ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 26, Number 4, Fall 1996

LOCAL ARTINIAN RINGS AND THE FRÖBERG RELATION

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0. Introduction. R will denote a local Artinian ring, $\mathbf{m}$ the unique maximal ideal and k the residue field. The *Poincaré series* is $P_R(t) = \sum_{i\geq 0} \dim(\operatorname{Tor}_i^R(k,k))t^i$ and the *Hilbert series* is $H_R(t) = \sum_{i\geq 0} \dim(\mathbf{m}^i/\mathbf{m}^{i+1})t^i$. $P_R(t)$ is a formal power series while $H_R(t)$ is a polynomial (but not the Hilbert polynomial) since R is Artinian.

We consider the *Fröberg relation*, first studied in [**7**]: $P_R(t) = H_R(-t)^{-1}$. R is a *Fröberg ring* if this relation holds. We are interested in determining when R is Fröberg, particularly in the critical case of $\mathbf{m}^3 = 0$ (cf., [**2**]). The Fröberg relation is a strong property; $P_R(t)$ need not even be rational [**1**]. Our main result is: If $\mathbf{m}^3 = 0$ and $\mathbf{m} \cdot \operatorname{ann} x = \mathbf{m}^2$ for all $x \in \mathbf{m}\backslash\mathbf{m}^2$, then R is a Fröberg ring.

This work was motivated by the classification problem for Artinian Witt rings (which are necessarily local) with the unique maximal ideal being the ideal I of even dimensional forms. We briefly review the problem. Let W_F be the class of Witt rings of nonsingular quadratic forms over fields L such that $|L^{\cdot}/L^{\cdot 2}|$ is finite and $-1 \in \sum L^2$. The class W_E of Artinian Witt rings of elementary type is defined inductively. Start with the fundamental Witt rings: $\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/4\mathbf{Z}$ and those of local type (Witt rings of certain P-adic fields, cf. [**13**, Chapter 3]). W_E consists of the Witt rings built from the fundamental Witt rings by a finite sequence of fiber products (over $\mathbf{Z}/2\mathbf{Z}$) and group ring extensions (by finite groups of exponent 2.) Lastly, W_A is the class of Artinian abstract Witt rings defined by Marshall [**13**]. Then $W_E \subset W_F \subset W_A$. It is conjectured that $W_E = W_A$, but neither inclusion is known to be an equality.

We show that if $R \in W_E$, that is, R is of elementary type, then R is a Fröberg ring. This motivated our search for sufficient conditions on R to be Fröberg that could easily be checked for abstract Witt rings.

Received by the editors on August 31, 1994, and in revised form on January 30, 1995.

In the first section we gather together examples of Fröberg rings that have appeared at least implicitly in the literature. To these examples we add Witt rings of elementary type. In the second section we again gather necessary and sufficient conditions that R be Fröberg that have appeared elsewhere. The third section contains the main theorem while the fourth discusses some partial converses.

1. Examples. We list examples of local Artinian rings which are Fröberg rings.

Example 1.1. If $\mathbf{m}^2 = 0$ then R is a Fröberg ring.

Proof. We use induction on $\mu(\mathbf{m})$, the minimal number of generators for $\mathbf{m}$. If $\mu(\mathbf{m}) = 1$, then a direct computation gives $P_R(t) = (1-t)^{-1}$ and $H_R(t) = 1 + t$. If $\mu(\mathbf{m}) > 1$, then choose a nonzero $x \in \mathbf{m}$ and set $S = R/(x)$. Then $P_R(t)^{-1} = P_S(t)^{-1} - t$ by [**9**, 3.4.4] and $H_R(t) = H_S(t) + t$. By induction, $P_S(t)^{-1} = H_S(-t)$ and so the Fröberg formula also holds for R. □

Example 1.2. If R is a Gorenstein with socle degree $\sigma(R) = 2$ and $\mu(\mathbf{m}) > 1$, then R is a Fröberg ring.

Proof. Let $S = R/(0:\mathbf{m})$. S is a Fröberg ring by (1.1), $P_R(t)^{-1} = P_S(t)^{-1} + t^2$ by [**11**, Theorem 2] and $H_R(t) = H_S(t) + t^2$. Hence, the Fröberg formula holds for R. □

Example 1.3. Suppose that $\mu(\mathbf{m}) \leq 2$ and R is not a complete intersection. Then R is a Fröberg ring if and only if $\mathbf{m}^2 = 0$. In particular, (1.2) fails if $\mu(\mathbf{m}) = 1$.

Proof. If $\mu(\mathbf{m}) = 1$, then a simple computation shows that $P_R(t) = (1-t)^{-1}$. This equals $H_R(-t)^{-1}$ if and only if $\mathbf{m}^2 = 0$. If $\mu(\mathbf{m}) = 2$, then by [**14**], $P_R(t) = (1+t^2)/(1-et^2-(e-1)t^3)$, where $e = \dim H_1(K)$, K the Koszul complex. If $P_R(t)^{-1}$ is a polynomial, then it has degree 1, and so R a Fröberg ring implies $\mathbf{m}^2 = 0$. The converse is Example 1.1. □

Example 1.4. Suppose that $\mu(\mathbf{m}) = 2$ and R is a complete intersection. Then R is a Fröberg ring if and only if $\mathbf{m}^3 = 0$ and $\dim(\mathbf{m}^2) = 1$.

Proof. In this case $P_R(t) = (1-t)^{-2}$. Thus, the Fröberg formula holds if and only if $H_R(t) = 1 + 2t + t^2$. □

Example 1.5. If R is Gorenstein with $\mu(\mathbf{m}) = 3$ and $\sigma(R) = 3$, then R is a Fröberg ring.

Proof. This is Case 3a in [**3**]. □

Example 1.6. Suppose that $R = k[x, y, z]/I$ where $(x, y, z)^3 \subset I$, $\mu(I) \geq 4$ and I is generated by forms of degree 2. Then R is a Fröberg ring.

Proof. These rings are Cases 4a, 5a and 6a in [**3**]. Since the cases are not explicitly defined in [**3**], we note that $\mu(I) \geq 4$ implies R is among cases 4, 5 and 6. Only in the "a" case is R Artinian. □

The rest of this section is concerned with adding one more example to the list, namely, (Artinian) Witt rings of elementary type.

Proposition 1.7. *Let $R = R_1 \sqcap R_2$ be the fiber product (over k) of R_1 and R_2. If R_1 and R_2 are Fröberg rings, then so is R.*

Proof. $P_R(t)^{-1} = P_{R_1}(t)^{-1} + P_{R_2}(t)^{-1} - 1$, by [**4**], and clearly $H_R(t) = H_{R_1}(t) + H_{R_2}(t) - 1$. □

Let C_p denote the cyclic group of prime order p. $R[C_p]$ denotes the group ring. $R[C_p]$ is again a local ring if and only if $\operatorname{char} k = p$ by [**10**, p. 153]. We will use the following only in the case $p = 2$, but the general case is no more difficult to prove.

