+\sqrt{5})/2)^{2K+1} + 1]$, $B(K) = 1 + \sqrt{5} + \sqrt{5}/[((3+\sqrt{5})/2)^{4K+2} - 1]$.

Proof. It is easily seen that $A(K)$ is an increasing function in K while $B(K)$ is a decreasing function in K.

Let $x = [a_0; a_1, a_2, \dots, a_i, \dots]$ be the expansion of x in simple continued fraction. We consider all possible cases on the partial quotients a_i. We first consider two possible cases.

Case A. There are infinitely many $a_{2i} \geq 3$. Then, by (2), we have

$$M_{2i-1} \geq a_{2i} \geq 3 > \sqrt{5} > A(K)$$

for any positive integer K. Hence, the lefthand inequality in (7) is correct by (3).

Case B. There are finitely many $a_{2i} \geq 3$. Then, for sufficiently large i, $a_{2i} \leq 2$. Consider two possible subcases.

Subcase 1. There are infinitely many $a_{2j+1} \geq 3$. Then, by (2), for sufficiently large j, we have

$$\begin{aligned} M_{2j} &\geq 3 + [0; a_{2j+2}, 1] + [0; a_{2j}, 1] \\ &\geq 3 + 2/3 > B(1) \geq B(K). \end{aligned}$$

Hence, the righthand side of (7) is correct by (3).

Subcase 2. There are finitely many $a_{2j+1} \geq 3$. Then for sufficiently large j, $a_{2j+1} \leq 2$. Consider two possible subcases.

Subcase a. There are infinitely many $a_{2i} = 2$. Then by (2), for sufficiently large i, we have

$$\begin{aligned} M_{2i-1} &\geq 2 + [0; a_{2i+1}, 1] + [0; a_{2i-1}, 1] \\ &\geq 2 + 2/3 > \sqrt{5} > A(K), \end{aligned}$$

for any positive integer K. Hence, the lefthand side of (7) is correct by (3).

Subcase b. There are finitely many $a_{2i} = 2$. Then for sufficiently large i, $a_{2i} = 1$. Consider two possible subcases.

Subcase i. There are finitely many $a_{2j+1} = 2$. Then for sufficiently large j, $a_{2j+1} = 1$. As a matter of fact, for any sufficiently large positive n, $a_n = 1$. Since for any simple continued fraction $[0; b_1, b_2, \dots, b_i]$, we have $[0; b_1, b_2, \dots, b_{2k}] < [0; b_1, b_2, \dots, b_i]$ if $2k < i$. Therefore, we have

$$M_{2j+1} \geq 1 + [0; 1, 1, \dots] + [0; a_{2j+1}, \dots, a_{2j-2K+2}].$$

We may pick up j so large that $a_{2j+1} = \cdots = a_{2j-2K+2} = 1$. Thus there are $2K$ consecutive 1s in the second continued fraction. By Lemma 1 we have

$$M_{2j+1} \geq 1 + (\sqrt{5} - 1)/2 + u_{2K}/u_{2K+1} \geq A(K).$$

Therefore, the left side of (7) is correct by (3).

Subcase ii. There are infinitely many $a_{2j+1} = 2$. Let N be the smallest positive integer such that $i, j > N$ imply $a_{2i} = 1$ and $a_{2j+1} \leq 2$. Let $S = \{2j_n + 1 \mid j > N, a_{2j_n+1} = 2\}$. Then S is an infinite set. There are two subcases to consider.

Subcase (a) There are infinitely many n such that $2j_n + 1 \in S$, $2j_{n+1} + 1 \in S$ and $j_{n+1} - j_n \leq 2K$, where K is a given positive integer. Then we can pick up an n satisfying these three properties and $2j_{n+1} - 2K > N$. Then we have, by (2),

$$\begin{aligned} M_{2j_{n+1}} \geq 2 &+ [0; a_{2j_{n+1}+2}, a_{2j_{n+1}+3}, \dots] \\ &+ [0; a_{2j_{n+1}}, \dots, a_{2j_{n+1}-4K+1}]; \end{aligned}$$

the first continued fraction is greater than $[0,1,1,\dots]$ because there are some $a_{2j_{n+2}}+1=2$. Since $2j_{n+1}-2j_n \le 4k$ or $2j_{n+1}-4k+1 \le 2j_n+1$, the second continued fraction is no less than $[0;1,1,\dots,2]$ with $4K-1$ consecutive 1s followed by a 2. Therefore, by Lemmas 2 and 5, we have

$$M_{2j_n+1} \ge 2+(\sqrt{5}-1)/2+u_{4K+1}/u_{4K+2}=B(K).$$

The righthand side of (7) is correct.

Subcase (b) There are infinitely many n such that $2j_{n+1} \in S$, $2j_{n+1}+1 \in S$ and $j_{n+1}-j_n \le 2K$. Then $(2j_{n+1}+1)-(2j_n+1) \ge 4K+2$ and $a_m=1$ for all m: $2j_n+1<m<2j_n+4K$. Therefore,

$$\begin{aligned} M_{2j_n+2K-1} \ge 1 &+ [0;1,1,\dots,1,a_{2j_n+2K}] \\ &+ [0;1,1,\dots,1,a_{2j_n-2K+2}]. \end{aligned}$$

Two of the above continued fractions contain $2k$ consecutive 1s; therefore, by Lemmas 1 and 4, we have

$$M_{2j_n+2K-1} > 1+2u_{2k}/u_{2k+1}=A(K)$$

for any positive integer K. The lefthand side of (7) is correct.

From the discussion above, we know that (7) is correct. □

Now we give a corollary.

Corollary 1. *Let x be an irrational number, ε a small positive real number. Then there are infinitely many convergents p_i/q_i of x satisfying*

$$(7) \qquad -1/[(\sqrt{5}-\varepsilon)q_i^2] < x-p_i/q_i < 1/[(\sqrt{5}+1+\varepsilon^2/9)q_i^2].$$

Proof. Let ε be a given small positive real number. Since $((3+\sqrt{5})/2)^{2K+1}+1 \to \infty$, we can pick up K so large such that $2\sqrt{5}/[(3+\sqrt{5})/2)^{2K+1}+1] < \varepsilon$. Then $A(K) > \sqrt{5}-\varepsilon$ and $B(K)=1+\sqrt{5}+\sqrt{5}/[((3+\sqrt{5})/2)^{2K+1}+1)((3+\sqrt{5})/2)^{2K+1}-1] > 1+\sqrt{5}+\sqrt{5}/[((3+\sqrt{5})/2)^{2K+1}+1]^2 > 1+\sqrt{5}+\varepsilon^2/4\sqrt{5}) > 1+\sqrt{5}+\varepsilon^2/9$. □

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