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THE BAER-KAPLANSKY THEOREM FOR A CLASS OF GLOBAL MIXED GROUPS

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1. Introduction. This paper considers the case of isomorphic endomorphism rings $E(G)$ and $E(G')$ of global, mixed abelian groups G and G'. Throughout, all groups are understood to be abelian, and all isomorphisms $E(G) \cong E(G')$ ring isomorphisms. Before describing new results, we recount some of the history of our general problem. According to the Baer-Kaplansky theorem [**4**, Theorem 28], if G and G' are torsion groups, then every isomorphism $E(G) \cong E(G')$ is induced by an isomorphism of the groups themselves. Kaplansky's proof of this result plays off the abundance of cyclic direct summands of reduced p-groups: primitive idempotents in $E(G)$ correspond to direct summands of G' under the ring isomorphism, and an isomorphism $G \to G'$ can be constructed by utilizing a carefully chosen set of such idempotents. By observing that Kaplansky's method also works in the torsion-free case if each group possesses a cyclic, nonzero direct summand, Wolfson [**16**] subsequently proved a similar theorem for torsion-free modules over the p-adic integers. Not surprisingly, the proofs of such theorems in the mixed case require quite different techniques, because idempotents in the endomorphism rings no longer suffice to recover the full structure of the underlying groups and modules. Suitable methods were developed in a sequence of papers by May [**9**, **5**–**8**], and a wide variety of Baer-Kaplansky type theorems were put forth by him for local, mixed groups and modules in these accounts. This author's framework now seems indispensable for dealing with the problem of isomorphic endomorphism rings of many classes of mixed groups, even in the little-explored global case. The method we will use of embedding both groups in the completion of a single torsion group (Lemma 1) is due to him.

The class $\mathcal{G}$ of global mixed groups that we shall consider is described below. We show that many nonisomorphic groups $G' \in \mathcal{G}$ with $E(G') \cong E(G)$ are possible for groups G in the class. Nevertheless,

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we determine a number of structural features of the groups $G \in \mathcal{G}$ that ensure $G \cong G'$ whenever G' is an arbitrary group of the same torsion-free rank as G and $E(G) \cong E(G')$. Our results are apparently the first examples of Baer-Kaplansky type theorems for global, reduced mixed groups of torsion-free rank greater than one.

For a mixed group G we henceforth let $T = T(G)$ denote the torsion subgroup and, for a prime p, let $T_p = T_p(G)$ be the p-torsion subgroup of G. For $G \in \mathcal{G}$ the direct sums and products $\oplus T_p$ and ΠT_p are understood to be taken over the (necessarily infinite) set of primes p such that $T_p \neq 0$. We put $E = E(G)$. We use the work "rank" for torsion-free rank and "dim" for dimension over Q. When we say G is mixed we mean that G is "honestly mixed" in the sense that $0 \neq T \neq G$. We consider mixed groups in a class $\mathcal{G}$ defined as follows.

Definition 1. The *class* $\mathcal{G}$ is the class of all reduced mixed groups G such that

1) G has finite rank and

2) G/T is divisible and

3) G is self-small ($\mathrm{Hom}\,(G, \oplus G_i) = \oplus_i \mathrm{Hom}\,(G, G_i)$ for any family of groups $\{G_i\}$ such that each $G_i \cong G$).

The following theorem provides a useful alternate characterization of the class $\mathcal{G}$.

Theorem 1 [**1**, Section 2]. *For a group G the following are equivalent*:

i) $G \in \mathcal{G}$

ii) a) *The group G/T is a nonzero finite dimensional Q-vector space and*

b) *each T_p is finite and*

c) *the inclusion map $i : \oplus T_p \to \Pi T_p$ can be extended to a pure embedding of G into ΠT_p and*

d) *if π_q is the projection of ΠT_p onto any fixed T_q and F is some maximal rank free subgroup of G, then $\pi_q(iF) = T_q$ for almost all primes q.*

Henceforth, for $G \in \mathcal{G}$, we suppress the pure embedding i and simply regard $\oplus T_p \subset G \subset \Pi T_p$. Since each endomorphism of G will have a unique extension to an endomorphism of ΠT_p, we also regard $\oplus E_p \subset E \subset \Pi E_p$, where $E_p = \mathrm{End}\,(T_p)$. In this context E is the pure subring of ΠE_p consisting of those $\lambda = (\lambda_p) \in \Pi E_p$ such that $\lambda(G) \subset G$.

We note for future reference that if $\lambda = (\lambda_p) \in \mathrm{Hom}\,(G, T) \subset E$ and F is a maximal rank free subgroup of $G \in \mathcal{G}$, then $k\lambda(F) = 0$ for some $k > 0$. By ii)d) $k\lambda_p(T_p) = 0$ for almost all p. Then, by ii)b), $k'k\lambda(T) = 0$ for some $k' > 0$. Since G/T is divisible and G is reduced, $k'k\lambda = 0$. Thus, for $G \in \mathcal{G}$, the ideal $\mathrm{Hom}\,(G, T) \subset E$ coincides with $T(E)$, the torsion subgroup (ideal) of E.

The groups in $\mathcal{G}$ were first studied in [**3**] in connection with the problem of finding mixed groups G such that the endomorphism ring E is a von Neumann regular ring or a right principal projective ring. The paper [**3**] was motivated by earlier papers of Rangaswamy [**11, 12, 13**] and of Fuchs and Rangaswamy [**2**] which investigated Baer, right principal projective and von Neumann regular endomorphism rings. In [**3**] and [**1**] the following rational algebra played a key role.

Definition 2. Let $G \in \mathcal{G}$, and let $V = G/T$. Define

$$A(G) = \{\bar{\alpha} \in \mathrm{End}_Q(V) \mid \bar{\alpha} \text{ is induced by } \alpha \in E\}.$$

It is easy to check that, for each $G \in \mathcal{G}$, $A(G)$ will be a Q-subalgebra of $\mathrm{End}_Q(V)$. For $G \in \mathcal{G}$, $A(G) \cong E/\mathrm{Hom}\,(G, T) = E/T(E)$.

In [**3**] it was shown that for $G \in \mathcal{G}$ if $A = A(G)$ is semisimple then E is regular. In [**1**] the flat dimension of groups in $\mathcal{G}$ as E-modules was computed. Specifically, the flat dimension of the module ${}_EG$ was proved to be equal to the flat dimension of the module ${}_AV$. For an arbitrary finite dimensional Q-vector space V, the problem of characterizing those Q-subalgebras $A \subset \mathrm{End}_Q(V)$ which can be realized as $A = A(G)$ for some $G \in \mathcal{G}$ is discussed in [**14**]. In [**14**], this mixed group realization problem is shown to be equivalent to the torsion-free realization problem studied in [**10**].

For $G' \in \mathcal{G}$ denote $T' = T(G')$ and $V' = G'/T'$.

In [**15**] the groups in $\mathcal{G}$ were put into a categorical setting.

Definition 3. The *category* $Q\mathcal{G}$ is the category with objects groups in $\mathcal{G}$ and morphisms $\mathrm{Hom}_{Q\mathcal{G}}(G, G') = \{\bar{\alpha} \in \mathrm{Hom}_Q(V, V') \mid \bar{\alpha}$ is induced by $\alpha \in \mathrm{Hom}_Z(G, G')\}$.

