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ON THE NEAR-RING COUNTERPART OF THE MATRIX RING ISOMORPHISM $\mathbf{M}_{mn}(R) \cong \mathbf{M}_n(\mathbf{M}_m(R))$

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1. Introduction. When R is a ring with identity, it is well known (and easy to show) that the $n\times n$ matrix ring over the $m\times m$ matrix ring over R is isomorphic to the $mn \times mn$ matrix ring over R. The near-ring situation is somewhat different to handle because matrices over near-rings are defined in a functional way (Meldrum and van der Walt [**7**]) which has very little, if any, resemblance to the traditional way of portraying matrices. However, when the near-ring happens to be a ring, the matrix near-ring is isomorphic to the familiar matrix ring.

The introduction of the concept of a matrix near-ring (in 1984) was soon followed by a series of papers covering a variety of basic results on this concept (cf., e.g., [**1, 2, 3, 4, 5, 6**, **8, 9, 10, 11, 12**, **13, 14, 15, 16, 17, 18**]). Notably absent among these results is an answer to the natural question on the existence or non-existence of the matrix near-ring isomorphism under consideration. This problem has to date withstood a considerable amount of effort–at one stage it was even strongly believed that the two matrix near-rings are isomorphic if and only if R is a ring. The question is finally settled in this paper.

2. Preliminaries. Let R be a right near-ring (not necessarily zero-symmetric) with identity 1. For any natural number n, R^n denotes the direct sum of n copies of the group $(R,+)$. The elements of R^n will be written as $\langle r_1, r_2, \dots , r_n\rangle$, where $r_1, r_2, \dots , r_n \in R$. The ith coordinate projection and injection functions are denoted by

$$\pi_i^{(n)} : R^n \longrightarrow R \quad \text{and} \quad \iota_i^{(n)} : R \longrightarrow R^n,$$

respectively.

Furthermore, for any (additively written) group G, $M(G)$ will denote the near-ring of all mappings of G into itself. The $n\times n$ matrix near-ring

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over R, denoted $\mathbf{M}_n(R)$, is defined to be the subnear-ring of $M(R^n)$ generated (as a near-ring) by the set

$$\{f_{ij}^r : R^n \longrightarrow R^n \mid r \in R \text{ and } 1 \leq i, j \leq n\}$$

of elementary $n \times n$ matrices. For $r \in R$ and $1 \leq i$, $j \leq n$, an elementary $n \times n$ matrix is defined as the function $f_{ij}^r : R^n \to R^n$ which maps each $\langle r_1, r_2, \dots, r_n\rangle \in R^n$ onto $\iota_i^{(n)}(rr_j)$. It follows that $\mathbf{M}_n(R)$ is a right near-ring with identity $I = f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1$.

In order to handle these matrices, we need a way of expressing them in terms of the elementary matrices. We therefore introduce the set of $n \times n$ matrix expressions $\mathbf{E}_n(R)$ over R, namely, the subset of the free semigroup over the alphabet

$$\{f_{ij}^r \mid r \in R \text{ and } 1 \leq i, j \leq n\} \cup \{(,), +\},$$

recursively defined as follows:

(a) $f_{ij}^r \in \mathbf{E}_n(R)$ for all $r \in R$ and $1 \leq i$, $j \leq n$;

(b) If E_1 and E_2 are elements of $\mathbf{E}_n(R)$, then $E_1 + E_2 \in \mathbf{E}_n(R)$;

(c) If $E \in \mathbf{E}_n(R)$, then $f_{ij}^r(E) \in \mathbf{E}_n(R)$ for all $r \in R$ and $1 \leq i$, $j \leq n$.

It follows immediately that every element of $\mathbf{E}_n(R)$ represents a matrix in $\mathbf{M}_n(R)$ and, conversely, every matrix in $\mathbf{M}_n(R)$ can be represented by (infinitely many) elements of $\mathbf{E}_n(R)$. The *length* $\ell(E)$ of an expression $E \in \mathbf{E}_n(R)$ is defined to be the number of f_{ij}^r's in it. The *weight* $w(U)$ of a matrix $U \in \mathbf{M}_n(R)$ is the length of an expression $E \in \mathbf{E}_n(R)$ of minimal length representing the matrix U.

