ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 26, Number 2, Spring 1996

ZETA REGULARIZED PRODUCTS AND FUNCTIONAL DETERMINANTS ON SPHERES

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1. Introduction. For $n \geq 2$, let S^n be the n-dimensional sphere with the standard metric, and let Δ_{S^n} be the Laplacian operator on the space of smooth functions. The eigenvalues of this operator are known to be $l(l+n-1)$ with multiplicity β_l^n, where

$$(1) \qquad \beta_l^n = \binom{n+l}{n} - \binom{n+l-2}{n} = \frac{(2l+n-1)}{(n-1)!} \prod_{k=1}^{n-2} (l+k).$$

This paper is concerned with using a factorization theorem for zeta regularized products to compute functional determinants of operators associated with Δ_{S^n}. These determinants are defined by the process of zeta regularization. The most important are $\det' \Delta_{S^n}$ and the determinant of the conformal Laplacian, $\det(\Delta_{S^n} + n(n-2)/4)$. The prime indicates the omission of the zero eigenvalue in the zeta regularized product. In general, the conformal Laplacian is defined to be $\Delta + (n-2)K/4(n-1)$, where Δ is the Laplacian and K is the scalar curvature. For the sphere, $K = n(n-1)$. Computation of the above determinants is equivalent to computing derivatives at $s = 0$ of the zeta function $\sum_{l=1}^{\infty} \beta_l^n [l(l+n-1)]^{-s}$ for the Laplacian and $\sum_{l=0}^{\infty} \beta_l^n [(l+n/2)(l+n/2-1)]^{-s}$ for the conformal Laplacian.

For simplicity, since we are concerned mainly with illustrating the factorization theorem, we restrict our discussion of the conformal Laplacian to the case when n is even. We consider the more general zeta function, $Z_n(s,a) = \sum_{l=1}^{\infty} \beta_l^n [(l+a)(l+n-1-a)]^{-s}$ for integers a, $0 \leq a \leq n-1$, with $a = 0$ corresponding to the Laplacian and $a = n/2$ to the conformal Laplacian. If $a(n-1-a) \neq 0$, then $\det(\Delta_{S^n} + a(n-1-a)) = a(n-1-a)\exp(-Z_n'(0,a))$, and if $a(n-1-a) = 0$, then $\det' \Delta_{S^n} = \exp(-Z_n'(0,a))$.

Received by the editors on June 11, 1993.

We show below (Theorem 1) that for integers a, $\det(\Delta_{S^n}+a(n-1-a))$ if $a(n-1-a)\neq 0$ and $(n-1)\det'\Delta_{S^n}$ are both of the form

$$\exp\left(r_n(a)+\sum_{k=1}^{n}\tau_k^n(a)\zeta'(-k+1)\right) \tag{2}$$

where ζ is the Riemann zeta function, and the numbers $r_n(a)$ and $\tau_k^n(a)$ are rational numbers for which we give explicit expressions in terms of coefficients of the Taylor expansion of β_l^n about $l=-a$ and $l=-(n-1)/2$. Using (2) and the functional equation for the Riemann zeta function, it is easy to compute numerical values for the constants involved. Our computations show that $r_n(a)=0$ for n odd and $\tau_k^n(a)=0$ if n and k have opposite parity.

Computations of determinants of Laplacians and conformal Laplacians have been done previously, by using a variety of techniques. Vardi [**8**] presented a method for computing $\det'\Delta_{S^n}$. Computation of $\det'(\Delta_{S^4}+2)$ has been done by Branson and Ørsted [**3**], and this result agrees with ours. Techniques used there are due to Weisberger [**12**]. Our technique has the advantage of offering a unified approach.

The values of these determinants are of interest since in many cases they give extremal values of the determinant for a class of metrics. See [**3, 2, 6**].

The technique that we introduce here for dealing with this computation, which is different from other approaches, is a lemma on zeta regularized products (Lemma 1), which may be useful in simplifying and understanding other computations of this type. This gives us, in addition, a way to factor our functional determinant $\det(\Delta_{S^n}+a(n-1-a))$ into multiple gamma functions, generalizing an equation of Voros [**11**], and giving an alternate approach to computing it for integral a, $0\leq a\leq n-1$ (Section 5).