It is routine to check that $Q\mathcal{G}$ is an additive category. In this setting the algebra $A(G)$ is just the $Q\mathcal{G}$-endomorphism ring of G. The category $Q\mathcal{G}$ can be regarded as a full subcategory of the category WALK. Moreover, there is an equivalence between natural subcategories of $Q\mathcal{G}$ and subcategories of the category of torsion-free finite rank groups and quasi-homomorphisms (see [**15**] for details).

We need the following result, giving some alternate characterizations of isomorphism in $Q\mathcal{G}$. List the primes in their natural order. For $G \in \mathcal{G}$ and $k \geq 1$ denote $G_k = \oplus_{1 \leq i \leq k} T_{p_i}$ and $G_k^* = G \cap \Pi_{p > p_k} T_p$. Since each T_p is finite, we have, for each k, $G = G_k \oplus G_k^*$.

Theorem 2 [**15**]. *Let G, $G' \in \mathcal{G}$. Then the following are equivalent*:

a) *$G \cong G'$ in $Q\mathcal{G}$*

b) *There exists k such that $G_k^* \cong (G')_k^*$ as abelian groups.*

c) *There exist subgroups $H \subset G$, $H' \subset G'$, each of bounded index, such that $H \cong H'$ as abelian groups.*

Let $G \in \mathcal{G}$ and G' be a group of finite rank such that $E \cong E'$, where $E' = \mathrm{End}\,(G')$. We show in Lemma 1 that $G' \in \mathcal{G}$. Indeed, we can regard G and G' to have a common torsion subgroup $T = \oplus T_p$ and each to be contained as a pure subgroup of ΠT_p. This fact, taken together with the equivalent a) $\leftrightarrow$ b) of Theorem 2, shows that to prove that $G \cong G'$ as abelian groups it will suffice to prove that $G \cong G'$ in the category $Q\mathcal{G}$. That is, we need to find inverse vector space isomorphisms $\tilde{\alpha} : V \to V'$, $\bar{\beta} : V' \to V$ such that $\bar{\alpha}$ and $\bar{\beta}$ are induced by group homomorphisms $\alpha : G \to G'$, $\beta : G' \to G$.

2. A_0-cyclic groups.

In view of the central role played by the algebra $A = A(G)$ for groups $G \in \mathcal{G}$, it seems natural, in our investigation of the possibilities for a Baer-Kaplansky theorem, to look for some sort of condition involving A. For the remainder of the paper we consider $G \in \mathcal{G}$ satisfying the condition of the following definition.

Definition 4. A group $G \in \mathcal{G}$ is called *A_0-cyclic* if there exists a vector $v \in V$ and a commutative subalgebra $A_0 \subset A$ such that $A_0 v = V$.

For $x \in G \subset \Pi T_p$ we henceforth denote $x_q = \pi_q(x)$, where, for any fixed prime q, $\pi_q : \Pi T_p \to T_q$ is the natural projection map.

If $G, G' \in \mathcal{G}$ are of rank one with $E \cong E'$ it is routine to check that $G \cong G'$. Indeed, by Lemma 1 (to be proved in Section 3), we can regard G and G' to be pure subgroups of ΠT_p with common torsion subgroup $\oplus T_p$. For any torsion-free elements $x \in G$, $x' \in G'$ the projections x_p and x'_p each are generators of T_p for almost all p. Thus, after possibly modifying x, x' in a finite number of p-components, we can construct an automorphism λ of ΠT_p carrying x to x'. Since the groups G/Zx, G'/Zx' are torsion and G' is pure in ΠT_p, it follows that $\lambda G = G'$.

In the rank two case, the following result provided our original motivation for considering A_0-cyclic groups.

Theorem 3. *Let $G \in \mathcal{G}$ be of rank two with G not A_0-cyclic. Then there exists $G' \in \mathcal{G}$ of rank two with $E = E'$ but G not isomorphic to G'.*

Proof. Let G be as in our hypothesis. Suppose there exist $a \in A$ and $v \in V$ such that $\{v, av\}$ is rationally independent. Then, since $\dim V = 2$, G is A_0-cyclic with A_0 the subalgebra generated by $\{1, a\}$, a contradiction. Hence, for each pair of elements $a \in A$, $v \in V$, av is a rational multiple of v. It follows immediately that A is semisimple with no proper idempotents. By elementary ring theory, A is a division ring. Further, suppose there exists an element $a \in A \backslash Q$. Then, for any $0 \neq v \in V$, the set $\{v, av\}$ is independent, a contradiction. Thus, we must have $A = Q$.

We divide the proof into two cases.

Case 1. The set of primes $P = \{p \mid T_p = (u_p) \oplus (v_p)$ with order $u_p >$ order $v_p > 0\}$ is infinite. Let x, y be independent torsion-free elements of G. Since the elements x_p and y_p generate T_p for almost all p, we can choose an infinite subset P' of P such that one of these elements, say x_p, is of maximal order. Thus, for $p \in P'$, we can write $T_p = (x_p) \oplus (w_p)$ with order w_p =order $v_p <$ order x_p. Moreover, we can choose w_p such that $y_p = c_p x_p \oplus w_p$ for some integer c_p.

In this case we construct G' as the pure subgroup of ΠT_p generated by T and the torsion-free elements x', y'. We construct x', y' by specifying their p-components for each prime p. First, for $p \notin P'$, set $x'_p = x_p$, $y'_p = y_p$. Next write P' as the disjoint union of infinite subsets $P' = U_{1 \leq i \leq 3} P_i$. For $p \in P_1$ we will define $x'_p = x_p$, $y'_p = d_p x_p \oplus w_p$ in such a way as to ensure that G' will not be isomorphic to G. To determine the integers d_p, suppose that G were isomorphic to G' via an isomorphism θ. Let

$$\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

be the rational matrix of the induced isomorphism from $V = G/T$ to $V' = G'/T$ with respect to the bases $\{\bar{x}, \bar{y}\}, \{\bar{x}', \bar{y}'\}$. (Here and henceforth a bar over an element will always denote a coset mod the appropriate torsion subgroup.) Then, for almost all primes p, it follows that $\theta(x_p) = \alpha x'_p + \beta y'_p$, $\theta(y_p) = \gamma x'_p + \delta y'_p$. Substitution of the definition of y'_p implies that, for almost all $p \in P_1$,

$$\theta(x_p) = (\alpha + \beta d_p) x_p \oplus \beta w_p, \qquad \theta(y_p) = (\gamma + \delta d_p) x_p \oplus \delta w_p.$$

Since $y_p = c_p x_p + w_p$, it follows that

$$\theta(y_p) = c_p \theta(x_p) + \theta(w_p) = c_p[(\alpha + \beta d_p) x_p \oplus \beta w_p] + \theta(w_p).$$

Comparison of the two computed expressions for $\theta(y_p)$ yields the equation

$$\theta(w_p) = [(\gamma + \delta d_p) x_p \oplus \delta w_p] - c_p[(\alpha + \beta d_p) x_p \oplus \beta w_p].$$

Since w_p is of strictly smaller order than x_p, for almost all $p \in P_1$ we must have

$$(\dagger)_p \qquad \begin{aligned} (\gamma + \delta d_p) - c_p(\alpha + \beta d_p) &= (\delta - c_p \beta) d_p + (\gamma - c_p \alpha) \\ &\equiv 0 \pmod p \end{aligned}$$

Suppose $(\delta - c_p \beta) \equiv 0 \pmod p$ for infinitely many $p \in P_1$. Then $(\gamma - c_p \alpha) \equiv 0 \pmod p$ on this same infinite subset. But this implies

$\alpha\delta - \gamma\beta = 0$, a contradiction. Thus, for almost all $p \in P_1$, $(\delta - c_p\beta)$ is not congruent to zero $\pmod p$. But then, for almost all $p \in P_1$, we can choose d_p to violate the congruence $(\dagger)_p$. Now, we enumerate the set of invertible rational matrices $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$ and construct the d_p's so that, for each candidate for the matrix of a possible induced isomorphism from V to V', $(\dagger)_p$ will fail for infinitely many $p \in P_1$. At this stage we know that, however we define x'_p, y'_p on $P_2 \cup P_3$, the group we construct cannot be isomorphic to G. For future needs we also choose each d_p not congruent to zero $\operatorname{mod} p$.