We will use the notation $\operatorname{mat}(E)$ to denote the matrix represented by the expression E. Note that mat sometimes refers to the function $\operatorname{mat} : \mathbf{E}_n(R) \to \mathbf{M}_n(R)$ and sometimes to the function $\operatorname{mat} : \mathbf{E}_n(\mathbf{M}_m(R)) \to \mathbf{M}_n(\mathbf{M}_m(R))$ for some natural number m. This will never cause any ambiguity, since it will always be clear from the context which function is actually referred to. The following lemma is immediate and is stated for future reference:

Lemma 1. *For any two expressions E_1 and E_2 in $\mathbf{E}_n(R)$, we have that*

$$\operatorname{mat}(E_1 + E_2) = \operatorname{mat}(E_1) + \operatorname{mat}(E_2)$$

and

$$\mathrm{mat}\,(f_{ij}^r(E_1)) = \mathrm{mat}\,(f_{ij}^r)\mathrm{mat}\,(E_1)$$

for all $r \in R$ *and* $1 \leq i,\, j \leq n$.

3. Matrices over $\mathbf{M}_m(R)$. Henceforth m and n will denote arbitrarily chosen, but fixed natural numbers. Our aim is to define a near-ring isomorphism

$$\phi : \mathbf{M}_{mn}(R) \longrightarrow \mathbf{M}_n(\mathbf{M}_m(R)).$$

This will be done *via* a map

$$\theta : \mathbf{E}_{mn}(R) \longrightarrow \mathbf{E}_n(\mathbf{M}_m(R))$$

which is constructed as follows: Let $f_{ij}^r \in \mathbf{E}_{mn}(R)$. Then there are uniquely determined natural numbers k_1, k_2, l_1, l_2, where $1 \leq k_1,\, k_2 \leq n$ and $1 \leq l_1,\, l_2 \leq m$ such that $i = m(k_1 - 1) + l_1$ and $j = m(k_2 - 1) + l_2$. Let

$$\theta(f_{ij}^r) := f_{k_1 k_2}^{f_{l_1 l_2}^r}.$$

For an arbitrary expression $E \in \mathbf{E}_{mn}(R)$, just replace each occurrence of f_{ij}^r in it by $\theta(f_{ij}^r)$ to obtain $\theta(E)$. Then θ is well-defined because each $f_{l_1 l_2}^r \in \mathbf{M}_m(R)$, and we also have the following immediate result which will be referred to later on.

Lemma 2. *For any two expressions E_1 and E_2 in $\mathbf{E}_{mn}(R)$ we have that*

$$\theta(E_1 + E_2) = \theta(E_1) + \theta(E_2)$$

and

$$\theta(f_{ij}^r(E_1)) = \theta(f_{ij}^r)\theta(E_1)$$

for all $r \in R$ *and* $1 \leq i,\, j \leq mn$.

Before defining the map ϕ, we need the following lemma which plays a crucial role throughout the remainder of this article.

Lemma 3. *Let* $E \in \mathbf{E}_{mn}(R)$. *Take any* $\langle V_1, V_2, \dots, V_n\rangle \in (\mathbf{M}_m(R))^n$ *and any* $\langle r_1, r_2, \dots, r_m\rangle \in R^m$. *Suppose that* $V_l\langle r_1, r_2, \dots, r_m\rangle = \langle s_{l1}, s_{l2}, \dots, s_{lm}\rangle$ *for each* $l = 1, 2, \dots, n$. *Then*

$$\mathrm{mat}\,(\theta(E))\langle V_1, V_2, \dots, V_n\rangle = \langle W_1, W_2, \dots, W_n\rangle \in (\mathbf{M}_m(R))^n$$

where

$$W_k\langle r_1, r_2, \dots, r_m\rangle = \left(\sum_{t=1}^{m} \iota_t^{(m)} \pi_{m(k-1)+t}^{(mn)}\right) \cdot \mathrm{mat}\,(E)\langle s_{11}, s_{12}, \dots, s_{1m}, s_{21}, \dots, s_{nm}\rangle$$

for all $k = 1, 2, \dots, n$.