Proposition 1.8. *Suppose* $\operatorname{char} k = p > 0$. *Let $S = R[C_p]$. Then $P_S(t) = (1-t)^{-1} P_R(t)$.*

Proof. Let g denote a generator of C_p, set $\lambda = 1+g+\cdots+g^{p-1}$ and let $\mathbf{n}$ denote the maximal ideal of S. Then $\mathbf{n} = S\mathbf{m} + S(1-g)$. Let (X, d) be a minimal R-resolution of k, with augmentation map $\alpha : X_0 \to k$. Let (Y, d) be a minimal S-resolution of $S/S(1-g)$ with augmentation map β. Here we may take $Y_i = S$ for all i and $d_i(1) = 1-g$ if i is odd, $d_i(1) = \lambda$ if i is even.

Consider $U = (X \otimes_R S) \otimes_S Y$, with $U_n = \prod_{i+j=n}(X_i \otimes S) \otimes Y_j$ and $d((x \otimes s) \otimes y) = (d(x) \otimes s) \otimes y + (-1)^{a+b}(x \otimes s) \otimes d(y)$. Here $a = \deg x$, $b = \deg y$ and we define $d(X_0) = 0 = d(Y_0)$. Then U is a complex, $d(U) \subset \mathbf{n}U$ and U has the projection $\gamma : (R \otimes S) \otimes S \to S/\mathbf{n}$ as the augmentation map.

We check that U is exact. Suppose $z = \sum_{i+j=n}(\sum_{k=1}^{p} a_{ik} \otimes g^k) \otimes 1_j$ is a cycle, where each $a_{ik} \in X_i$ and 1_j denotes the unity in $Y_j = S$. Then

$$(*) \qquad d(a_{ik}) + (-1)^n(a_{i-1\,k} - a_{i-1\,k-1}) = 0,$$

if $n-i$ is even, while if $n-i$ is odd, then

$$d(a_{ik}) + (-1)^n \sum_{k=1}^{p} a_{i-1\,k} = 0.$$

For each cycle $z \in U_n$, let $\pi(z)$ be the largest index i such that some $a_{ik} \neq 0$ (set $\pi(0) = -1$). We prove exactness by showing that if $\pi(z) \geq 0$ then there is a boundary b with $\pi(z+b) < \pi(z)$. First suppose that $\pi(z) = n$. Summing the equations $(*)$ for $i = n$ yields $\sum_k d(a_{nk}) = 0$. Choose $\beta \in X_{n+1}$ with $d(\beta) = \sum a_{nk}$. The boundary of

$$(\beta \otimes 1) \otimes 1_0 + (-1)^n\left(\left(\sum_{k=1}^{p-1} a_{nk} \otimes (g^{k-1} + \cdots + g + 1)\right) \otimes 1_1\right)$$

is $b = (\sum a_{nk} \otimes g^k) \otimes 1_0 + w$, where $w \in \Pi(X_i \otimes S) \otimes Y_j$ over (i, j) with $i + j = n$ and $j > 0$. Thus $\pi(z-b) < n = \pi(z)$.

Next suppose $\pi(z) = i < n$, $z \neq 0$; that is, $a_{i+1\,k} = 0$ for all k. If $n-i$ is odd, then $a_{i1} = \cdots = a_{ip}$. Then $a_{ip} \otimes \lambda = \sum a_{ik} \otimes g^k$ and $b \equiv d((a_{ip} \otimes 1) \otimes 1_{n-i+1}) = (a_{ip} \otimes d) \otimes 1_{n-i} + (-1)^{n+1}(d(a_{ip}) \otimes 1) \otimes 1_{n-i+1}$.

Thus $\pi(z-b) < \pi(z)$. Lastly, if $n-i$ is even then $\sum_k a_{ik} = 0$ and $\sum a_{ik} \otimes g^k = \sum_{k=1}^{p-1} a_{ik} \otimes (g^k - 1)$. As before, the boundary of $(\sum_{k=1}^{p-1} a_{ik} \otimes (g^{k-1} + \cdots + g + 1)) \otimes 1_{n-i+1}$ is $b = (\sum a_{ik} \otimes g^k) \otimes 1_{n-i} + w$, where $w \in \Pi(X_i \otimes S) \otimes Y_j$ summed over (i,j) with $i+j = n$ and $j > n-i$. Thus, $\pi(z-b) < \pi(z)$ as desired.

Clearly $\operatorname{rank}(U_n) = \sum_{i=0}^{n} \operatorname{rank}(X_i)$, so that $P_S(t) = (1-t)^{-1} P_R(t)$. □

When $S = R[C_p]$ the relation between H_S and H_R is not as simple as that for the Poincaré series. For example, if $R_1 = \mathbf{Z}/27\mathbf{Z}$ and $R_2 = (\mathbf{Z}/3\mathbf{Z})[C_3]$, then $H_{R_1}(t) = 1 + t + t^2 = H_{R_2}(t)$. However, if $S_i = R_i[C_3]$, then

$$H_{S_1}(t) = H_{R_1}(t) \cdot (1 + t + t^4)$$
$$H_{S_2}(t) = H_{R_2}(t) \cdot (1 + t + t^2).$$

But we do have the following result.

Proposition 1.9. *Suppose that R is a Fröberg ring and* $\operatorname{char} k = p > 0$. *$S = R[C_p]$ is a Fröberg ring if and only if $p = 2$.*

Proof. $(\rightarrow)$. We have $H_S(t) = (1+t)P_R(-t)^{-1} = (1+t)H_R(t)$ by (1.8). Suppose that $\mathbf{m}^{q-1} \neq 0$ and $\mathbf{m}^q = 0$. The maximal ideal $\mathbf{n}$ of S is $S\mathbf{m} + S(1-g)$, where g is a generator of C_p. We must have $\mathbf{n}^{q+1} = 0$, as $\deg H_S = 1 + \deg H_R$. But if $p \geq 3$ and $x \in \mathbf{m}^{q-1} \backslash \{0\}$ then $x(1-g)^2 \in \mathbf{n}^{q+1} \backslash \{0\}$, a contradiction. So $p = 2$.