We next employ the primes in $P_2 \cup P_3$ to make $E' = E$. Let $G' \in \mathcal{G}$ be such that $\oplus T_p \subset G' \subset \oplus T_p$ and regard $E' \subset \Pi E_p$. The ring $E_0 = \{\lambda = (\lambda_p) \in \Pi E_p \mid$ there exists $q = q(\lambda) \in Q$ such that, for almost all p, $\lambda_p =$ left multiplication by $q\}$ will be a subring of $E' = E(G')$. Furthermore, if $A' = A(G') \cong Q$, then E' will coincide with E_0. (The ring E_0 is simply the inverse image of $Q \subset A'$ under the ring epimorphism $E' \to E'/T(E') \cong A'$.) Thus, since $A = A(G) \cong Q$, to make $E' = E_0 = E$ it suffices to make $A' \cong Q$.

For $p \in P_2$ set $x'_p = x_p$, $y'_p = w_p$; for $p \in P_3$ set $x'_p = w_p$, $y'_p = x_p$. Regard $A' \subset \operatorname{End}_Q(V')$ and identify $\operatorname{End}_Q(V')$ with $M_2(Q)$ by associating each map with its matrix with respect to the basis $\{\bar{x}', \bar{y}'\}$ of V'. Now, looking at the set of primes $P_2 \cup P_3$ and recalling that order $x_p >$ order w_p, shows that any element $a' \in A'$ must have an associated matrix of the form $\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$. Say that a' is induced by $\theta \in E'$. Then, for almost all $p \in P_1$,

$$\theta(y'_p) = d_p\theta(x'_p) + \theta(w_p) = d_p\alpha x'_p + \theta(w_p) = \delta y'_p = \delta d_p x_p \oplus \delta w_p.$$

Arguing as above, we see that $d_p(\alpha - \delta) \equiv 0 \pmod p$ for almost all $p \in P_1$. Since no d_p is divisible by p, we have $\alpha = \delta$. Thus, $A' \cong Q$ and the proof in Case 1 is complete.

Case 2. The set P is finite. In this case we must have $L = \{p \mid T_p = (u_p) \neq 0\}$ is infinite. Otherwise T_p would be a rank two homogeneous p-group for almost all p. But then $T_p = (x_p) \oplus (y_p)$ for almost all p. This would imply that $A \cong M_2(Q)$, a contradiction. We construct G', exactly as in Case 1, by utilizing the set of primes L to construct suitable independent torsion-free elements x', y'. We just sketch the similar but easier arguments for this case. Write $L = U_{1 \leq i \leq 4} L_i$, a disjoint union of infinite subsets. Assume, without loss of generality, that, for $p \in L_1$, $x_p = u_p$, $0 \neq y_p = c_p u_p$. On L_1 set $x'_p = u_p$, $y'_p = d_p u_p$, the d_p's chosen, in a similar manner to that of Case 1, to ensure that G' cannot be isomorphic to G. On L_2 set $x'_p = u_p$, $y'_p = 0$. On L_3 set $y'_p = u_p$, $x'_p = 0$. On L_4 set $x'_p = y'_p = u_p$. The definition of x'_p, y'_p on the set of primes $\cup_{2 \leq i \leq 4} L_i$ will ensure $A' \cong Q$, hence $E' = E_0 = E$. $\square$

3. Reduction lemmas and first results. In this section we prove three lemmas that simplify our work in proving isomorphism theorems for A_0-cyclic groups $G \in \mathcal{G}$. We get some immediate results as direct applications.

Lemma 1. *Assume $G \in \mathcal{G}$ has torsion $T = \oplus T_p$ and G' is a group such that $E \cong E'$. Then there is a pure embedding $\varphi : G' \to \Pi T_p$ with $T \subset \varphi G'$ such that $E = E(\varphi G')$. Moreover, $G' \in \mathcal{G}$ if G' has finite rank.*

Proof. First we note that G' contains no $Z(p^\infty)$ or Q for then E' would contain a copy of $\hat{Z}_p$ or Q. But, since G is pure in ΠT_p with each T_p finite, E contains neither $\hat{Z}_p$ nor Q. Therefore, G' is reduced. Let $\phi : E' \to E$ be an isomorphism and $T' = \oplus T'_p$ be the torsion subgroup of G'. For each p, ϕ induces an isomorphism $\phi_p : T_p(E') \to T_p(E) = E_p$. Since $G \in \mathcal{G}$ each $E_p = E(T_p)$ is finite, hence so is $T_p(E')$. It follows that T'_p must also be finite and, hence, that $T_p(E') = E(T'_p)$. Thus, $\phi_p : E(T'_p) \cong E_p$. By the Baer-Kaplansky theorem, ϕ_p is induced by an isomorphism $\varphi_p : T'_p \to T_p$. The isomorphisms φ_p in turn induce an isomorphism $\varphi : \Pi T'_p \to \Pi T_p$ such that $\phi(\alpha')$ and $\varphi\alpha'\varphi^{-1}$ agree on T for all $\alpha' \in E'$. Since $\Pi T_p/T$ is divisible and ΠT_p is reduced $\phi(\alpha') = \varphi\alpha'\varphi^{-1}$ (as endomorphisms of ΠT_p), for all $\alpha' \in E'$. Thus, if $\alpha \in E$, then $\alpha = \varphi(\alpha') = \varphi\alpha'\varphi^{-1}$ for some $\alpha' \in E'$. It follows that $E \subset E(\varphi G')$. Let $\beta \in E(\varphi G')$. Then $\varphi^{-1}\beta\varphi \in E'$ so that $\beta = \phi(\varphi^{-1}\beta\varphi) \in E$. Hence $E(\varphi G') = E$ and φ is the desired embedding of G' in ΠT_p.

If $\pi : G' \to G'/T'$ is the natural map, then ϕ induces an isomorphism $\mathrm{Hom}\,(G'/T', T') \to \mathrm{Hom}\,(G/T, T)$ via $\alpha \to \phi(\alpha\pi)$ because $\phi(\alpha\pi)(T) = \varphi\alpha\pi\varphi^{-1}(T) = 0$ if $\alpha \in \mathrm{Hom}\,(G'/T', T')$. Since $\mathrm{Hom}\,(G/T, T)$ is zero, so is $\mathrm{Hom}\,(G'/T', T')$. If $T'_p = 0$, then $T_p = 0$, so G is a pure subgroup of $\Pi T_{q \neq p} T_q$. It follows that $1/p \in E$, hence $1/p \in E'$ as well, so that $pG' = G'$. Thus, if G'/T' is not p-divisible for some prime p, then $T'_p \neq 0$. But this would yield a nonzero composition of homomorphisms $G'/T' \to (G'/T')/p(G'/T') \to T'_p$, contrary to fact. Hence G'/T' is divisible. Since each T'_p is finite, G'/T' is divisible and G' is reduced, the inclusion map $\oplus T'_p \to \Pi T'_p$ can be extended to a pure embedding of G' into $\Pi T'_p$.