Proof. We use induction on $\ell(E)$, the length of E. Let $\ell(E) = 1$, i.e., $E = f_{ij}^r$ for some $r \in R$ and $1 \le i,\ j \le mn$. Then there are unique numbers k_1, k_2, l_1, l_2 where $1 \le k_1,\ k_2 \le n$ and $1 \le l_1,\ l_2 \le m$ such that $i = m(k_1 - 1) + l_1$ and $j = m(k_2 - 1) + l_2$. Hence $\theta(E) = f_{k_1k_2}^{f_{l_1l_2}^r}$. It follows that

$$\begin{aligned}\mathrm{mat}\,(\theta(E))\langle V_1, V_2, \dots, V_n\rangle &= f_{k_1k_2}^{f_{l_1l_2}^r}\langle V_1, V_2, \dots, V_n\rangle \\ &= \langle W_1, W_2, \dots, W_n\rangle\end{aligned}$$

where

$$W_k = \begin{cases} f_{l_1l_2}^r V_{k_2} & \text{if } k = k_1 \\ 0 & \text{if } k \ne k_1. \end{cases}$$

Now

$$\begin{aligned}W_{k_1}\langle r_1, r_2, \dots, r_m\rangle &= f_{l_1l_2}^r V_{k_2}\langle r_1, r_2, \dots, r_m\rangle \\ &= f_{l_1l_2}^r \langle s_{k_21}, s_{k_22}, \dots, s_{k_2m}\rangle \\ &= \iota_{l_1}^{(m)}(r s_{k_2l_2}).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\mathrm{mat}\,(E)\langle s_{11}, s_{12}, \dots, s_{1m}, s_{21}, \dots, s_{nm}\rangle \\ &\qquad = f_{ij}^r\langle s_{11}, s_{12}, \dots, s_{1m}, s_{21}, \dots, s_{nm}\rangle \\ &\qquad = \iota_i^{(mn)}(r s_{k_2l_2}),\end{aligned}$$

which implies that

$$\begin{aligned}\left(\sum_{t=1}^{m}\iota_t^{(m)}\pi_{m(k_1-1)+t}^{(mn)}\right)&\mathrm{mat}\,(E)\langle s_{11}, s_{12}, \dots, s_{1m}, s_{21}, \dots, s_{nm}\rangle\\ &= \left(\sum_{t=1}^{m}\iota_t^{(m)}\pi_{m(k_1-1)+t}^{(mn)}\right)\iota_i^{(mn)}(rs_{k_2l_2})\\ &= \iota_{l_1}^{(m)}\pi_i^{(mn)}\iota_i^{(mn)}(rs_{k_2l_2})\\ &= \iota_{l_1}^{(m)}(rs_{k_2l_2})\\ &= W_{k_1}\langle r_1, r_2, \dots, r_m\rangle.\end{aligned}$$

If $k \neq k_1$, then

$$\begin{aligned}\left(\sum_{t=1}^{m}\iota_t^{(m)}\pi_{m(k-1)+t}^{(mn)}\right)&\mathrm{mat}\,(E)\langle s_{11}, s_{12}, \dots, s_{1m}, s_{21}, \dots, s_{nm}\rangle\\ &= \left(\sum_{t=1}^{m}\iota_t^{(m)}\pi_{m(k-1)+t}^{(mn)}\right)\iota_i^{(mn)}(rs_{k_2l_2})\\ &= \langle 0, 0, \dots, 0\rangle\\ &= W_k\langle r_1, r_2, \dots, r_m\rangle\end{aligned}$$

because $i = m(k_1-1)+l_1 \neq m(k-1)+t$ for any t, $1 \leq t \leq m$, if $k \neq k_1$.