$(\leftarrow)$. Let $1 \neq g \in C_2$. Note that $(1-g)^2 = 2(1-g) \in \mathbf{m}(1-g)$, as $\operatorname{char} k = 2$. Again the maximal ideal of S is $\mathbf{n} = S\mathbf{m} + S(1-g)$. Then $\mathbf{n}^2 = S\mathbf{m}^2 + S\mathbf{m}(1-g)$ and, in general, $\mathbf{n}^q = S\mathbf{m}^q + S\mathbf{m}^{q-1}(1-g)$. We thus have a k-space isomorphism $(\mathbf{m}^q/\mathbf{m}^{q+1}) \times (\mathbf{m}^{q-1}/\mathbf{m}^q) \rightarrow \mathbf{n}^q/\mathbf{n}^{q+1}$. Thus $H_S(t) = (1+t)H_R(t) = (1+t)P_R(-t)^{-1} = P_S(-t)^{-1}$, by (1.8). □

Corollary 1.10. *If R is an (Artinian) Witt ring of elementary type, then R is a Fröberg ring.*

Proof. If R is $\mathbf{Z}/2\mathbf{Z}$ or $\mathbf{Z}/4\mathbf{Z}$ this follows from (1.1). If R is of local type, then R is Gorenstein of socle degree 2 (cf. [**6**]) and so R is a Fröberg ring by (1.2). Then (1.7) and (1.9) imply the result. □

2. Equivalent conditions. We gather conditions on R equivalent to R being a Fröberg ring.

Notation. We let $b_i = \dim \operatorname{Tor}_i^R(k,k)$ and $h_i = \dim(\mathbf{m}^i/\mathbf{m}^{i+1})$. Let c_i be defined by $H_R(-t)^{-1} = \sum_{i\geq 0} c_i t^i$.

Lemma 2.1. (1) $b_0 = c_0 = 1$.

(2) $b_1 = h_1 = c_1 = \mu(\mathbf{m})$.

(3) $c_{n+1} = \sum_{i=1}^{n+1}(-1)^{i+1} h_i c_{n+1-i}$.

Proof. Elementary. □

Definition. Suppose (F, D) is a minimal resolution of k. Let:

$$A(i,j) = \dim(\mathbf{m}^i F_j \cap Z_j)/\mathbf{m}^{i-1}Z_j), \qquad i \geq 1$$
$$B(i,j) = \dim(\mathbf{m}^i F_j \cap \mathbf{m}^{i-2}Z_j/\mathbf{m}^{i-1}Z_j), \qquad i \geq 2.$$

Note that $A(1,j) = 0$ and $B(2,j) = A(2,j)$.

Lemma 2.2. Let (F, d) be a minimal resolution of k. Then for $i \geq 1, j > 1$:

$$\mu(\mathbf{m}^{i-1}Z_j) = h_i b_j - A(i,j) + B(i+1,j) - \mu(\mathbf{m}^i Z_{j-1}).$$

Proof. Consider the sequence

$$\frac{\mathbf{m}^{i-1}Z_j}{\mathbf{m}^i Z_j} \xrightarrow{\pi} \frac{\mathbf{m}^i F_j}{\mathbf{m}^{i+1}F_j} \xrightarrow{e} \frac{\mathbf{m}^i Z_{j-1}}{\mathbf{m}^{i+1}Z_{j-1}},$$

where e is induced by d_j and π by inclusion. Then e is surjective and $\ker(e) = (\mathbf{m}^i F_j \cap Z_j)/\mathbf{m}^{i+1}F_j$, since if $d_j(v) \in \mathbf{m}^{i+1}Z_{j-1}$ where $v \in \mathbf{m}^i F_j$, then $d_j(v) = \sum a_k d_j(w_k)$ for some $a_k \in \mathbf{m}^{i+1}$ and $w_k \in F_j$. Thus $v \equiv v - \sum a_k w_k \bmod m^{i+1}F_j$ and $d(v - \sum a_k w_k) = 0$.

We also have $\mathrm{im}\,(\pi) = \mathbf{m}^{i-1}Z_j/\mathbf{m}^{i+1}F_j$. Thus $\dim\,(\ker(e)/\mathrm{im}\,(\pi)) = A(i,j)$. And $\ker(\pi) = \mathbf{m}^{i-1}Z_j \cap \mathbf{m}^{i+1}F_j/\mathbf{m}^i Z_j$ which has dimension $B(i+1,j)$. We obtain

$$\begin{aligned}\mu(\mathbf{m}^{i-1}Z_j) &= \dim\,(\mathrm{im}\,(\pi)) + \dim\,(\ker(\pi))\\ &= B(i+1,j) - A(i,j) + \dim\,(\ker(e))\\ &= B(i+1,j) - A(i,j) + h_i b_j - \mu(\mathbf{m}^i Z_{j-1}). \qquad \square\end{aligned}$$

Theorem 2.3. *Let (F,d) be a minimal resolution of k. For $n \geq 1$,*

$$\begin{aligned}b_{n+1} = \sum_{i=1}^{n+1}(-1)^{i+1}h_i b_{n-i+1} &+ \sum_{i=2}^{n+1}(-1)^i B(i,n-i+2)\\ &- \sum_{i=2}^{n}(-1)^i A(i,n-i+1).\end{aligned}$$

Proof. $b_{n+1} = \mu(Z_n)$ so it suffices to prove

$$\begin{aligned}\mu(\mathbf{m}^{n-p}Z_p) = &\sum_{i=n-p+1}^{n+1}(-1)^{n-p+1-i}h_i b_{n-i+1}\\ &+ \sum_{i=n-p+2}^{n+1}(-1)^{n-p-i}B(i,n-i+2)\\ &- \sum_{i=n-p+1}^{n}(-1)^{n-p-i}A(i,n-i+1).\end{aligned}$$

This follows from (2.2) by induction on p. $\square$

Corollary 2.4. *R is Fröberg ring if and only if for all $n \geq 1$,*

$$\sum_{i=2}^{n+1}(-1)^i B(i,n-i+2) = \sum_{i=2}^{n}(-1)^i A(i,n-i+1).$$

Proof. The Fröberg relation is $b_n = c_n$ for all $n \geq 1$. If the relation holds, then $b_{n+1} = \sum(-1)^{n-i+1} h_i b_{n+1-i}$ by (2.1) and so (2.3) implies the result. Conversely, if the two sums in the statement are equal, then $b_{n+1} = \sum(-1)^{n-i+1} h_i b_{n-i+1}$. As $b_0 = c_0$ and $b_1 = c_1$, a simple induction argument shows that $b_{n+1} = c_{n+1}$ for all n. □

Tate [**15**] constructed an R-algebra, that is, a graded skew commutative differential algebra, U resolution of k which can be assumed minimal [**8**]. Löfwall [**12**] calls U an S-R-algebra if $du \in \mathbf{m}^2 U$ implies $u \in \mathbf{m}U$ or $u \in U_0$.

Proposition 2.5. *Suppose that* $\mathbf{m}^3 = 0$. *The following are equivalent*:

(1) $P_R(t) = H_R(-t)^{-1}$.