To finish the proof, set $\varphi G = G'$ and suppose now that $E = E'$ as subrings of ΠE_p, and that G' has finite rank. Note that we cannot have $\mathrm{rank}\, G' = 0$, for then $G' = \oplus T_p$ and $E' = \Pi E_p \neq E$. For the sake of contradiction, assume that $G' \neq \mathcal{G}$. Then condition ii) d) of Theorem 1 must fail for G', for we have already verified that the group G' satisfies ii a)–c). Let F' be a maximal free subgroup of G'. Then $\pi_p(F') \neq T_p$ for infinitely many p, so there exists $\psi \in \Pi E_p$ such that $\psi(F') = 0$ and $\psi(T_p) \neq 0$ for infinitely many p. Then $\psi(g)$ must be torsion-free for some $g \in G$, otherwise $\psi \in \mathrm{Hom}\,(G, T) = T(E)$. Since G has finite rank it is easy to construct a $\lambda \in \Pi E_p$ such that $\lambda\psi(g) \notin G$. But since $\psi(F') = 0$ then $\lambda\psi \in \mathrm{Hom}\,(G', T) \subset E' = E$, and we have the desired contradiction. □

Lemma 2. *Assume that $G, G' \in \mathcal{G}$ are purely embedded in ΠT_p, each containing $T = \oplus T_p$, with $E = E'$ and $\mathrm{rank}\, G' \leq \mathrm{rank}\, G$. If G is A_0-cyclic, then $G \cong G'$ if there exists an element $v' \in V' = G'/T$ whose annihilator in A_0 is zero.*

Proof. By the remark after Definition 2, identify A with $E/T(E)$. We claim there is a commutative, free subring $C \subset E$ such that $A_0 = Q\overline{C}$, where $\overline{C} = [C + T(E)]/T(E)$. If $F = \oplus Z x_i$ is a maximal free subgroup of G and $\bar{\alpha}, \bar{\beta} \in A_0$, then for each i we have $(\alpha\beta - \beta\alpha)(x_{ip}) = 0$ for almost all p. Hence $(\alpha\beta - \beta\alpha)T_p = 0$ for almost all p by Theorem 1(d). Thus, we can obtain commuting preimages of $\bar{\alpha}$ and $\bar{\beta}$ in E. It follows that there exists a commutative set of preimages in E of any fixed Q-basis for A_0. The subring of E generated by any such set is the desired C.

Now let $x \in G$ be such that $A_0\bar{x} = V$. If $a \in A_0$ with $a\bar{x} = 0$, then $aV = aA_0\bar{x} = A_0 a\bar{x} = 0$, thus $a = 0$. Since C is isomorphic to $\overline{C} \subset A_0$, it follows that the annihilator of x in C is zero. Then $\operatorname{rank} Cx = \operatorname{rank} C = \operatorname{rank} \overline{C} = \dim A_0 = \dim V$ so that Cx is a maximal free subgroup of G. Hence $\pi_p(Cx) = T_p$ for almost all p. It follows that $C_p x_p = T_p$ for almost all p, where C_p is the image of C under the natural projection map $\nu_p : \Pi E_q \to E_p$. Assume that there exists $v' \in V'$ with zero annihilator in A_0, and choose a preimage x' of v' such that $x'_p = 0$ whenever $C_p x_p \neq T_p$. For every p, we can obtain $c^{(p)} \in C_p$ such that $c^{(p)} x_p = x'_p$. If $\lambda = (\lambda_p) \in \Pi E_p$ is defined by setting $\lambda_p = c^{(p)}$ for each prime p, then λ centralizes C, and $\lambda(x) = x'$. Thus $\lambda Cx = C\lambda x = Cx' \subset G'$. Since G/Cx is torsion and G' is pure in ΠT_p, it follows that $\lambda(G) \subset G'$. The following chain of (in)equalities

$$\dim V' = \operatorname{rank} G' \leq \operatorname{rank} G = \dim V = \dim A_0 = \dim A_0 v' \leq \dim V'$$

shows that $A_0 v' = V'$. Let $\bar{\lambda} : V \to V'$ be the map induced by λ. We have

$$\bar{\lambda} V = Q\bar{\lambda}\overline{C}\bar{x} = Q\overline{C}\bar{x}' = A_0 \bar{x}' = A_0 v' = V',$$

hence $\bar{\lambda}$ is an isomorphism. Because λ_p is an automorphism of T_p for almost all p (those p such that $C_p x_p = C_p x'_p = T_p$), there is a homomorphism $\delta : G' \to G$ such that $\bar{\delta}$ is inverse to $\bar{\lambda}$. Thus, $G \cong G'$ in $Q\mathcal{G}$, as desired. □

Lemma 3. *Assume that $G, G' \in \mathcal{G}$ are embedded in ΠT_p with $E = E'$, and suppose that G is A_0-cyclic. Let $A_0 = \oplus_{1 \leq i \leq k} F_i \oplus J$ be a Wedderburn decomposition for the commutative Artinian algebra A_0, where each F_i is a field and J is the radical of A_0. For $1 \leq i \leq k$, let $f_i \in F_i$ be the identity element. Then*

a) *We can choose the subring C of Lemma 2 to contain a set of orthogonal idempotents $\{e_i \mid 1 \leq i \leq k\}$ with $\bar{e}_i = f_i$ for each i and*

b) *each $e_i G, e_i G' \in \mathcal{G}$ and*

c) *$G \cong G'$ if each $e_i G \cong e_i G'$ in $Q\mathcal{G}$ and*

d) *$E(e_i G) = E(e_i G')$ for all i and*

e) *$A(e_i G)$ can be identified with $f_i A f_i$ and each $e_i G$ is $f_i A_0$-cyclic where $f_i A_0$ is the direct sum of a field and a nilpotent ideal.*

Proof. Parts a) and b) are each an easy exercise. For c) note that in $Q\mathcal{G}$, $G \cong \oplus e_i G$, $G' \cong \oplus e_i G'$. To show that $G \cong G'$ in $Q\mathcal{G}$ (and therefore $G \cong G'$ as abelian groups), it suffices to show each $e_iG \cong e_iG'$ in $Q\mathcal{G}$. Part d) follows since $E(e_iG) = E(e_iG') = e_iEe_i$. For e), note that $A(e_iG) \cong E(e_iG)/T[E(e_i(G)] \cong f_iAf_i$. The algebra f_iAf_i has commutative subalgebra f_iA_0. If $A_0v = V$ each e_iG is f_iA_0-cyclic with $f_iA_0(f_iv) = f_iV$. Finally, for each i, $f_iA_0 = F_i \oplus f_iJ$ as stated. □

We obtain our first isomorphism theorem as an immediate consequence of our lemmas.

