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TORSION OF DIFFERENTIALS OF AFFINE QUASI-HOMOGENEOUS HYPERSURFACES

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ABSTRACT. In this paper we prove that the torsion modules of the module of Kaehler differentials of affine hypersurfaces defined by a reduced quasi-homogeneous polynomial with an isolated singularity at the origin are cyclic. We give explicit expressions for generators. Moreover, we exhibit an isomorphism between the torsion submodule of $\Omega^{N-1}_{A/K}$ and $\Omega^{N}_{A/K}$ for such hypersurfaces. $A-D-E$ singularities provide examples of such hypersurfaces.

0. Introduction. Let K be a field of characteristic zero. We consider reduced affine quasi-homogeneous hypersurfaces in $\mathbf{A}_K^N$ with an isolated singularity at the origin. In the local analytic case we already know by Theorem 4(1) of [**11**] that for reduced hypersurfaces with an isolated singularity at the origin only $\Omega^{N-1}_{A/K}$ and $\Omega^{N}_{A/K}$ have nonzero torsion, where $N-1$ is the dimension of our hypersurface. The proof extends to the algebraic case as well. Since $\Omega^{N}_{A/K}$ is clearly cyclic on generator $\omega_1\wedge\cdots\wedge\omega_N$, it remains to consider the torsion submodule of $\Omega^{N-1}_{A/K}$.

The main result of this paper is an elementary proof that the torsion submodule of $\Omega^{N-1}_{A/K}$, henceforth denoted by $T(\Omega^{N-1}_{A/K})$, is a cyclic A-module and an explicit formula for its generator. We show

Theorem 1. *If A is a reduced affine hypersurface with an isolated singularity at the origin defined by a quasi-homogeneous polynomial F with weights λ_i and of (total) degree n, then $T(\Omega^{N-1}_{A/K})$ is a cyclic A-module generated by*

$$w_0=\sum(-1)^{i+1}(\lambda_i/n)x_i\,dx_1\wedge\cdots\wedge\hat{d}x_i\wedge\cdots\wedge dx_N.$$

We then proceed to show

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Theorem 2. *Let A be a reduced hypersurface in A_K^N with an isolated singularity at the origin defined by a quasi-homogeneous polynomial F with weights λ_i and of* (*total*) *degree n. Then we have the following A-module isomorphisms*:

$$T(\Omega_{A/K}^{N-1}) \simeq Aw_0 \simeq \frac{A}{(\partial f/\partial x_1, \partial f/\partial x_2, \dots, \partial f/\partial x_N)} \simeq \Omega_{A/K}^N,$$

where $w_0 = \sum(-1)^{i+1}(\lambda_i/n)x_i\, dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_N$.

The fact that for local analytic algebras $\dim_{\mathbf{C}} T(\Omega_{X,x}) = \dim_{\mathbf{C}} \mathcal{O}_{X,x}/(\partial F/\partial X_1, \dots, \partial F/\partial X_N)_x)$ was already shown by [**4**, Proposition 1.11], or by [**21**, Proposition 2.8]. My result holds for (global) reduced hypersurfaces with only isolated singularities. Moreover, it uses very elementary methods. With the help of Macaulay it is possible to implement this algorithm on a computer.

In the last section we use the isomorphism of Theorem 2 to compute the Milnor number for $A - D - E$ singularities; for a definition and further details on $A - D - E$ singularities, we refer to [**5**] or [**7**].

1. Preliminaries. Let K be a field of characteristic 0, $R = K[X_1, \dots, X_N]$, and consider the hypersurface $A = R/(F)$, where $F \in R$ is a reduced quasi-homogeneous polynomial with weights λ_i. Recall (cf. [**1**]) that a nonvanishing polynomial $F \in K[X_1, \dots, X_N]$ of degree n is quasi-homogeneous with weights λ_i if and only if it satisfies the generalized Euler equation:

$$\sum_{i=1}^{N} \lambda_i X_i \frac{\partial F}{\partial X_i} = nF.$$

To ease notation, let $\partial f/\partial x_i = \partial F/\partial X_i + (F)$ for $i = 1, \dots, N$. The singular locus $\operatorname{Sing}(A)$ of a hypersurface in $\mathbf{A}^N$ defined by a quasi-homogeneous polynomial is defined by the ideal $(\partial F/\partial X_1, \dots, \partial F/\partial X_N)$ in R. A has an isolated singularity at $P \in \operatorname{Spec}(A)$ if there exists $s \in A - P$ such that $\operatorname{Sing}(A_s) = \{P_s\}$, cf. [**6**].