Now suppose $\ell(E) = p > 1$ and that the result is true for all expressions in $\mathbf{E}_{mn}(R)$ of length less than p. There are two possibilities:

(a) $E = E_1 + E_2$, where $E_1, E_2 \in \mathbf{E}_{mn}(R)$ and $\ell(E_1), \ell(E_2) < p$. Because $\mathrm{mat}\,(\theta(E)) = \mathrm{mat}\,(\theta(E_1)) + \mathrm{mat}\,(\theta(E_2))$ by Lemmas 1 and 2, the result follows immediately in this case.

(b) $E = f_{ij}^{r}E_1$ for some $r \in R$, $1 \leq i$, $j \leq mn$, and where $E_1 \in \mathbf{E}_{mn}(R)$ with $\ell(E_1) < p$. Here we have $\theta(E) = f_{k_1k_2}^{f_{l_1l_2}^{r}}\theta(E_1)$ where $i = m(k_1-1)+l_1$ and $j = m(k_2-1)+l_2$ for unique numbers $1 \leq k_1$, $k_2 \leq n$ and $1 \leq l_1$, $l_2 \leq m$, as before. By the induction hypothesis, it follows that

$$\begin{aligned}\mathrm{mat}\,(\theta(E))\langle V_1, V_2, \dots, V_n\rangle &= f_{k_1k_2}^{f_{l_1l_2}^{r}}\mathrm{mat}\,(\theta(E_1))\langle V_1, V_2, \dots, V_n\rangle\\ &= f_{k_1k_2}^{f_{l_1l_2}^{r}}\langle W_1, W_2, \dots, W_n\rangle\end{aligned}$$

where

$$W_k\langle r_1, r_2, \dots, r_m\rangle = \left(\sum_{t=1}^{m} \iota_t^{(m)} \pi_{m(k-1)+t}^{(mn)}\right) \operatorname{mat}(E_1)\langle s_{11}, s_{12}, \dots, s_{1m}, s_{21}, \dots, s_{nm}\rangle$$

for all $k = 1, 2, \dots, n$. Hence

$$\operatorname{mat}(\theta(E))\langle V_1, V_2, \dots, V_n\rangle = f_{k_1 k_2}^{f_{l_1 l_2}^{r}}\langle W_1, W_2, \dots, W_n\rangle = \langle W_1', W_2', \dots, W_n'\rangle$$

where

$$W_k' = \begin{cases} f_{l_1 l_2}^{r} W_{k_2} & \text{if } k = k_1 \\ 0 & \text{if } k \neq k_1. \end{cases}$$

Now, if we suppose that

$$\operatorname{mat}(E_1)\langle s_{11}, s_{12}, \dots, s_{1m}, s_{21}, \dots, s_{nm}\rangle = \langle s_{11}', s_{12}', \dots, s_{1m}', s_{21}', \dots, s_{nm}'\rangle,$$

it follows that

$$\begin{aligned} W_{k_1}'\langle r_1, r_2, \dots, r_m\rangle &= f_{l_1 l_2}^{r}\left(\sum_{t=1}^{m} \iota_t^{(m)} \pi_{m(k_2-1)+t}^{(mn)}\right) \\ &\quad \cdot \operatorname{mat}(E_1)\langle s_{11}, s_{12}, \dots, s_{1m}, s_{21}, \dots, s_{nm}\rangle \\ &= f_{l_1 l_2}^{r}\left(\sum_{t=1}^{m} \iota_t^{(m)} \pi_{m(k_2-1)+t}^{(mn)}\right)\langle s_{11}', s_{12}', \dots, s_{1m}', s_{21}', \dots, s_{nm}'\rangle \\ &= f^{r} l_1 l_2 \langle s_{k_2 1}', s_{k_2 2}', \dots, s_{k_2 m}'\rangle \\ &= \iota_{l_1}^{(m)}(r s_{k_2 l_2}') \\ &= \left(\sum_{t=1}^{m} \iota_t^{(m)} \pi_{m(k_1-1)+t}^{(mn)}\right) \\ &\quad \cdot \operatorname{mat}(f_{ij}^{r} E_1)\langle s_{11}, s_{12}, \dots, s_{1m}, s_{21}, \dots, s_{nm}\rangle \\ &= \left(\sum_{t=1}^{m} \iota_t^{(m)} \pi_{m(k_1-1)+t}^{(mn)}\right) \\ &\quad \cdot \operatorname{mat}(E)\langle s_{11}, s_{12}, \dots, s_{1m}, s_{21}, \dots, s_{nm}\rangle. \end{aligned}$$