(2) *If* (F, d) *is a minimal resolution of* k, *then* $\mathbf{m}^2 F \subset \mathbf{m}Z$.

(3) *There is one* (*every*) *minimal* R-*algebra resolution f* k *that is an* S-R-*algebra.*

(4) *The Yoneda algebra* $\mathrm{Ext}_R(k, k)$ *is generated by* $\mathrm{Ext}^1_R(k, k)$.

Proof. (1) is equivalent to $B(2, 1) = 0$ and, for $n > 1$, $B(2, n) = A(2, n-1)$ by (2.4). Since $B(2, n-1) = A(2, n-1)$, (1) is equivalent to $B(2, n) = 0$ for all $n \geq 1$. That is, (1) holds if and only if $\mathbf{m}^2 F \cap Z \subset \mathbf{m}Z$.

(2) → (3). If $d(x) \in \mathbf{m}^2 F$, then $d(x) \in \mathbf{m}Z$ and so $d(x) = d(\sum a_i y_i)$ for some $a_i \in \mathbf{m}$ and $y_i \in F$. Then $x - \sum a_i y_i \in Z \subset \mathbf{m}F$ and so $x \in \mathbf{m}F$. Thus F is an S-R-algebra.

(3) → (2). Let $x \in \mathbf{m}^2 F$. Then $\mathbf{m}^3 = 0$ implies that $x \in Z$ and so $x = d(y) \in \mathbf{m}^2 F$ for some $y \in F$. Thus, $y \in \mathbf{m}F$ and so $x = d(y) \in \mathbf{m}Z$.

(4) ↔ (1) is [**12**, Theorem 2.3]. □

3. The case $\mathbf{m}^3 = 0$. Throughout this section we assume that $\mathbf{m}^3 = 0$ and set $g = \dim(\mathbf{m}/\mathbf{m}^2)$ and $h = \dim(\mathbf{m}^2)$. For a finitely generated R-module M we abuse notation and write $\dim M$ for $\dim_k(M/\mathbf{m}M)$, the cardinality of a minimal generating set for M.

Lemma 3.1. *Let A and B be R-modules with A finitely generated and free. Let $f : A \to B$ be an R-module map. If*

(i) $f(A) \subset \mathbf{m}B$,

(ii) $\ker(f) \subset \mathbf{m}A$ *and*

(iii) $\mathbf{m}^2 A = \mathbf{m}\ker(f)$,

then $\dim(\ker(f)) + \dim(f(\mathbf{m}A)) = g\dim(A)$.

Proof. The following is exact:

$$0 \longrightarrow \frac{\ker(f)}{\mathbf{m}\ker(f)} \longrightarrow \frac{\mathbf{m}A}{\mathbf{m}^2 A} \longrightarrow f(\mathbf{m}A) \longrightarrow 0. \qquad \square$$

For any ideal $I \not\subset \mathbf{m}^2$ let $(F(I), d(I))$ be a minimal resolution of R/I. We note that $F(I)_0 = R$ and $d(I)_0(r) = r + I$. If $\dim I = p$, then we may take $F(I)_1 = R^p$ and $d(I)_1(r_1, \dots, r_p) = \sum r_i x_i$, where $x_1, \dots, x_p$ is a minimal generating set for I.

Lemma 3.2. *Let $I \not\subset \mathbf{m}^2$ be an ideal with* $\dim I = 1$. *Let* $J = \operatorname{ann} I$. *Then*

(a) *for $p \geq 0 : F(I)_{p+1} = F(J)_p$ and*

(b) *for $p \geq 1 : d(I)_{p+1} = d(J)_p$.*

Proof. Let $x \in \mathbf{m}/\mathbf{m}^2$ generate I. Then $F(I)_1 = R$ and $\ker d(I)_1 = \operatorname{ann}(x) = J$. Now $F(J)_0 = R$ and $\ker d(J)_0 = J$ also. Hence, we may assume that the resolutions are the same. $\square$

Definition. For an ideal $I \not\subset \mathbf{m}^2$ we define inductively:

$$\begin{aligned} \beta_0(I) &= 1 \\ \beta_1(I) &= \dim(I) \\ \beta_2(I) &= g\dim(I) - \dim(\mathbf{m}I) \\ \beta_{i+2}(I) &= g\beta_{i+1}(I) - h\beta_i(I), \quad \text{for } i \geq 1. \end{aligned}$$

Lemma 3.3. *For $q \geq 2$ and all $1 \leq i \leq q-1$, suppose we have*

(i) $\mathbf{m} \cdot \ker(d(I)_i) = \mathbf{m}^2 F(I)_i$ *and*

(ii) $\dim\left(\ker(d(I)_i)\right) = \beta_{i+1}(I)$.

Suppose further that a set B *generates* $\ker d(I)_q$ *and* $|B| = \beta_{q+1}(I)$. *Then*

(a) B *is a minimal generating set for* $\ker d(I)_q$ *and*

(b) $\mathbf{m} \ker d(I)_q = \mathbf{m}^2 F(I)_q$.

Proof. We consider $\bar{d}(I)_q : \mathbf{m}F(I)_q/\mathbf{m}^2F(I)_q \to \mathbf{m}^2F(I)_{q-1}$. By (i) this is surjective. So $\dim(\ker \bar{d}(I)_q) = g\dim(F(I)_q) - h\dim(F(I)_{q-1}) = g\beta_q(I) - h\beta_{q-1}(I) = \beta_{q+1}(I)$ by (ii). The map $\ker d(I)_q/\mathbf{m}\cdot\ker d(I)_q \to \ker \bar{d}(I)_q$ is surjective. Hence,

$$|B| \geq \dim(\ker d(I)_q) \geq \dim(\ker \bar{d}(I)_q) = \beta_{q+1}(I),$$

giving equality throughout. Equality in the first place gives (b), while equality in the second gives (a). □

Lemma 3.4. *Suppose that* $x_1, \dots, x_p \in \mathbf{m}\backslash\mathbf{m}^2$, $p \geq 2$, *is a minimal generating set for an ideal* I. *Set* $J = (x_1, \dots, x_{p-1})$ *and* $K = (x_p)$. *Let* $L = L^0, L^1, L^2, \dots$ *be a sequence of ideals such that*

(i) $\dim L = \dim(\mathbf{m}J \cap \mathbf{m}K)$ *and*

(ii) $\dim L^i = \dim(\mathbf{m}L^{i-1})$, $i \geq 1$.