We remark that choosing a to be an integer makes things considerably simpler because of the fact that $\beta_l^k=0$ for $l=-1,\dots,-(n-1)$, and the computation reduces to computation of derivatives of the Riemann zeta function. The same techniques could be used for noninteger values of a but would involve a more complicated expression involving derivatives of the Hurwitz zeta function.

2. A lemma on zeta regularized products. If λ_k is a sequence of nonzero complex numbers, consider the zeta function

$$Z(s) = \sum_{k=1}^{\infty} \lambda_k^{-s}.$$

If $Z(s)$ converges for $\operatorname{Re} s > s_0$ and has meromorphic continuation to a function meromorphic in $\operatorname{Re} s > s_1$, $s_1 < 0$ with at most simple poles, then we say the sequence is *zeta regularizable.* We define $\lambda_k^{-s} = \exp(-s \log \lambda_k)$, and the definition of Z depends on the choice of $\arg \lambda_k$.

For a zeta regularizable sequence, we define the zeta regularized product

$$\prod{}_z \lambda_k = \exp(-Z'(0)).$$

We also define the product $\prod_z (\lambda_k - \lambda)$, where for $\lambda \neq \lambda_k$ we adopt the convention that $\arg(\lambda_k - \lambda) \approx \arg \lambda_k$ for $|\lambda_k|$ large.

For λ_k the sequence of nonzero eigenvalues of the Laplacian on a manifold, $\prod_z \lambda_k$ is called the *determinant of the Laplacian* and $\prod_z (\lambda_k + \lambda)$ the *functional determinant*, $\det'(\Delta + \lambda)$. This concept was introduced in [**5**].

Information on the formal properties of zeta regularized products can be found in [**7**] and [**11**]. The use of zeta regularized products can be traced back as far as Barnes [**1**]. Our approach is to try to factor a zeta regularized product into simpler ones. One sees that the equality of $\prod_z (\lambda_k^2 - \lambda^2)$ and $\prod_z (\lambda_k - \lambda) \prod_z (\lambda_k + \lambda)$ is untrue without the introduction of an exponential factor. This factor can be computed from the relationship between the zeta regularized product and the Weierstrass product found in the references mentioned. We give a confirmation independent of this in the proof below.

Lemma 1. *Let λ_j be a zeta regularizable sequence, and let h be an integer such that $\sum_{j=1}^{\infty} |\lambda_j|^{-h-1} < \infty$. For $\operatorname{Re} s$ sufficiently large, let*

$$Z(s) = \sum_{j=1}^{\infty} \lambda_j^{-s} \tag{3}$$

and for $\lambda \neq \lambda_j$, let

$$F(s) = \sum_{j=1}^{\infty} ((\lambda_j^2 - \lambda^2)^{-s} - (\lambda_j - \lambda)^{-s} - (\lambda_j + \lambda)^{-s}). \tag{4}$$

Then we have

$$-F'(0) = \sum_{j=1}^{[h/2]} \operatorname*{Res}_{s=2j} Z(s)\left(1+\frac{1}{3}+\cdots+\frac{1}{2j-1}\right)\frac{\lambda^{2j}}{j}. \tag{5}$$

In terms of zeta regularized products, the result can be stated as

$$\prod\nolimits_z (\lambda_j^2-\lambda^2) = e^{F'(0)} \prod\nolimits_z (\lambda_j-\lambda) \prod\nolimits_z (\lambda_j+\lambda)$$

where $F'(0)$ *is given by* (5).

Proof. Define $G_k(s)$ by subtracting terms of the binomial expansion of the summand of (4),

$$\begin{aligned} G_k(s) &= (\lambda_k^2-\lambda^2)^{-s} - (\lambda_k-\lambda)^{-s} - (\lambda_k+\lambda)^{-s} \\ &\quad - \sum_{j=0}^{[h/2]} \left(\binom{-s}{j} \lambda_k^{-2s-2j} - 2\binom{-s}{2j} \lambda_k^{-s-2j} \right) \lambda^{2j}. \end{aligned} \tag{6}$$