Also, if $k \neq k_1$, then

$$\left(\sum_{t=1}^{m} \iota_t^{(m)} \pi_{m(k-1)+t}^{(mn)}\right) \mathrm{mat}\,(f_{ij}^{r} E_1)\langle s_{11}, s_{12}, \dots, s_{1m}, s_{21}, \dots, s_{nm}\rangle$$
$$= \langle 0, 0, \dots, 0\rangle$$
$$= W_k'\langle r_1, r_2, \dots, r_m\rangle$$

because $i = m(k_1 - 1) + l_1 \neq m(k-1) + t$ for any $t, 1 \leq t \leq m$, if $k \neq k_1$.

This completes the proof. □

We can now proceed to define the map ϕ. Let $U \in \mathbf{M}_{mn}(R)$. Take any $E \in \mathbf{E}_{mn}(R)$ such that $\mathrm{mat}\,(E) = U$. Define $\phi(U)$ to be the matrix $\mathrm{mat}\,(\theta(E))$ in $\mathbf{M}_n(\mathbf{M}_m(R))$.

Before we can say anything further, we need to show that ϕ is a well-defined function. So suppose $E_1, E_2 \in \mathbf{M}_{mn}(R)$ such that $\mathrm{mat}\,(E_1) = \mathrm{mat}\,(E_2) = U$. Take any $\langle V_1, V_2, \dots, V_n\rangle \in (\mathbf{M}_m(R))^n$ and $\langle r_1, r_2, \dots, r_m\rangle \in R^m$ and suppose that $V_l\langle r_1, r_2, \dots, r_m\rangle = \langle s_{l1}, s_{l2}, \dots, s_{lm}\rangle$ for each $l = 1, 2, \dots, n$. Then, by Lemma 3, it follows that

$$\mathrm{mat}\,(\theta(E_1))\langle V_1, V_2, \dots, V_n\rangle = \langle W_1, W_2, \dots, W_n\rangle \in (\mathbf{M}_m(R))^n$$

where

$$W_k\langle r_1, r_2, \dots, r_m\rangle = \left(\sum_{t=1}^{m} \iota_t^{(m)} \pi_{m(k-1)+t}^{(mn)}\right) \cdot \mathrm{mat}\,(E_1)\langle s_{11}, s_{12}, \dots, s_{1m}, s_{21}, \dots, s_{nm}\rangle$$

for all $k = 1, 2, \dots, n$. Similarly, we have

$$\mathrm{mat}\,(\theta(E_2))\langle V_1, V_2, \dots, V_n\rangle = \langle W_1', W_2', \dots, W_n'\rangle \in (\mathbf{M}_m(R))^n$$

where

$$W_k'\langle r_1, r_2, \dots, r_m\rangle = \left(\sum_{t=1}^{m} \iota_t^{(m)} \pi_{m(k-1)+t}^{(mn)}\right) \cdot \mathrm{mat}\,(E_2)\langle s_{11}, s_{12}, \dots, s_{1m}, s_{21}, \dots, s_{nm}\rangle$$

for all $k = 1, 2, \ldots, n$. But since $\mathrm{mat}\,(E_1) = \mathrm{mat}\,(E_2)$, we deduce that $W_k = W_k'$ for all $k = 1, 2, \ldots, n$ and hence that $\phi(U) = \mathrm{mat}\,(\theta(E_1)) = \mathrm{mat}\,(\theta(E_2))$, i.e., ϕ is well-defined.

That ϕ is indeed an isomorphism is formalized in the following theorem.