Then, for $q \geq 1$,

$$\beta_q(I) = \beta_q(J) + \beta_q(K) + \sum_{i=0}^{q-2} \beta_{q-i-1}(L^i).$$

Proof. The first two cases must be done separately.

$$\begin{aligned}
\beta_1(I) &= \dim(I) \\
&= \dim(J) + \dim(K) = \beta_1(J) + \beta_1(K) \\
\beta_2(I) &= g\dim(I) - \dim(\mathbf{m}I) \\
&= g(\dim(J) + \dim(K)) \\
&\quad - (\dim(\mathbf{m}J) + \dim(\mathbf{m}K) - \dim(\mathbf{m}J \cap \mathbf{m}K)) \\
&= \beta_2(J) + \beta_2(K) + \beta_1(L^0).
\end{aligned}$$

For $q \geq 3$, we have

$$\begin{aligned}\beta_q(I) &= g\beta_{q-1}(I) - h\beta_{q-2}(I)\\ &= g(\beta_{q-1}(J) + \beta_{q-1}(K) + \beta_{q-2}(L^0) + \cdots + \beta_1(L^{q-3}))\\ &\quad - h(\beta_{q-2}(J) + \beta_{q-2}(K) + \beta_{q-3}(L^0) + \cdots + \beta_1(L^{q-4}))\\ &= \beta_q(J) + \beta_q(K) + \beta_{q-1}(L^0) + \cdots + \beta_3(L^{q-4}) + g\beta_1(L^{q-3}).\end{aligned}$$

Since $g\beta_1(L^{q-3}) = g\dim(L^{q-3}) = \beta_2(L^{q-3}) + \dim(\mathbf{m}L^{q-3}) = \beta_2(L^{q-3}) + \beta_1(L^{q-4})$, the result holds. □

Construction. Let $J_1, \dots, J_s$ be ideals of R with no $J_i \subset \mathbf{m}^2$. Let $M = \prod J_i \subset R^s$. We write $F(M)_q$ for $\prod F(J_i)_q$ and $d(M)_q$ for $\prod d(J_i)_q$.

Let $y_{i1}, \dots, y_{ia(i)}$ be a minimal generating set for J_i. Let $T_{ij} \subset \mathbf{m}y_{ij}$ be such that the image of T_{ij} is a basis for $\mathbf{m}y_{ij}/((\mathbf{m}y_{i1} + \cdots + \mathbf{m}y_{ij-1}) \cap \mathbf{m}y_{ij})$ or simply $\mathbf{m}y_{i1}$ if $j = 1$. Let $S_{ij} \subset \mathbf{m}$ be a set with $y_{ij}S_{ij} = T_{ij}$. Let J_{ij} be the ideal generated on S_{ij}. Define

$$J_i^1 = \prod_j J_{ij}, \qquad M^1 = \prod_i J_i^1.$$

We will also write M^0 for M, M^2 for $(M^1)^1$, M^3 for $(M^2)^1$, etc.

Lemma 3.5. *With the notation above,*

(a) $\dim(M^1) = \dim(\mathbf{m}M)$.

(b) $F(M^1)_0 = F(M)_1$.

(c) $d(M)_1(V)$ *is a minimal generating set for* $d(M)_1(\mathbf{m}F(M)_1)$, *where*

$$V_i = \{z_{ij}e_j \mid z_{ij} \in S_{ij},\ 1 \leq j \leq a(i)\} \subset J_i^1$$

and

$$V = \{v_ie_i \mid v_i \in V_i,\ 1 \leq i \leq s\} \subset \mathbf{m}F(M)_1.$$

Proof. (a) $\cup_j T_{ij}$ is a minimal generating set for $\mathbf{m}J_i$. Hence, $\dim(\mathbf{m}J_i^1) = \sum |S_{ij}| = \sum |T_{ij}| = \dim(\mathbf{m}J_i)$.

(b) $F(M^1)_0$ has rank $\sum a(i) = \dim M = \operatorname{rank} F(M)_1$. Hence, we may assume that $F(M^1)_0 = F(M)_1$.

(c) $d(J_i)_1(\mathbf{m}F(J_i)_1) = \mathbf{m}y_{i1}+\cdots+\mathbf{m}y_{i\,a(i)} = \mathbf{m}J_i$. And $d(J_i)_1(V_i) = \cup y_{ij}S_{ij} = \cup T_{ij}$ which is a minimal generating set for $\mathbf{m}J_i$. □

The following is our main theorem. It is equivalent to: If $\mathbf{m}^3 = 0$ and $\mathbf{m}^2 = \mathbf{m}\cdot \operatorname{ann} x$ for all $x \in \mathbf{m}\backslash\mathbf{m}^2$, then $R \to R/p$ is a Golod map (cf. [**10**]) for all $p \in \mathbf{m}^2$. We have not worked with Golod maps since a direct proof is no more difficult.

Theorem 3.6. *Suppose that* $\mathbf{m}^3 = 0$. *If* $\mathbf{m}\cdot \operatorname{ann} x = \mathbf{m}^2$ *for all* $x \in \mathbf{m}\backslash\mathbf{m}^2$*, then* R *is a Fröberg ring.*

Proof. Let I be an ideal not contained in $\mathbf{m}^2$. We will inductively construct a minimal resolution of R/I, $(F(I), d(I))$, such that

$$\mathbf{m}^2 F(I)_q = \mathbf{m}\cdot \ker d(I)_q, \tag{3.7}$$

and

$$\dim(\ker d(I)_q) = \beta_{q+1}(I). \tag{3.8}$$

Condition (3.7) applied to $I = \mathbf{m}$ shows that R is Fröberg by (2.5). Let $p = \dim(I)$.

We begin with the case $q = 1$. $F(I)_1 = R^p$ and $d(I)_1(r_1,\dots,r_p) = \sum r_i x_i$, where $x_1,\dots,x_p$ is a minimal generating set for I. We will use induction on p. If $p = 1$, then $\ker d(I)_1 = \operatorname{ann} x_1$ and (3.7) holds by assumption. And $\beta_2(I) = gp - \dim(\mathbf{m}x_1) = \dim(\operatorname{ann} x_1)$ by (3.1) applied to multiplication by x_1, $\mu: R \to R$.

Now suppose that $p > 1$. Write $J = (x_1,\dots,x_{p-1})$ and $K = (x_p)$. Then $F(I)_1 = F(J)_1 \times F(K)_1$ and $d(I)_1(r,s) = d(J)_1(r) + d(K)_1(s)$. We construct (as in Section 2) a minimal generating set for $\ker d(I)_1$. Let S_1 be a minimal generating set for $\ker d(J)_1$ and S_2 the same for $\ker d(K)_1$. Set $B_1 = \{(z,0) \mid z \in S_1\}$ and $B_2 = \{(0,z) \mid z \in S_2\}$. Let $w_1,\dots,w_t$ be a minimal generating set for $\mathbf{m}J \cap \mathbf{m}K$. Choose $z_1,\dots,z_t \in \mathbf{m}\backslash\mathbf{m}^2$ such that $z_i x_p = w_i$. Set $L = Rz_1 + \cdots + Rz_t$.