Elementary estimates show that

$$G(s) = \sum_{k=1}^{\infty} G_k(s)$$

converges for $\operatorname{Re} s \geq 0$. Since $G_k'(0)=0$, term by term differentiation gives

$$G'(0) = 0. \tag{7}$$

Now summing (6) over the index k gives

$$F(s) = \sum_{j=0}^{[h/2]} \left[\binom{-s}{j} Z(2s+2j) - 2\binom{-s}{2j} Z(s+2j) \right] + G(s). \tag{8}$$

Differentiating (8) at $s=0$ and using (7) gives the result. □

3. Main theorem.

Theorem 1. *For* $n = 2, 3, \dots$ *and* a *an integer such that* $0 \le a \le n-1$, *and* $\operatorname{Re} s > n$, *let*

$$Z(s,a) = \sum_{l=1}^{\infty} \beta_l^n [(l+a)(l+n-1-a)]^{-s} \tag{9}$$

where

$$\beta_l^n = \binom{n+l}{n} - \binom{n+l-2}{n}.$$

For $k = 1, 2, \dots, n$, *define functions* $t_k^n(a)$ *by the expansion*

$$\beta_{l-a}^n = \sum_{k=1}^{n} t_k^n(a) l^{k-1}. \tag{10}$$

The derivative at zero of (9) *is given by*

$$-Z'(0,a) = r_n(a) - \sum_{k=1}^{n} (1 + (-1)^{n+k}) t_k^n(a) \zeta'(1-k) - A, \tag{11}$$

where

$$r_n(a) = (1 + (-1)^n) \sum_{j=1}^{[n/2]} \frac{1}{2j} t_{2j}^n \left(\frac{n-1}{2}\right) \left(1 + \frac{1}{3} + \cdots + \frac{1}{2j-1}\right) \left(\frac{n-1}{2} - a\right)^{2j}, \tag{12}$$

and where $A = \log(n-1)$ *if* $a(n-1-a) = 0$ *and* $A = \log(a(n-1-a))$ *otherwise.*

Proof. Let

$$F(s) = \sum_{l=1}^{\infty} \beta_l^n \{((l+a)(l+n-1-a))^{-s} - (l+a)^{-s} - (l+n-1-a)^{-s}\}. \tag{13}$$

Substituting $\lambda = (n-1)/2 - a$, $h = n$, gives us the situation described in Lemma 1. Hence, if for $\operatorname{Re} s > n$, we set

$$\tilde{Z}(s,\lambda) = \sum_{l=1}^{\infty} \beta_l^n \left(l + \frac{n-1}{2} + \lambda \right)^{-s}, \tag{14}$$

we have

$$F'(0) = \sum_{j=1}^{[n/2]} \operatorname*{Res}_{s=2j} \tilde{Z}(s,0) \left(1 + \frac{1}{3} + \cdots + \frac{1}{2j-1} \right) \frac{\lambda^{2j}}{j}. \tag{15}$$

We now show that

$$\operatorname*{Res}_{s=k} \tilde{Z}(s,0) = t_k^n \left(\frac{n-1}{2} \right) \tag{16}$$

for $1 \le k \le n-1$, k an integer. Recall that the Hurwitz zeta function is defined for $\operatorname{Re} s > 1$ by

$$\zeta(s,x) = \sum_{n=1}^{\infty} (n+x)^{-s}.$$

For every x, $\zeta(s,x)$ can be continued to be analytic in s except for a pole at $s = 1$ with residue 1. Using (11), we may write

$$\begin{aligned} \tilde{Z}(s,0) &= \sum_{l=1}^{\infty} \sum_{j=1}^{n} t_j^n \left(\frac{n-1}{2} \right) \left(l + \frac{n-1}{2} \right)^{-s+j-1} \\ &= \sum_{j=1}^{n} t_j^n \left(\frac{n-1}{2} \right) \zeta\left(s - j + 1, \frac{n-1}{2} \right). \end{aligned}$$

Now (16) easily follows.