Theorem 4. *Let $G \in \mathcal{G}$ be A_0-cyclic with A_0 semi-simple. Suppose that G' is a group with* $\operatorname{rank} G' = \operatorname{rank} G$ *and $E' \cong E$. Then $G' \cong G$.*

Proof. Using Lemma 1 we can regard $\oplus T_p \subset G$, $G' \subset \Pi T_p$, and $E' = E$. Adopting the notation of Lemma 3, we need only prove that $e_iG \cong e_iG'$ in $Q\mathcal{G}$ for each i. Since A_0 is semi-simple, we have $f_iA_0 = F_i$, a field. Say $A_0v = V$. Then

$$(\mathcal{F}) \qquad \begin{aligned} \operatorname{rank} e_iG &= \dim f_iV = \dim F_i(f_iv) \\ &= \dim F_i \le \dim f_iV' = \operatorname{rank} e_iG'. \end{aligned}$$

The sole inequality in $(\mathcal{F})$ holds since f_iV' is a nonzero F_i-vector space. But $\operatorname{rank} G = \sum \operatorname{rank} e_iG$, $\operatorname{rank} G' = \sum \operatorname{rank} e_iG'$. Since $\operatorname{rank} G = \operatorname{rank} G'$, we must have $\operatorname{rank} e_iG = \operatorname{rank} e_iG'$ for each i. By Lemma 3, each e_iG is f_iA_0-cyclic. By Lemma 2, to show that $e_iG \cong e_iG'$ in $Q\mathcal{G}$, it suffices to find $v_i' \in f_iV'$ with zero annihilator in $f_iA_0 = F_i$. Because each F_i is a field, any nonzero $v_i' \in f_iV'$ will do, and the proof is complete. □

We mention for future reference that Theorem 4 can be proved under the weaker assumption that $\operatorname{rank} G' \le \operatorname{rank} G$. In this case we would choose a fixed i such that $\operatorname{rank} e_iG' \le \operatorname{rank} e_iG$. Then $(\mathcal{F})$ would show that, for this i, $\operatorname{rank} e_iG = \operatorname{rank} e_iG'$. As above, we would conclude that, for this i, $e_iG' \cong e_iG$. An induction on $\operatorname{rank} G$ would complete the proof.

Arguing in a similar way as in the proof of Theorem 4, we can show the following:

Theorem 5. *Let $G \in \mathcal{G}$ be such that the module ${}_AV$ is simple with* *commutative centralizer. Let G' be a group with* $\operatorname{rank} G' = \operatorname{rank} G$ *and* $E' \cong E$. *Then* $G' \cong G$.

Proof. Identify A with $E/T(E)$. Regard $\oplus T_p \subset G$, $G' \subset \Pi T_p$ with $E = E'$. Here the centralizer of V is a field F and A is a full $d \times d$ matrix ring over F. Let $\{e_i \mid 1 \leq i \leq d\}$ be a set of preimages in E of a set of matrix units $\{f_i \mid 1 \leq i \leq d\} \subset A$. In the category $Q\mathcal{G}$ there are associated decompositions $G \cong \oplus_{1 \leq i \leq d} e_i G$, $G' \cong \oplus_{1 \leq i \leq d} e_i G'$. For each i, the algebra $f_i A f_i = A(e_i G)$ is isomorphic to F. Furthermore, $\operatorname{rank} e_i G = \dim f_i V = \dim F$. Thus, each $e_i G$ is A_0-cyclic for $A_0 = A(e_i G)$. Now we proceed as in the previous proof. □

4. Projection properties. In this section we consider the Baer-Kaplansky problem for A_0-cyclic groups with non-semisimple A_0. We can get positive results by assuming that G has a set of primes of one of the two types described as follows.

Definition 5. An infinite set of primes $\mathcal{S}$ is called a *weak projective set* (respectively, *strong projective set*) for the group $G \subset \Pi T_p$, if for each $g \in G$ of infinite order, $g_p \neq 0$ (respectively $g_p \notin pT_p$) for almost all $p \in \mathcal{S}$.

Plainly, if $\mathcal{S}$ is a strong or weak projective set for G, so is any infinite subset of $\mathcal{S}$. Also, to check a set of primes $\mathcal{S}$ is a strong or weak projective set, it is enough to check the appropriate condition for each $g \in F$, F a maximal free subgroup of G. This makes it easy to construct numerous examples of groups $G \in \mathcal{G}$ with weak or strong projective sets.

Theorem 6. *Let $G \in \mathcal{G}$ be A_0-cyclic with a strong projective set $\mathcal{S}$. Let G' be a group with* $\operatorname{rank} G' = \operatorname{rank} G$ *and* $E' \cong E$. *Then* $G' \cong G$.

Proof. By Lemma 1, regard $\oplus T_p \subset G$, $G' \subset \Pi T_p$ with $E' = E$. Choose a set of orthogonal idempotents $\{e_1, \dots, e_n\}$ with $A_0 = \oplus \bar{e}_i A_0$, $\bar{e}_i A_0 = F_i \oplus J_i$, as in Lemma 3. By replacing G, G' with G_k^*, $(G')_k^*$ for a suitable k, we can assume that, as abelian groups $G = \oplus e_i G$,

$G' = \oplus e_i G'$. Since this replacement will affect neither the hypotheses nor the conclusion of Theorem 6, it will suffice to prove that our new $G = G_k^*$, $G' = (G')_k^*$ are isomorphic. (Refer back to Theorem 2 and the comments following it. Also note that if $E(G) = E(G')$ then $E(G_k^*) = E[(G')_k^*]$ for any k.)

Having made this replacement, we next observe that, since $\operatorname{rank} G' = \operatorname{rank} G$, we must have $\operatorname{rank} e_i G' \leq \operatorname{rank} e_i G$ for some i. We will prove that, for this i, $e_i G \cong e_i G'$. A rank induction argument then will show that the remaining parts, $\oplus_{j \neq i} e_j G$ and $\oplus_{j \neq i} e_j G'$ are also isomorphic. It is easy to check that these remaining parts satisfy the hypotheses of Theorem 6. In particular, $\oplus_{j \neq i} e_j G$ is $\oplus_{j \neq i} \bar{e}_j A_0$-cyclic, $E(\oplus_{j \neq i} e_j G) = E(\oplus_{j \neq i} e_j G')$ and the set $\mathcal{S}$ will still be a strong projective set for $\oplus_{j \neq i} e_j G$.

Note that, since $E(e_i G) = e_i E e_i = E(e_i G')$ and $\operatorname{rank} e_i G' \leq \operatorname{rank} e_i G$, the second part of Lemma 1 tells us that $e_i G' \in \mathcal{G}$. Rename the summands $e_i G, e_i G'$ with the names G, G'. By Lemma 3, we have reduced our problem to the case $A_0 = F \oplus J$. Our goal will be to apply Lemma 2. To do this, we need to find an element $v' \in V' = G'/T$ with zero annihilator in A_0. As in the proof of Lemma 2, we choose a free subring $C \subset E$ such that $\overline{C} = [C + T(E)]/T(E)$ is a full free subring of A_0. Here we construct C in the form $C = S \oplus N$ where S is a free subring of E with $\overline{S} = [S + T(E)]/T(E)$ the ring of integers in the algebraic number field F and N is a free subring of E with $\overline{N} = [N + T(E)]/T(E)$ a full free subring of the radical J. As in the proof of Lemma 2, if $A_0 \bar{x} = V$ then $C x_p = T_p$ for almost all p.