We note that $\dim L = t = \dim(\mathbf{m}J \cap \mathbf{m}K)$, namely, that $\{z_1, \ldots, z_t\}$ is a minimal generating set for L. For, suppose that $z_1 = \sum_{i \geq 2} r_i z_i$. Then multiplying by x_p gives $w_1 = \sum_{i \geq 2} r_i w_i$, which contradicts the minimality of the w,s. For each z_i choose $\pi(z_i) \in \mathbf{m}F(J)_1$ such that $d(J)_1(\pi(z_i)) = -w_i = -z_i x_p$. Finally, set $B_3 = \{(\pi(z_i), z_i) \mid 1 \leq i \leq t\}$.

Now $B_1 \cup B_2 \cup B_3$ generates $\ker d(I)_1$. Namely, if $d(I)_1(r,s) = 0$ for $r \in F(J)_1$ and $s \in F(K)_1$, then set $\alpha = -d(J)_1(r) = d(K)_1(s)$. Now $\alpha \in \mathbf{m}J \cap \mathbf{m}K$ so we may write $\alpha = \sum r_i w_i$ for some $r_i \in R$. Then $r - \sum r_i \pi(z_i) \in \ker d(J)_1$ and $s - \sum r_i z_i \in \ker d(K)_1$. Thus $(r,s) - \sum r_i(\pi(z_i), z_i)$ is in the span of $B_1 \cup B_2$ and (r,s) is in the span of $B_1 \cup B_2 \cup B_3$. Further, $|B_1 \cup B_2 \cup B_3| = \dim(\ker d(J)_1) + \dim(\ker d(K)_1) + \dim L = \beta_2(I)$ by (3.3). Hence, by (3.2), $B_1 \cup B_2 \cup B_3$ is a minimal generating set for $\ker d(I)_1$ and $\mathbf{m} \cdot \ker d(I)_1 = \mathbf{m}^2 F(I)_1$. This shows both (3.7) and (3.8) for $q = 1$.

Now suppose that $q > 1$. We again verify (3.7) and (3.8) by induction on p. If $p = 1$, then we are done by induction on q and (3.2). So assume that $p > 1$. Write J, K and L as before.

Claim. *For* $1 \leq s \leq q$,

$$F(I)_s = F(J)_s \times F(K)_s \times \prod_{i=3}^{s+1} F(L^{i-3})_{s-i+2}$$

$$d(I)_s = \begin{bmatrix} d(J)_s & 0 & * & \cdots & & * \\ & d(K)_s & 0 & * & & \cdot \\ & & d(L^0)_{s-1} & 0 & & \cdot \\ & & & \ddots & * & \cdot \\ & 0 & \ddots & & 0 & * \\ & & & & d(L^{s-3})_2 & d(L^{s-2})_1 \end{bmatrix}$$

where each $*$ *maps to* $\mathbf{m}$ *times the appropriate value space.* (*The meaning of the claim is*: *these* $F(I)_s, d(I)_s$ *form the first* q *terms of a minimal resolution of* R/I).

When $s = 1$, the claim is true as $F(I)_1 = F(J)_1 \times F(K)_1$ and $d(I)_1 = d(J)_1 \times d(K)_1$. We will also verify the claim for $s = 2$ to help the reader understand the general case. We use the notation set in the proof of the $q = 1$, $p > 1$ case. We have $\ker d(I)_1 = \ker d(J)_1 \times \ker d(K)_1 + \sum R(\pi(z_i), z_i)$. Hence we may take $F(I)_2 = F(J)_2 \times F(K)_2 \times F(L)_1$ and

$$d(I)_2 = \begin{pmatrix} d(J)_2 & 0 & * \\ 0 & d(K)_2 & d(L)_1 \end{pmatrix}$$

where $*$ maps a basis element e_i of $F(L)_1 = R^t$ to $\pi(z_i)$ and, as usual, $d(L)_1$ maps e_i to z_i. Then $\operatorname{im} d(I)_2 = \ker d(I)_1 \subset \mathbf{m}F(I)_1$.

To prove the claim in general, we fix s, $1 \leq s < q$, where the claim holds and prove the claim for $s+1$. The heart of the proof is constructing a minimal generating set for $\ker d(I)_s$.

Let e_j, $1 \leq j \leq s+1$, map $F(J)_s$, $F(K)_s$, $F(L^{j-3})_{s-j+2}$, respectively, into $F(I)_s$. Let S_j, $1 \leq j \leq s+1$, be a minimal generating set for $\ker d(J)_s$, $\ker d(K)_s$, $\ker d(L^{j-3})_{s-j+2}$, respectively.

For $i = 1, 2$, set $B_i = \{e_i(z) \mid z \in S_i\}$. Note that, by induction on p, (3.8) gives $|B_1| = \beta_{s+1}(J)$ and $|B_2| = \beta_{s+1}(K)$. Next suppose that $3 \leq i \leq s+1$ and $z \in S_i$. Then

$$\begin{aligned} d(I)_s(e_i(z)) &= (*, *, \dots, *, 0, d(L^{i-3})_{s-i-2}(z), 0, 0, \dots, 0) \\ &= (*, *, \dots, *, 0, 0, \dots, 0), \end{aligned}$$

with the second zero in the ith coordinate and a $*$ in the jth coordinate representing an element of $\mathbf{m}^2$ times the jth summand of $F(I)_{s-1}$. Now, by (3.7) and induction on p for $d(J)_s$, $d(K)_s$ and induction on s for $d(L^{j-3})_{s-j+2}$, $d(J)_s$, $d(K)_s$ and each $d(L^{j-3})_{s-j+2}$ map onto $\mathbf{m}^2$ times the jth summand of $F(I)_{s-1}$. In particular, $d(I)_s(e_i(z))$ is in the image of the submatrix of $d(I)_s$ consisting of the first $i-2$ rows and columns. Hence, there is a vector $v(z)$ of the form

$$(*, *, \dots, *, 0, z, 0, \dots, 0)$$

in $\ker d(I)_s$, where z is in the ith coordinate. Set $B_i = \{v(z) \mid z \in S_i\}$. Note that by induction $|B_i| = \dim(\ker d(L^{i-3})_{s-i+2}) = \beta_{s-i+3}(L^{i-3})$.

We construct a final set B_{s+2}. As in (3.5), choose $W \subset \mathbf{m}F(L^{s-2})_1$ such that $d(L^{s-2})_1(W)$ is a minimal generating set for the image of $\mathbf{m}F(L^{s-2})_1$. For $w \in W$,

$$d(I)_s(e_{s+1}(w)) = (*, \dots, *, d(L^{s-2})_1(w))$$

which is in $\mathbf{m}^2F(I)_{s-1}$. Again, we may find a vector $v(w) = (*, \dots, *, w)$ that is in $\ker d(I)_s$. Set $B_{s+2} = \{v(w) \mid w \in W\}$. Note that $|B_{s+2}| = |W| = \dim(d(L^{s-2})_1(\mathbf{m}F(L^{s-2})_1))$.