Lemma 1 (special case of [**11**, Satz 4(1)]. *For reduced hypersurfaces A with an isolated singularity at the origin:*

$$T(\Omega_{A/K}^i) = 0 \quad \textit{for } i < N-1 \quad \textit{and } i > N.$$

The following result characterizes hypersurfaces defined by a quasi-homogeneous polynomial with a single isolated singularity at the origin by a condition on the partial derivatives of F.

Lemma 2. *Let R, A and F be as before, and assume that A has at most isolated singularities. Then the following conditions are equivalent*:

(i) $(\partial F/\partial X_1, \dots, \partial F/\partial X_N)$ *is an R-sequence*,

(ii) *let* $I = (\partial f/\partial x_1, \dots, \partial f/\partial x_N) \subset A$, *then* $\dim_K(A/I) < \infty$,

(iii) $\operatorname{Spec}(A)$ *has an isolated singularity at the origin*,

(iv) $\operatorname{Spec}(A)$ *has only a single isolated singularity at the origin.*

We have the following well-known (cf. [**11**] or [**6**]) description of $\Omega^i_{A/K}$.

Lemma 3. *For all positive integers i, we get surjections* $\pi_i : \Omega^i_{R/K} \to \Omega^i_{A/K}$. $\operatorname{Ker}(\pi_i)$ *is the R-submodule* $F\Omega^i_{R/K} + \Omega^{i-1}_{R/K} \wedge dF$.

Moreover, we have the following description of $T(\Omega^j_{A/K})$ for reduced affine hypersurfaces A, cf.

Lemma 4 [**9**, Lemma 4.11]. *For all integers $j > 0$, there is an A-module isomorphism* $\operatorname{Ker}(\theta_j) \simeq T(\Omega^j_{A/K})$, *where* $\theta_j : \Omega^j_{A/K} \to A \otimes \Omega^{j+1}_{R/K}$ *is defined by* $\theta_j(da_1 \wedge \cdots \wedge da_j) = (1/j!) 1 \otimes dr_1 \wedge \cdots \wedge dr_j \wedge dF$, *where r_i are elements of R that reduce to a_i modulo (F).*

If the hypersurface is nonsingular, then all the $\Omega^j_{A/K}$'s are torsion-free and the maps θ_j are all injective.

2. Proofs of the main theorems.

Proof of Theorem 1. Let us first show that $w_0 \in T(\Omega^{N-1}_{A/K})$. We have

$$\begin{aligned}\theta_{N-1}(w_0) &= \frac{(-1)^{N-1}}{(N-1)!} \otimes_R \left(\sum_{i=1}^{N} \frac{\lambda_i}{n} X_i \frac{\partial F}{\partial X_i} \right) dX_1 \wedge \cdots \wedge dX_N \\ &= \frac{(-1)^{N-1}}{(N-1)!} \otimes_R F\, dX_1 \wedge \cdots \wedge dX_N.\end{aligned}$$

Since θ_{N-1} is A-linear, we get $T(\Omega^{N-1}_{A/K}) \supseteq Aw_0$. So we need to prove the opposite inclusion. Let $w = \sum u_i\, dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_N$ be an arbitrary element of $T(\Omega^{N-1}_{A/K})$. That means $w \in \operatorname{Ker}(\theta_{N-1})$. Equivalently,

$$1 \otimes_R \left(\sum (-1)^{i+1} U_i \frac{\partial F}{\partial X_i} \right) dX_1 \wedge \cdots \wedge dX_N = 0,$$

where $u_i = U_i + (F)$, $i = 1, \ldots, N$. So we have

$$\sum_i (-1)^{i+1} U_i \frac{\partial F}{\partial X_i} = PF,$$

for some polynomial $P \in R$. So expressing F in terms of its partial derivatives, we get

$$\sum_{i=1}^{N} (-1)^{i+1} \left(U_i - (-1)^{i+1} \frac{\lambda_i}{n} P X_i \right) \frac{\partial F}{\partial X_i} = 0.$$

Let $C_i = U_i - (-1)^{i+1}(\lambda_i/n) P X_i$. Then

$$(N*) : \sum_{i=1}^{N} (-1)^{i+1} C_i \frac{\partial F}{\partial X_i} = 0.$$

We define a Koszul complex $K(\underline{x}, R/J)$ with $\underline{x} = [\ldots, (-1)^{i+1} \partial F/\partial X_i, \ldots]$ by

$$K_0 = R \quad \text{and} \quad K_p = \oplus_{i_1 < \cdots < i_p} R e_{i_1 \cdots i_p},$$

where

$$e_{i_1 \cdots i_p} = dX_1 \wedge \cdots d\hat{X}_{i_1} \wedge \cdots \wedge d\hat{X}_{i_p} \wedge \cdots dX_N.$$