Let F' be a maximal free subgroup of G'. By deleting a finite subset from $\mathcal{S}$, if necessary, we can assume that for all $p \in \mathcal{S}$: (1) $C x_p = T_p$, (2) $\pi_p(F') = T_p$ and (3) S/pS is semisimple. Let $p \in \mathcal{S}$ and say $pS = P_p^1 \cdots P_p^{j_p}$ is a decomposition of pS, where $\{P_p^1, P_p^2, \ldots, P_p^{j_p}\}$ is a set of j_p-many distinct prime ideals in S. Such a decomposition exists since $S \cong \overline{S}$, a Dedekind domain, and S/pS is semisimple. Plainly, $j_p \leq \operatorname{rank} S$. Thus, there exists a positive integer $r \leq \operatorname{rank} S$ and an infinite subset $\mathcal{S}_1 \subset \mathcal{S}$ with pS decomposing into exactly r distinct primes in S for all $p \in \mathcal{S}_1$. Note that $\mathcal{S}_1$ is still a strong projective set for G.

For each $p \in \mathcal{S}_1$, let $\{e(1,p), \ldots, e(r,p)\}$ be the set of orthogonal idempotents in the ring S/pS such that $S/pS = \oplus_{1 \leq j \leq r} e(j,p) S/pS \cong$

$\oplus_{1\le j\le r} S/P_p^j$. Regarding C/pC as an S/pS module, we have a similar decomposition $C/pC = \oplus_{1\le j\le r} e(j,p)C/pC$. Here each $e(j,p)C/pC \cong S/P_p^j \oplus e(j,p)N/pN$ is a local ring. For convenience, for each $p \in \mathcal{S}_1$ and $1 \le j \le r$, denote the ring $e(j,p)C/pC$ by C_p^j and the coset $e(j,p)[x_p + pT_p]$ by x_p^j. Since, for all $p \in \mathcal{S}_1$, $Cx_p = T_p$, it follows that $C/pC(x_p + pT_p) = T_p/pT_p$ and hence that:

$$(d_p) \qquad T_p/pT_p = \bigoplus_{1\le j\le r} C_p^j x_p^j.$$

(In each decomposition (d_p) some of the x_p^j's may be zero.)

Call a subset $I \subset \{1,\dots,r\}$ a support set for an element $y \in F'$ if there exists an infinite subset $\mathcal{S}_2 = \mathcal{S}_2(y,I) \subset \mathcal{S}_1$ such that, for all $p \in \mathcal{S}_2$, $y_p + pT_p = \oplus_{1\le j\le r} c_p^j x_p^j$ with $x_p^j \ne 0$ and c_p^j a unit in C_p^j for all $j \in I$. (A single element y could have more than one support set, corresponding to different choices for $\mathcal{S}_2$). Choose an element $x' \in F'$ such that x' has a support set maximal in the collection $\{I \subset \{1,\dots,r\} \mid I \text{ is a support set for some } y \in F'\}$. To complete the proof of Theorem 6, we show that $v' = \bar{x}'$ is our desired element of V' with zero annihilator in A_0.

Say that I is a support set for x' that is maximal in the collection of support sets. Let $\mathcal{S}_2 = \mathcal{S}_2(x',I)$ be as above. Then, for each $j \in \{1,\dots,r\}\backslash I$, either $x_p^j = 0$ or the x_p^j coordinate of x' in (d_p) is a nonunit, for all but finitely many primes in $\mathcal{S}_2$. This holds since, otherwise, $I \cup \{j\}$ would be a support set for x'. By deleting finitely many primes from $\mathcal{S}_2$ we can assume that, for all $p \in \mathcal{S}_2$ and for all $j \notin I$, either $x_p^j = 0$ or the x_p^j coordinate of x' in (d_p) is a nonunit. Also assume that $p \in \mathcal{S}_2 \to p > r$.

We show that if $j \notin I$, then $x_p^j = 0$ for almost all $p \in S_2$. Suppose, by way of contradiction, that there exists a $j \notin I$ and an infinite subset $\mathcal{S}_3 \subset \mathcal{S}_2$ such that $x_p^j \ne 0$ for all $p \in \mathcal{S}_3$. Let $F' = \oplus_{1\le k\le m} Zy_k$ be the maximal free subgroup of G' chosen above. Then, since $\pi_p(F') = T_p$, the set $\{e(j,p)[y_{kp} + pT_p] \mid 1 \le k \le m\}$ generates $e(j,p)T_p/pT_p$. Because each C_p^j is local, for each $p \in \mathcal{S}_3$ there must exist $y(p) \in \{y_1,\dots,y_m\}$ such that $[y(p) + pT_p]$ has an invertible x_p^j coefficient in decomposition (d_p). Thus, we can choose an infinite subset $\mathcal{S}_4 \subset \mathcal{S}_3$ and a fixed $y \in \{y_1,\dots,y_m\}$ such that, for all $p \in \mathcal{S}_4$, $[y + pT_p]$ has an invertible x_p^j coefficient in decomposition (d_p). We consider the set of elements $H = \{tx' + y \mid 0 \leq t \leq r\}$. Any element of H will have an invertible x_p^j coefficient in (d_p) for all $p \in \mathcal{S}_4$ since its coefficient will be the sum of a nonunit and a unit in the local ring C_p^j. Fix $p \in \mathcal{S}_4$. Suppose, for some $i \in I$ and $0 \leq t,\ s \leq r$, the elements $tx' + y$ and $sx' + y$ each has a noninvertible x_p^i coefficient. Since C_p^i is local $(t-s)x'$ also has a noninvertible x_p^i coefficient. Since $i \in I$ and $p \in \mathcal{S}_4 \to p > r$, it follows that $t = s$. Thus, for each $i \in I$, at most one element of H can have a noninvertible x_p^i coefficient. Since cardinality $I \leq r$ and cardinality $H = r+1$, at least one element of H has invertible x_p^i coefficient in (d_p) for all $i \in I$ and our fixed $p \in \mathcal{S}_4$. Arguing as before, there exists an infinite subset $\mathcal{S}_5 \subset \mathcal{S}_4$ and an $h \in H$ such that h has an invertible x_p^i coefficient in (d_p) for all $i \in I$ and all $p \in \mathcal{S}_5$. But then $h \in H \subset F'$ will have support set $I \cup \{j\}$, contradicting the maximality of I in the collection of support sets. We have shown that $(j \notin I \to x_p^j = 0)$ for almost all $p \in \mathcal{S}_2$. Without loss, assume that $(j \notin I \to x_p^j = 0)$ for all $p \in \mathcal{S}_2$.

Suppose that the annihilator of $\bar{x}'$ in A_0 is nonzero. Since $\overline{C}$ is a full free subring of A_0 it follows that there exists $0 \neq \gamma \in C$ with $\gamma x' = 0$. For $p \in \mathcal{S}_2$ consider the decomposition (d_p) of x'

$$x'_p + pT_p = \bigoplus_{1 \leq j \leq r} c_p^j x_p^j.$$

Since $\gamma x'_p = 0$ for all p, for all $p \in \mathcal{S}_2$, we have

$$0 = \bigoplus_{1 \leq j \leq r} c_p^j [e(j,p)\gamma] x_p^j.$$

Here we are simply writing γ for the map induced by γ on T_p/pT_p.