The set $B = B_1 \cup \cdots \cup B_{s+2}$ generates $\ker d(I)_s$ as can be easily (if not quickly) checked. Further,

$$\begin{aligned}|B| &= \sum_{i=1}^{s} |B_i| + \dim(\ker d(L^{s-2})_1) \\ &\quad + \dim(d(L^{s-2})_1(\mathbf{m}F(L_1^{s-2}))) \\ &= \sum_{i=1}^{s} |B_i| + g\dim F(L^{s-2})_1,\end{aligned}$$

by (3.1). Hence,

$$\begin{aligned}|B| &= b_{s+1}(J) + \beta_{s+1}(K) + \sum_{i=3}^{s} \beta_{s-i+3}(L^{i-3}) \\ &\quad + \beta_2(L^{s-2}) + \dim(\mathbf{m}L^{s-2}) \\ &= \beta_{s+1}(J) + \beta_{s+1}(K) + \sum_{i=3}^{s+2} \beta_{s-i+3}(L^{i-3}) \\ &= \beta_{s+1}(I),\end{aligned}$$

by (3.4). Hence, B is a minimal generating set for $\ker d(I)_q$.

We now complete the proof of the claim. By induction on p, $d(J)_{s+1}$ maps $F(J)_{s+1}$ onto the span of B_1 and $d(K)_{s+1}$ maps $F(K)_{s+1}$ onto the span of B_2 (by induction on p). For $3 \le i \le s+1$, $F(L^{i-3})_{s-i+3}$ is mapped onto the span of S_i. Hence, $d(L^{i-3})_{s-i+3}$ combined with maps on $F(J)_{s+1}$, $f(K)_{s+1}$ and $F(L^{j-3})_{s-j+3}$, $3 \le j \le i-2$, will map onto the span of B_i. Lastly, W generates L^{s-1} by (3.5) so that $F(L^{s-1})_1$ maps onto the span of W. Thus, we may take $F(I)_{s+1}$ and $d(I)_{s+1}$ as in the claim.

We may now quickly prove (3.7) and (3.8) for $q > 1$ and $p > 1$. The preceding argument applied to the case $s = q$ gives a minimal generating set B for $\ker d(I)_q$ such that $|B| = \beta_{q+1}(I)$. This gives (3.8) and then (3.3) implies (3.7). $\square$

Corollary 3.9. *Let R be an abstract Witt ring with* $\mathbf{m}$ *the fundamental ideal. If* $\mathbf{m}^3 = 0$*, then* R *is a Fröberg ring.*

Proof. By (3.6) we need to check that

$$\mathbf{m}^2 \cap \operatorname{ann}\langle 1, -a\rangle = \mathbf{m} \cdot \operatorname{ann}\langle 1, -a\rangle$$

for any $a \in G(R)$. This is proven for Witt rings of fields in [**5**, 2.14]. The proof uses only $AP(2)$ and $\operatorname{ann}\langle 1, -a\rangle$ being a 1-Pfister ideal. These both hold for abstract Witt rings, the first by definition (see [**13**, p. 63]) and the second by [**13**, 4.23]. □

We remark that it is not known if every Witt ring with $\mathbf{m}^3 = 0$ is of elementary type. Thus, (3.9) cannot be deduced from (1.10).

4. Partial converses to the main theorem. The converse to (3.6) does not hold. Consider $R = (\mathbf{Z}/2\mathbf{Z})[X_1, X_2, X_3]/(X_1^3, X_1X_2, X_1X_3, X_1^2 - X_2^2, X_1^2 - X_3^2)$. Let $\mathbf{m} = (X_1, X_2, X_3)$ and write $x_i = X_i + \mathbf{m}$. Then $\mathbf{m}^3 = 0$ and R is a Fröberg ring [**3**]. Yet $\operatorname{ann} x_2 = Rx_1 + \mathbf{m}^2$ so that $\mathbf{m} \cdot \operatorname{ann} x_2 = Rx_1^2 \neq \mathbf{m}^2$ (as $x_2x_3 \notin Rx_1^2$).

However, we do have:

Proposition 4.1. *Suppose that* $\mathbf{m}^3 = 0$ *and* $\mu(\mathbf{m}^2) = 1$*. The following are equivalent.*

(1) *R is a Fröberg ring.*

(2) $\mathbf{m} \cdot \operatorname{ann} x = \mathbf{m}^2$ *for all* $x \in \mathbf{m}\backslash\mathbf{m}^2$.

(3) $R \approx R_1 \sqcap R_2$ *where* $(R_1, \mathbf{n}_1, k)$ *is Gorenstein of socle of degree* 2 *and* $\mu(\mathbf{n}_1) > 1$ *and* $(R_2, \mathbf{n}_2, k)$ *satisfies* $\mathbf{n}_2^2 = 0$.

Proof. (2) → (1) is (3.6) while (3) → (1) is the combination of (1.1), (1.2) and (1.7). We first check (1) → (2). Suppose that $\mathbf{m} \cdot \operatorname{ann} x \neq \mathbf{m}^2$ for some $x \in \mathbf{m}\backslash\mathbf{m}^2$. Then $\mathbf{m} \cdot \operatorname{ann} x = 0$. Note that $\mathbf{m}x = \mathbf{m}^2$ (else $\mathbf{m}x = 0$ and $\operatorname{ann} x = \mathbf{m}$) and $\mu(\mathbf{m}) > 1$ by (1.3). Set $e = \mu(\mathbf{m})$. Then $\dim(\operatorname{ann} x/\mathbf{m}^2) = e - 1$ and we may choose $y_1, \dots, y_{e-1} \in \operatorname{ann} x$ such that modulo $\mathbf{m}^2$. These form a basis for $\operatorname{ann} x/\mathbf{m}^2$. Now $\mathbf{m}y_i = 0$, $1 \le i \le e-1$ and $x \notin \operatorname{ann} x$, as $\mathbf{m}x \neq 0$ and $\mathbf{m} \cdot \operatorname{ann} x = 0$. Hence,

$y_1, \dots, y_{e-1}, x$ is a minimal generating set for $\mathbf{m}$. $\overline{R} = R/(y_1, \dots, y_{e-1})$ is a Fröberg ring by [**9**, 3.4.4]. But $\mu(\bar{\mathbf{m}}) = 1$ and $\bar{x}^2 \neq 0$, contradicting (1.3). Hence, (2) holds.