The differential $d_p : K_p \mapsto K_{p-1}$ is given by

$$d_p(e_{i_1 \cdots i_p}) = \sum_{j=1}^{p} (-1)^{j-1} x_{i_j} e_{i_1 \cdots \hat{i}_j \cdots e_{i_p}}.$$

So we recognize $(N*)$ as a Koszul relation, cf. [**8**, p. 127]. Since $\partial F/\partial X_1, \dots, \partial F/\partial X_N$ are an R-sequence, we know that the Koszul complex is exact. Thus, one has an equality of vectors:

$$[C_1, \dots, C_N] = \left[\dots, (-1)^{i+1} \frac{\partial F}{\partial X_i}, \dots \right] \cdot \sum_{1 \le l < k \le N} \alpha_{lk} A_{lk},$$

where $\alpha_{lk} \in R$ and A_{lk} with $1 \le l < k \le N$ denotes the alternating $N \times N$ matrix $\delta_{lk} - \delta_{kl}$ where $\delta_{..}$ is the Kronecker delta. From

$$\Omega^{N-1}_{A/K} = \frac{\Omega^{N-1}_{R/K}}{F\Omega^{N-1}_{R/K} + \Omega^{N-2}_{R/K} \wedge dF},$$

we get

$$\left[\dots, (-1)^{i+1} \frac{\partial F}{\partial X_i}, \dots \right] \cdot A_{lk} \cdot \begin{bmatrix} \vdots \\ e_i \\ \vdots \end{bmatrix} \equiv 0 \bmod \operatorname{Ker}(\pi_{N-1}),$$

where $e_i = dX_1 \wedge \cdots \wedge d\hat{X}_i \wedge \cdots \wedge dX_N$. Therefore,

$$\sum_{i=1}^{N} C_i dX_1 \wedge \cdots \wedge d\hat{X}_i \wedge \cdots \wedge dX_N \equiv 0 \bmod \operatorname{Ker}(\pi_{N-1}).$$

It then follows that $w = \overline{P} w_0$ in $\Omega^{N-1}_{A/K}$. We have shown that $T(\Omega^{N-1})$ is a cyclic A-module with generator $w_0 = \sum((-1)^{i+1}\lambda_i/n) x_i \, dx_1 \wedge \cdots \wedge d\hat{x}_i \wedge \cdots \wedge dx_N$. □

Proof of Theorem 2. Note that since our hypersurface is singular we have that $\Omega^N_{A/K} \neq 0$. From the definition of exterior product $\Omega^N_{R/K}$ is a cyclic R-module generated by $dX_1 \wedge \cdots \wedge dX_N$.

$$\Omega^N_{A/K} \simeq \frac{\Omega^N_{R/K}}{\operatorname{Ker}(\pi_N)}$$

and the R-submodule $\mathrm{Ker}\,(\pi_N)$ is generated by $\{(\partial F/\partial X_i)dX_1 \wedge \cdots \wedge dX_n\}_{1\le i\le N}$. By Lemma 2, $\Omega^N_{A/K}$ is a cyclic A-module on generator $dx_1 \wedge \cdots \wedge dx_N$, so

$$\Omega^N_{A/K} \simeq A\Big/\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N}\right).$$

Next we want to show: $\mathrm{Ann}\,(T(\Omega^{N-1}_{A/K})) = (\partial f/\partial x_1, \dots, \partial f/\partial x_N)$. For all $i = 1, \dots, N$,

$$\begin{aligned}\frac{\partial F}{\partial X_i} W_0 &= \sum_j (-1)^{j-1} \frac{\lambda_j}{n} \frac{\partial F}{\partial X_i} X_j\, dX_1 \wedge \cdots \wedge d\hat{X}_j \wedge \cdots \wedge dX_N \\ &= \sum_{j\neq i} (-1)^{j+1} \frac{\lambda_j}{n} \frac{\partial F}{\partial X_i} X_j\, dX_1 \wedge \cdots \wedge d\hat{X}_j \wedge \cdots \wedge dX_N \\ &\qquad + (-1)^{i+1} F\, dX_1 \wedge \cdots \wedge d\hat{X}_i \wedge \cdots \wedge dX_N \\ &\quad - \sum_{j\neq i} (-1)^{i+1} \frac{\lambda_j}{n} \frac{\partial F}{\partial X_j} X_j\, dX_1 \wedge \cdots \wedge d\hat{X}_i \wedge \cdots \wedge dX_N \\ &= (-1)^{i+1} F\, dX_1 \wedge \cdots \wedge d\hat{X}_i \wedge \cdots \wedge dX_N \\ &\quad + \sum_{j<i} (-1)^{N+i+j+1} \frac{\lambda_j}{n} X_j\, dX_1 \wedge \cdots \wedge d\hat{X}_j \wedge \cdots \wedge d\hat{X}_i \\ &\quad \wedge \cdots \wedge dX_N \wedge dF - \sum_{i<j} (-1)^{N+i+j+1} \frac{\lambda_j}{n} X_j\, dX_1 \wedge \\ &\quad \cdots \wedge d\hat{X}_i \wedge \cdots \wedge d\hat{X}_j \wedge \cdots \wedge dX_N \wedge dF \in \mathrm{Ker}\,(\pi_{N-1}).\end{aligned}$$