Fix $p \in \mathcal{S}_2$. If $j \notin I$, then $x_p^j = 0$. If $j \in I$ then c_p^j is invertible and $c_p^j[e(j,p)\gamma]x_p^j = 0$, hence $[e(j,p)\gamma]x_p^j = 0$. Thus, for all $1 \leq j \leq r$, $[e(j,p)\gamma]x_p^j = 0$. But then $\gamma(x_p + pT_p) = (\gamma x)_p + pT_p = 0 + pT_p$. Hence $(\gamma x)_p \in pT_p$. Since p is an arbitrary element of the infinite subset $\mathcal{S}_2 \subset \mathcal{S}_1 \subset \mathcal{S}$ and $\mathcal{S}$ is a strong projective set, then γx must be a torsion element. Thus, there exists a positive integer k with $0 \neq k\gamma \in C$ and $(k\gamma)x = 0$. This contradicts the previously established fact that the annihilator of x in C is zero. This final contradiction completes the proof of Theorem 6. $\square$

Recall that, for any fixed prime q, $\nu_q : \Pi E_p \to E_q$ is the natural projection map. We can replace the existence of a strong projective set for G with the existence of a weak projective set for G, provided we also require that the projections $C_p = \nu_p(C)$ be local subrings of $E_p = \operatorname{End}(T_p)$ for almost all $p \in \mathcal{S}$. We sketch the similar but much easier proof.

Theorem 7. *Let $G \in \mathcal{G}$ be A_0-cyclic with a weak projective set $\mathcal{S}$. Suppose that there exists a free subring $C \subset E$ with $\overline{C}$ a full free subring of A_0 such that $C_p = \nu_p(C)$ is a local ring for almost all primes $p \in \mathcal{S}$. Let G' be a group with $\operatorname{rank} G' = \operatorname{rank} G$ and $E' \cong E$. Then $G' \cong G$.*

Proof. Regard $G, G' \subset \Pi T_p$ with $E = E'$. Let $C \subset E$ and $\mathcal{S}$ be as in the hypothesis of Theorem 7. If $A_0 \bar{x} = V$, then $Cx_p = C_p x_p = T_p$ for almost all p. Let $F' = \oplus_{1 \le k \le m} Z y_k$ be a maximal free subgroup of G'. By the second part of Lemma 1, $G' \in \mathcal{G}$. Thus, for almost all p, the group T_p is generated by $\{y_{1p}, \dots, y_{mp}\}$. We can assume, without loss of generality, that $T_p = C_p x_p$ and that T_p is generated by $\{y_{1p}, \dots, y_{mp}\}$ for all $p \in \mathcal{S}$. For $p \in \mathcal{S}$ and each $1 \le k \le m$, choose $c_k^{(p)} \in C_p$ such that $y_{kp} = c_k^{(p)} x_p$. It is easy to check that for each of these p there must exist a $k(p)$ with $1 \le k(p) \le m$ such that $c_{k(p)}^{(p)}$ is a unit in C_p. Thus, there exists a fixed k with $1 \le k \le m$ and an infinite subset $\mathcal{S}_1 \subset \mathcal{S}$ such that, for all $p \in \mathcal{S}_1$, $c_k^{(p)}$ is a unit in the commutative local ring C_p. If there were a $0 \ne \gamma \in C$ with $\gamma y_k = 0$, then, for all p, $\pi_p(\gamma y_k) = \nu_p(\gamma) c_k^{(p)} x_p = 0$. Thus, we would have $\nu_p(\gamma) x_p = 0$ for all $p \in \mathcal{S}_1$. Hence $\pi_p(\gamma x) = \nu_p(\gamma) x_p = 0$ for all $p \in \mathcal{S}_1$. Since $\mathcal{S}_1 \subset \mathcal{S}$ it would follow that γx must be a torsion element. Since $C \cong Cx$ we obtain a contradiction. We have shown that the annihilator of y_k in C is zero. Thus, $\bar{y}_k \in V'$ has zero annihilator in A_0, and we apply Lemma 2 to complete the proof. □

The condition that $\nu_p(C)$ be local for almost all $p \in \mathcal{S}$ is independent of the choice of the free subring $C \subset E$ with $\overline{C}$ full in A_0. Since an infinite subset of a weak projective set for G is still a weak projective set for G, we could (trivially) modify our condition to the requirement that $\nu_p(C)$ be local on an infinite subset of $\mathcal{S}$.

5. Cyclic p-components and special J's. Suppose $G \in \mathcal{G}$ and almost all the p-components T_p of G are cyclic. If $\alpha, \beta \in E$, then $(\alpha\beta - \beta\alpha)_p = 0$ whenever T_p is cyclic, hence $(\alpha\beta - \beta\alpha)G$ is bounded. It follows that the algebra $A = A(G)$ is commutative. Note that in this case G is A_0-cyclic if and only if the module ${}_AV$ is cyclic. Moreover, for any subring $C \subset E$, almost all of the projections C_p must be local (being subrings of $E(T_p)$ for cyclic T_p). With these facts in mind, we can improve on Theorem 7.

Theorem 8. *Assume that almost all the p-components T_p of $G \in \mathcal{G}$ are cyclic, ${}_AV$ is cyclic, and G possesses a weak projective set. If G' is a group with* $\operatorname{rank} G = \operatorname{rank} G'$, *then every isomorphism $E' \cong E$ is induced by an isomorphism $G' \cong G$.*

Proof. Let $\phi : E' \to E$ be an isomorphism. As in the proof of Lemma 1, ϕ induces an embedding φ of G' in ΠT_p such that $E = E(\varphi G')$, and it is not hard to check that ϕ is induced by an isomorphism of the groups if $\alpha G = \varphi G'$ for α in the center of $E(\Pi T_p)$. Because the hypotheses of Theorem 7 are met, there is an automorphism β of ΠT_p with $\beta G = \varphi G'$. Define a second automorphism $\alpha = (\alpha_p)$ of ΠT_p by taking $\alpha_p = 1_{T_p}$ if T_p is not cyclic, and $\alpha_p = \beta_p$ otherwise. Then α centralizes $E(\Pi T_p)$, and $\alpha G = \varphi G'$. $\square$

Our next theorem is a modification of Theorem 8. We can drop the requirement that G have a projective set if we require that $J(A)$, the radical of A, is suitably small.

Theorem 9. *Assume that almost all the p-components T_p of $G \in \mathcal{G}$ are cyclic and ${}_AV$ is cyclic. Assume also that* $\dim J(A) \leq 2$. *If G' is a group with* $\operatorname{rank} G' = \operatorname{rank} G$, *then every isomorphism $E' \cong E$ is induced by an isomorphism $G' \cong G$.*

Proof. We first show that if $\operatorname{rank} G' = \operatorname{rank} G$ and $E' \cong E$, then $G' \cong G$. Applying Lemma 1, we regard $\oplus T_p \subset G$, $G' \subset \Pi T_p$ with $E = E'$ and note that $G' \in \mathcal{G}$. By using a set of orthogonal idempotents of A and repeating the steps that began the proof of Theorem 6, we may reduce our proof to the case $A = F \oplus J$ with F a field and $J = J(A)$, and $\operatorname{rank} G' \leq \operatorname{rank} G$. Note that in this reduction our new J, an idempotent times our old J, still has dimension no greater than two. Having made this reduction, we immediately conclude that $G' \cong G$ if $J = 0$ (Theorem 4) or if G has a weak projective set (Theorem 8). (Like Theorem 4, and with only a slight modification in the proof, Theorem 8 holds under the weaker assumption that $\operatorname{rank} G' \leq \operatorname{rank} G$.)