To prove (1) $\to$ (3), we note that for any local ring R with $\mathbf{m}^3 = 0$ and $\mu(\mathbf{m}^2) = 1$, we have $R \approx R_1 \sqcap R_2$ with R_1 Gorenstein and $\mathbf{n}_2^2 = 0$. Namely, let $\mathbf{m}_0 = \operatorname{ann} \mathbf{m}$. If $\mathbf{m}_0 = \mathbf{m}^2$, then R is Gorenstein. Otherwise, choose $y_1, \dots, y_t \in \mathbf{m}_0 \backslash \mathbf{m}^2$ such that the images form a basis for $\mathbf{m}_0/\mathbf{m}^2$. Extend by $x_1, \dots, x_{e-t}$ to a minimal generating set for $\mathbf{m}$. Set $\mathbf{m}_1 = (y_1, \dots, y_t)$ and $\mathbf{m}_2 = (x_1, \dots, x_{e-t})$. Then $\mathbf{m} = \mathbf{m}_1 \oplus \mathbf{m}_2$ and $R \approx R/\mathbf{m}_1 \sqcap R/\mathbf{m}_2$ with $R_1 = R/\mathbf{m}_1$ Gorenstein and $R_2 = R/\mathbf{m}_2$ satisfying $(\mathbf{m}/\mathbf{m}_2)^2 = 0$. Finally, (1) implies $R/\mathbf{m}_1$ is Fröberg and hence $\mu(\mathbf{m}/\mathbf{m}_1) > 1$ by (1.3). $\square$

When $\mathbf{m}^3 = 0$ and $\mu(\mathbf{m}^2) = 2$ we get only a partial converse to (3.6).

Proposition 4.2. *Suppose that $\mathbf{m}^3 = 0$ and $\mu(\mathbf{m}^2) = 2$. If R is a Fröberg ring, then*

(1) *For all $x \in \mathbf{m} \backslash \mathbf{m}^2$ either $\mathbf{m} \cdot \operatorname{ann} x = \mathbf{m}^2$ or $\mathbf{m}x = \mathbf{m}^2$.*

(2) *There exists an ideal $I \not\subset \mathbf{m}^2$ such that $\mathbf{m} \cdot \operatorname{ann} x = \mathbf{m}^2$ for all $x \in I \backslash \mathbf{m}^2$ and $\mu(I) = \mu(\mathbf{m}) - 2$.*

Proof. (1) Suppose that $\mathbf{m}x \neq \mathbf{m}^2$ and $\mathbf{m} \cdot \operatorname{ann} x \neq \mathbf{m}^2$ for some $\mathbf{m} \backslash \mathbf{m}^2$. If $\mathbf{m}x = 0$, then $\operatorname{ann} x = \mathbf{m}$ and $\mathbf{m} \cdot \operatorname{ann} x = \mathbf{m}^2$. We may thus assume that $\dim(\mathbf{m}x) = 1$ and so $\dim(\operatorname{ann} x/\mathbf{m}^2) = g - 1$ where $g = \dim(\mathbf{m}/\mathbf{m}^2)$. First consider the case where $\mathbf{m} \cdot \operatorname{ann} x = 0$. We may then choose a basis of $\mathbf{m}/\mathbf{m}^2$: $y_1 + \mathbf{m}^2, \dots, y_{g-1} + \mathbf{m}^2, z + \mathbf{m}^2$ with each $y_i \in \operatorname{ann} x$ and $z \in \mathbf{m}$. Then $\mathbf{m}$ is generated by $y_1, \dots, y_{g-1}, z$ and as $\mathbf{m}y_i \subset \mathbf{m} \cdot \operatorname{ann} x = 0$, we get $\mathbf{m}^2 = Rz^2$. This contradicts the assumption that $\mu(\mathbf{m}^2) = 2$.

We may thus assume that $\mathbf{m} \cdot \operatorname{ann} x = Rs$ for some $s \in \mathbf{m}^2$. Let $H = ((s) : \mathbf{m})$. Then $\mathbf{m}^2 \subset \operatorname{ann} x \subset H$ and $H \neq \mathbf{m}$ (else $\mathbf{m}^2 = Rs$). Thus $g - 1 = \dim(\operatorname{ann} x/\mathbf{m}^2) \leq \dim(H/\mathbf{m}^2) \leq g - 1$. Hence, $H = \operatorname{ann} x$. As before $\mathbf{m}$ may be generated by $y_1, \dots, y_{g-1}, z$ with each $y_i \in \operatorname{ann} x = H$ and $z \notin H$.

Now $d_1 : R^g \to R$ is $d_1(r_0, r_1, \dots, r_{g-1}) = r_0 z + \sum r_i y_i$. If $(r_0, r_1, \dots, r_{g-1}) \in Z_1$, then $r_0 z = -\sum r_i y_i \in Rs$ and so $r_0 \in$

$((s):(z))$. Clearly, $\mathbf{m}^2 \subset H \subset ((s):(z))$ and $((s):(z)) \neq \mathbf{m}$ (else $\mathbf{m}z \subset Rs$ and $z \in H$). Hence, $H = ((s):(z))$. Thus, if $(r_0, r_1, \dots, r_{g-1}) \in Z_1$, then $r_0 \in H$. Hence, if $t \in \mathbf{m}^2 \backslash Rs$, then $(t, 0, \dots, 0) \in \mathbf{m}^2 F_1 \backslash \mathbf{m} Z_1$. But R Fröberg implies that $\mathbf{m}Z_1 = \mathbf{m}^2 F_1$ (2.5), yielding a contradiction.

(2) Suppose that $\mathbf{m} \cdot \operatorname{ann} z \neq \mathbf{m}^2$ for some z. Then for all $x \in \operatorname{ann} z \backslash \mathbf{m}^2$ we have $\mathbf{m} \cdot \operatorname{ann} x = \mathbf{m}^2$ by (1). Also, by (1), $\mathbf{m}z = \mathbf{m}^2$ so that $\dim(\operatorname{ann} \mathbf{m}^2) = g - 2$. Take $I = \operatorname{ann} z$. □

In the example at the start of this section, $\mathbf{m}^3 = 0$, $\mathbf{m}^2 = (x_1^2, x_2 x_3)$ and $\mathbf{m} \cdot \operatorname{ann} x_i = \mathbf{m}^2$ only for $i = 1$. Thus the ideal of (4.2) is Rx_1. This shows that we cannot find an ideal I with $\mu(I) > \mu(\mathbf{m}) - 2$ and $\mathbf{m} \cdot \operatorname{ann} x = \mathbf{m}^2$ for all $x \in I \backslash \mathbf{m}^2$.

We know of no case where $\mathbf{m}^3 = 0$, $\mu(\mathbf{m}^2) = 2$ and (4.2) (1) holds, and yet R is not a Fröberg ring.

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