Now by (13) we have

$$Z'(0,a) = F'(0) + \tilde{Z}'(0,\lambda) + \tilde{Z}'(0,-\lambda), \tag{17}$$

and we need to compute the last two terms on the right side of (17). First note that $\beta_l^n = 0$ for $l = -1, -2, \dots, -(n-1)$ and $\beta_0^n = 1$. Hence, if $a \neq 0$, we have

$$\begin{aligned}\tilde{Z}(0, -\lambda) &= \sum_{l=1}^{\infty} \beta_l^n (l+a)^{-s} \\ &= \sum_{l=1}^{\infty} \beta_{l-a}^n l^{-s} - a^{-s} \\ &= \sum_{k=1}^{n} t_k^n(a)\zeta(s-k+1) - a^{-s}.\end{aligned}$$

Thus,

$$\tilde{Z}'(0, -\lambda) = \sum_{k=1}^{n} t_k^n(a)\zeta'(-k+1) + \log a, \qquad a \neq 0. \tag{18}$$

If $a = 0$ we have

$$\tilde{Z}'\left(0, -\frac{n-1}{2}\right) = \sum_{k=1}^{n} t_k^n(0)\zeta'(-k+1). \tag{19}$$

Similarly,

$$\begin{aligned}\tilde{Z}'(0, \lambda) = &\sum_{k=1}^{n} t_k^n(n-1-a)\zeta'(-k+1) \\ &+ \log(n-1-a), \qquad a \neq n-1,\end{aligned} \tag{20}$$

and for $a = n-1$,

$$\tilde{Z}'\left(0, \frac{n-1}{2}\right) = \sum_{k=1}^{n} t_k^n(0)\zeta'(-k-1). \tag{21}$$

From $\beta_l^n = (-1)^{n+1}\beta_{-l+n+1}^n$, we deduce

$$t_k^n(n-1-a) = (-1)^{n+k} t_k^n(a). \tag{22}$$

Now combining (15)–(22), we get (11). □

4. Computations for small n. The following is a table of $\det'\Delta_{S^n}$, for $n = 2, \ldots, 6$, and conformal Laplacians $\det'(\Delta_{S^n} + n(n-2)/4)$ for $n = 4, 6, 8$. The first expression in each set of equations is obtained from Theorem 1. The second expression is obtained by replacing the values of ζ' at negative integers by its equivalent expression in terms of values of ζ and ζ' at positive integers obtained by differentiating the functional equation

$$\zeta(1-s)G(s) = \zeta(s)G(1-s)$$
$$G(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}.$$

Numerical evaluation of $\zeta'(k)$, $k > 1$, is discussed in [**9**]. The computations below were done by using *Mathematica*.

$$\begin{aligned}
\det{}'\Delta_{S^2} &= \exp\left(\frac{1}{2} - 4\zeta'(-1)\right)\\
&= \exp\left(\frac{1}{6} + \frac{1}{3}(\gamma + \log(2\pi)) - \frac{2\zeta'(2)}{\pi^2}\right)\\
&= 3.19531\ldots\\
\det{}'\Delta_{S^3} &= \frac{1}{2}\exp(-2\zeta'(-2) - 2\zeta'(0))\\
&= \frac{1}{2}\exp\left(\log(2\pi) + \frac{\zeta(3)}{2\pi^2}\right)\\
&= 3.33885\ldots\\
\det{}'\Delta_{S^4} &= \frac{1}{3}\exp\left(\frac{15}{16} - \frac{2\zeta'(-3)}{3} - \frac{13\zeta'(-1)}{3}\right)\\
&= \frac{1}{3}\exp\left(\frac{1267}{2160} + \frac{16}{45}(\gamma + \log(2\pi)) - \frac{13\zeta'(2)}{6\pi^2} + \frac{\zeta'(4)}{2\pi^4}\right)\\
&= 1.73694\ldots\\
\det{}'\Delta_{S^5} &= \frac{1}{4}\exp\left(\frac{-\zeta'(-4)}{6} - \frac{23\zeta'(-2)}{6} - 2\zeta'(0)\right)\\
&= \frac{1}{4}\exp\left(\log(2\pi) + \frac{23\zeta(3)}{24\pi^2} - \frac{\zeta(5)}{8\pi^4}\right)\\
&= 1.76292\ldots
\end{aligned}$$