Theorem 4. *The mapping $\phi : \mathbf{M}_{mn}(R) \to \mathbf{M}_n(\mathbf{M}_m(R))$ is a near-ring isomorphism.*

Proof. It follows directly from Lemmas 1 and 2 that ϕ is a homomorphism.

To show that ϕ is injective, let $U \in \mathbf{M}_{mn}(R)$ be a nonzero matrix with

$$U\langle r_{11}, r_{12}, \ldots, r_{1m}, r_{21}, \ldots, r_{nm}\rangle = \langle s_{11}, s_{12}, \ldots, s_{1m}, s_{21}, \ldots, s_{nm}\rangle$$

where (say) $s_{11} \neq 0$. Now define $V_l = f_{11}^{r_{l1}} + f_{22}^{r_{l2}} + \cdots + f_{mm}^{r_{lm}}$ for each $l = 1, 2, \ldots, n$. Then $V_l \in \mathbf{M}_m(R)$ and $V_l\langle 1, 1, \ldots, 1\rangle = \langle r_{l1}, r_{l2}, \ldots, r_{lm}\rangle$ for all $l = 1, 2, \ldots, n$. By Lemma 3 it follows that $\phi(U)\langle V_1, V_2, \ldots, V_n\rangle = \langle W_1, W_2, \ldots, W_n\rangle$ where

$$\begin{aligned} W_1\langle 1, 1, \ldots, 1\rangle &= \left(\sum_{t=1}^{m} \iota_t^{(m)} \pi_t^{(mn)}\right) \\ &\quad \cdot U\langle r_{11}, r_{12}, \ldots, r_{1m}, r_{21}, \ldots, r_{nm}\rangle \\ &= \left(\sum_{t=1}^{m} \iota_t^{(m)} \pi_t^{(mn)}\right) \\ &\quad \cdot \langle s_{11}, s_{12}, \ldots, s_{1m}, s_{21}, \ldots, s_{nm}\rangle \\ &= \langle s_{11}, s_{12}, \ldots, s_{1m}\rangle \\ &\neq \langle 0, 0, \ldots, 0\rangle. \end{aligned}$$

This shows that W_1 is a nonzero matrix in $\mathbf{M}_m(R)$, implying $\mathrm{Ker}\,\phi = \{0\}$.

The surjectivity of ϕ follows easily by an induction argument on the weight of the matrices in $\mathbf{M}_n(\mathbf{M}_m(R))$, starting with matrices of weight 1, i.e., matrices of the form $f_{k_1 k_2}^{V}$, where $V \in \mathbf{M}_m(R)$ and $1 \leq k_1$,

$k_2 \leq n$. A separate induction argument on the weight of V ensures the existence of an element of $\mathbf{M}_{mn}(R)$ which maps onto $f_{k_1k_2}^V$.

(The routine character of this latter part of the proof is illustrated through considering an example.

Take

$$f_{k_1k_2}^V = f_{k_1k_2}^{f_{11}^{x_1}(f_{11}^{x_2}+f_{12}^{x_3})+f_{22}^{x_4}},$$

which can be written as

$$f_{k_1k_2}^V = f_{k_1k_2}^{f_{11}^{x_1}}\left(f_{k_2k_2}^{f_{11}^{x_2}} + f_{k_2k_2}^{f_{12}^{x_3}}\right) + f_{k_1k_2}^{f_{22}^{x_4}}.$$

Then $\phi(U) = f_{k_1k_2}^V$, where

$$\begin{aligned} U = f^{x_1}_{m(k_1-1)+1,m(k_2-1)+1}&(f^{x_2}_{m(k_2-1)+1,m(k_2-1)+1} \\ &+ f^{x_3}_{m(k_2-1)+1,m(k_2-1)+2}) \\ + f^{x_4}_{m(k_1-1)+2,m(k_2-1)+2}& \end{aligned}$$

is an element of $\mathbf{M}_{mn}(R)$.) $\square$

Corollary 5. *For any natural numbers m and n it follows that*

$$\mathbf{M}_m(\mathbf{M}_n(R)) \cong \mathbf{M}_n(\mathbf{M}_m(R)).$$

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