So we have, by Theorem 1,

$$\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N}\right) \subseteq \mathrm{Ann}\,(T(\Omega^{N-1}_{A/K})).$$

Since $d(w_0) = (\sum \lambda_i)/n\, dx_1 \wedge \cdots \wedge dx_N \neq 0$, we know that $T(\Omega^{N-1}_{A/K}) \neq 0$. Clearly d is K-linear and by Lemma 1, $\Omega^N_{A/K}$ is finite dimensional, so to show that d is a vector space isomorphism it suffices to show that d is surjective. Let $\{x_1^{i_1} x_2^{i_2} \cdots x_N^{i_N}\, dx_1 \wedge \cdots \wedge dx_N : i_k \in \mathbf{N}\}$ be a generating set for $\Omega^N_{A/K}$. We need to show that $x_1^{i_1} x_2^{i_2} \cdots x_N^{i_N}\, dx_1 \wedge \cdots \wedge dx_N \in d(T(\Omega^{N-1}_{A/K}))$ for all possible $i_k \in \mathbf{N}$, $k = 1, \dots, N$. Since

$d(x_1^{i_1} x_2^{i_2} \cdots x_N^{i_N} w_0) = \sum_k (i_k + 1)(\lambda_k/n) x_1^{i_1} x_2^{i_2} \cdots x_N^{i_N} \, dx_1 \wedge \cdots \wedge dx_N$, and $\sum_k (i_k+1)(\lambda_k/n) \neq 0$, we have shown that

$$d : T(\Omega^{N-1}_{A/K}) \longrightarrow \Omega^N_{A/K} \quad \text{is a vector space isomorphism.}$$

Hence $\mathrm{Ann}\,(T(\Omega^{N-1}_{A/K})) = (\partial f/\partial x_1, \dots, \partial f/\partial x_N)$, and so

$$T(\Omega^{N-1}_{A/K}) \simeq \Omega^N_{A/K}. \qquad \square$$

3. Examples. Let K be an algebraically closed field of char 0, e.g., the complex numbers. The so-called $A-D-E$ singularities are defined by the following quasi-homogeneous polynomials:

$$\begin{aligned}
A_k &: F(A,k) = X_1^{k+1} + X_2^2 + X_3^2 + \cdots + X_N^2 \\
D_k &: F(D,k) = X_1(X_1^{k-2} + X_2^2) + X_3^2 + \cdots + X_N^2 \\
E_6 &: F(E,6) = X_1^4 + X_2^3 + X_3^2 + \cdots + X_N^2 \\
E_7 &: F(E,7) = X_2(X_1^3 + X_2^2) + X_3^2 + \cdots + X_N^2 \\
E_8 &: F(E,8) = X_1^5 + X_2^3 + X_3^2 + \cdots + X_N^2.
\end{aligned}$$

The name $A-D-E$ singularity stems from the fact that the Dynkin diagram obtained from successive blow-ups is the Dynkin-diagram of a Lie algebra of the form A_k, D_k or E_6, E_7, E_8, where k denotes the number of vertices. For more information on the characterization of $A-D-E$ singularities, see the paper by G. Greuel [**5**]. Using the fact that $T(\Omega^{N-1}_{A/K}) \simeq \Omega^N_{A/K}$, we get that

$$\dim{}_K T(\Omega^{N-1}_{A/K}) = \dim{}_K \left(\frac{K[X_1, X_2, \dots, X_N]}{(\partial F/\partial X_1, \dots, \partial F/\partial X_N)} \right),$$

where F has to be replaced by any of the quasi-homogeneous polynomials from above. So, for $A-D-E$ singularities corresponding to a Dynkin diagram with k vertices, we have

$$\dim{}_K T(\Omega^{N-1}_{A/K}) = k.$$

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