For the remaining case assume that $J \neq 0$ and that G does not possess a weak projective set. We will see that $\dim J = 2$ in this case. Since ${}_AV$ is cyclic there exists a torsion-free element $x \in G$ such that Ex is of full rank in G. Choose $0 \neq \bar{\alpha} \in J$, and note that support $(\alpha x) = \{\text{primes } p \mid (\alpha x)_p \neq 0\}$ is infinite but is not a projective set for G. Since Ex is full-rank in G, it follows that we may choose $\beta \in E(G)$ so that βx is torsion-free and $(\beta x)_p = 0$ for all p in an infinite set $P \subset$ support (αx). Then $\bar{\beta}$ cannot be a unit of A, so $\bar{\beta} \in J$. Because of the supports of the elements αx, βx no nontrivial combination of $\bar{\alpha}$ and $\bar{\beta}$ can annihilate $\bar{x}$. Thus, $\bar{\alpha}$ and $\bar{\beta}$ are independent, wherefore they span J.

Since $G' \in \mathcal{G}$, it is not hard to see that there exists $x' \in G'$ such that x'_p generates the cyclic group T_p for infinitely many p in both P and support (βx). We claim $\bar{x}'$ has zero annihilator in $A = F \oplus J$. To establish the claim, suppose that $(\bar{\gamma} + n\bar{\alpha} + m\bar{\beta})\bar{x}' = 0$ where $\bar{\gamma} \in F$ and $n, m \in Z$. First note that $\bar{\gamma} = 0$ because otherwise $\bar{\gamma} + n\bar{\alpha} + m\bar{\beta}$ is invertible in A. Thus there is an integer $k > 0$ with $k(n\alpha + m\beta)x' = 0$. For $p \in P$ it follows that $kn\alpha_p x'_p = 0$. We must have $n = 0$, otherwise, for infinitely many $p \in P$ with x'_p a generator of T_p, $\alpha_p x'_p = 0$. Thus $\alpha_p = 0$ on this infinite subset of $P \subset$ support (αx), a contradiction. Similarly, $m = 0$.

By Lemma 2, we have $G' \cong G$. As in the proof of Theorem 8, one can adjust the isomorphism to obtain one which centralizes $E(\Pi T_p)$. This shows that the original ring isomorphism $E' \cong E$ is induced and completes the proof. □

Corollary. *Assume that almost all the components T_p of $G \in \mathcal{G}$ are cyclic, ${}_AV$ is cyclic and* $\operatorname{rank} G \leq 3$. *If* $\operatorname{rank} G' = \operatorname{rank} G$, *then every isomorphism $E' \cong E$ is induced by an isomorphism $G' \cong G$.*

Proof. As in the proof of Lemma 2, since A is commutative and ${}_AV$ is faithful, the cyclic module ${}_AV$ is isomorphic to A. Thus, $\dim A = \dim V = \operatorname{rank} G \leq 3$. Hence $\dim J \leq 2$ and we can apply Theorem 9. □

Let I be a nilpotent ideal of a rational algebra A. In the statement of our final theorem, the nilpotency of I is the smallest positive integer n such that $I^n = 0$. It is straightforward to check that if $n > d = \dim I$ then $n = d + 1$ and I has a Q-basis of the form $\{\beta_1, \beta_1\beta_2, \dots, \beta_1\beta_2\cdots\beta_{n-1}\}$ for some $\{\beta_i \mid 1 \leq i \leq n-1\} \subset I$.

Theorem 10. *Assume that $G \in \mathcal{G}$ is A_0-cyclic. Suppose that the nilpotency of $J(A_0)$ is greater than $\dim J(A_0)$. If G' is a group with $\operatorname{rank} G' = \operatorname{rank} G$ and $E' \cong E$, then $G' \cong G$.*

Proof. As usual, assume that $\oplus T_p \subset G$, $G' \subset \Pi T_p$ with $E' = E$. By the comments preceding the statement of our theorem, $J = J(A_0)$ has a basis of the form $\{\beta_1, \beta_1\beta_2, \dots, \beta_1\beta_2\cdots\beta_{n-1}\}$. Again, we now proceed as in the beginning of the proof of Theorem 6 to reduce to the case: $A_0 = F \oplus J$, $G' \in \mathcal{G}$, $\operatorname{rank} G' \leq \operatorname{rank} G$. Having made this reduction, if our new J (an idempotent $\bar{e}$ times our old J) is zero, we can conclude that $G' \cong G$, by Theorem 4. Otherwise, our new J will have a Q-basis $\{\gamma_1, \gamma_1\gamma_2, \dots, \gamma_1\gamma_2\cdots\gamma_{k-1}\}$ with $\gamma_j = \bar{e}\beta_j$, $1 \leq j \leq k-1 \leq n-1$. In this case, let $\theta \in E$ be a preimage of $\gamma_1\gamma_2\cdots\gamma_{k-1}$ under the map $E \to E/\operatorname{Hom}(G,T) \cong A$. Since $\gamma_1\gamma_2\cdots\gamma_{k-1} \neq 0$, then $\theta \notin \operatorname{Hom}(G,T)$. Since $E = E'$, in view of the remark in the second paragraph after Theorem 1, we have that $\operatorname{Hom}(G,T) = T(E) = T(E') = \operatorname{Hom}(G',T)$. Thus $\theta \notin \operatorname{Hom}(G',T)$. Hence, there exists an element $v' \in V'$ with $\gamma_1\gamma_2\cdots\gamma_{k-1}v' \neq 0$.

We show that the annihilator of v' in A_0 is zero. An application of Lemma 2 will then complete the proof. To this end, suppose that $a \in A_0$ with

$$av' = (\gamma + q_1\gamma_1 + q_2\gamma_1\gamma_2 + \cdots + q_{k-1}\gamma_1\gamma_2\cdots\gamma_{k-1})v' = 0,$$

where $\gamma \in F$ and the q_i's are rationals. As before, $\gamma = 0$, otherwise the element $a \in A_0$ would be invertible. Thus, we have

$$\gamma_1(q_1 + q_2\gamma_2 + \cdots + q_{k-1}\gamma_2\cdots\gamma_{k-1})v' = 0.$$

If $q_1 \neq 0$, the element in parentheses is a unit of the commutative ring A_0, hence $\gamma_1 v' = 0$. But v' was chosen so that $\gamma_1\gamma_2\cdots\gamma_{k-1}v' \neq 0$, a contradiction. It follows that $q_1 = 0$. We can now repeat this argument to conclude that $q_2 = \cdots = q_{k-1} = 0$. Thus $a = 0$, and the proof is complete. □

Corollary. *Let $G \in \mathcal{G}$ be A_0-cyclic with* $\operatorname{rank} G = 2$. *If G' is a group with* $\operatorname{rank} G' = \operatorname{rank} G$ *and $E' \cong E$, then $G' \cong G$.*

Taken together, the above corollary and Theorem 3 show that, for rank two groups $G \in \mathcal{G}$, the class of A_0-cyclic groups coincides with the class of groups for which our version of the Baer-Kaplansky theorem holds.

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