$$
\begin{aligned}
\det{}'\Delta_{S^6} &= \frac{1}{5}\exp\left(\frac{455}{432} - \frac{\zeta'(-5)}{30} - 2\zeta'(-3) - \frac{149\zeta'(-1)}{30}\right) \\
&= \frac{1}{5}\exp\left(\frac{303733}{453600} + \frac{751}{1890}(\gamma + \log(2\pi)) \right. \\
&\qquad\qquad \left. - \frac{149\zeta'(2)}{60\pi^2} + \frac{3\zeta'(4)}{2\pi^4} - \frac{\zeta'(6)}{8\pi}^{6}\right) \\
&= 1.29002\ldots \\
\det(\Delta_{S^4} + 2) &= \exp\left(-\frac{1}{144} - \frac{2\zeta'(-3)}{3} - \frac{\zeta'(-1)}{3}\right) \\
&= \exp\left(-\frac{53}{2160} + \frac{1}{45}(\gamma + \log(2\pi)) \right. \\
&\qquad\qquad \left. - \frac{\zeta'(2)}{6\pi^2} + \frac{\zeta'(4)}{2\pi^4}\right) \\
&= 1.04562\ldots \\
\det(\Delta_{S^6} + 6) &= \exp\left(\frac{1}{1350} - \frac{\zeta'(-5)}{30} + \frac{\zeta'(-1)}{30}\right) \\
&= \exp\left(\frac{1459}{453600} - \frac{1}{378}(\gamma + \log(2\pi)) \right. \\
&\qquad\qquad \left. + \frac{\zeta'(2)}{60\pi^2} - \frac{\zeta'(6)}{8\pi^6}\right) \\
&= .995257\ldots \\
\det(\Delta_{S^8} + 12) &= \exp\left(-\frac{5497}{50803200} - \frac{\zeta'(-7)}{1260} \right. \\
&\qquad\qquad \left. + \frac{\zeta'(-5)}{360} + \frac{\zeta'(-3)}{360} - \frac{\zeta'(-1)}{210}\right) \\
&= \exp\left(-\frac{391481}{76204800} + \frac{23}{56700}(\gamma + \log(2\pi)) \right. \\
&\qquad\qquad \left. - \frac{\zeta'(2)}{420\pi^2} - \frac{\zeta'(4)}{480\pi^4} + \frac{\zeta'(6)}{96\pi^6} + \frac{\zeta'(8)}{32\pi^8}\right) \\
&= 1.00069\ldots.
\end{aligned}
$$

5. Multiple gamma functions.

From the above analysis, $\det(\Delta_{S^n} + a(n-1-a))$ can be related to multiple gamma functions.

We recall that the multiple gamma functions of Barnes, see [**1, 8, 10**], are defined in terms of zeta regularized products associated with the zeta function

$$Z_{n,B}(a,x) = \sum_{l=0}^{\infty} \alpha_l^n (l+x)^{-s} \tag{23}$$

where $\alpha_l^n = \binom{n-1-l}{n-1}$. Define

$$P_n(x) = \exp(-Z'_{n,B}(0,x)).$$

Now there are polynomials ϕ_n of degree n such that $\exp(\phi_n(x))P_n(x)^{-1}$ is Barnes function $\Gamma_n(x)$ [**8**].

We prefer to work with the functions $P_n(x)$. Since $\alpha_{l-1}^n + \alpha_l^{n-1} = \alpha_l^n$, we have the functional equation

$$P_n(x+1)P_{n-1}(x) = P_n(x). \tag{24}$$

Also, it is easy to compute by the methods in Section 3 that

$$P_n(1) = \exp\left(\frac{-1}{(n-1)!}\sum_{j=1}^{n-1} \zeta'(-j)s_j^{n-1}\right) \tag{25}$$

where s_j^n are the Stirling numbers defined by $\prod_{j=1}^n (x+j-1) = \sum_{j=1}^n s_j^n x^j$.

We note now that $\beta_k^n = 2\alpha_k^n - \alpha_k^{n-1}$, so that the function $\tilde{Z}$ in Theorem 1 can be written in terms of $Z_{n,B}$ and $Z_{n-1,B}$. We get

$$\det(\Delta_{S^n} + a(n-1-a)) = e^{r_n(a)} \frac{[P_n(a)]^2[P_n(n-1-a)]^2}{P_{n-1}(a)P_{n-1}(n-1-a)} \tag{26}$$

where $r_n(a)$ is given by (12).

The equation (26) gives a factorization of the lefthand side into multiple gamma functions and an exponential factor. Computation of this expression for integral a can then be done using (24) and (25). For $n = 2$, this is equivalent to the factorization (6.41) of [**11**], and so (26) is a generalization of this formula.

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