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UNIQUENESS OF SMALL SOLUTIONS TO THE DIRICHLET PROBLEM FOR THE HIGHER DIMENSIONAL H-SYSTEM

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ABSTRACT. Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, $u, v : \Omega \to \mathbf{R}^{n+1}$ be two solutions of the constant mean curvature equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|\nabla u|^{n-2} \frac{\partial}{\partial x_i} u \right) = n^{n/2} H(u_{x_1} \times \cdots \times u_{x_n}).$$

Assume $u = v$ on $\partial\Omega$ and $|H| \max(\sup_\Omega |u|, \sup_\Omega |v|) < 1/n$. Then u and v coincide in Ω.

The Dirichlet problem for the equation of surfaces of prescribed mean curvature in $\mathbf{R}^3$

$$\Delta u = 2H(u_{x_1} \times u_{x_2})$$

has been frequently studied, and great progress has been achieved in the last decades. We only mention the important contributions of E. Heinz [**3, 4**], S. Hildebrandt [**5**] and H.C. Wente [**9**]. A far more extensive bibliography can, e.g., be found in [**1**]. Under reasonable assumptions on H and the prescribed boundary values, existence, uniqueness and regularity of "small" solutions have been proven.

F. Duzaar and M. Fuchs [**1, 2**] have studied the higher dimensional analogue

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|\nabla u|^{n-2} \frac{\partial}{\partial x_i} u \right) = n^{n/2} H(u_{x_1} \times \cdots \times u_{x_n}) \quad \text{in } \Omega \subset \mathbf{R}^n, \qquad u = u_0 \quad \text{on } \partial\Omega. \tag{1}$$

Here $\Omega \subset \mathbf{R}^n$, $n \geq 2$, is a smoothly bounded domain, $u : \overline{\Omega} \to \mathbf{R}^{n+1}$ is the unknown vector function, H is a constant and u_0 a sufficiently

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smooth boundary datum. $|\nabla u|^2 := \sum_{i=1}^n |u_{x_i}|^2$, $|\cdot|$ is the usual Euclidean norm. The cross product $a_1 \times \cdots \times a_n$ of n vectors in $\mathbf{R}^{n+1}$ is defined in such a way that the relation $(a_1 \times \cdots \times a_n) \cdot b := (a_1, \ldots, a_n, b) := \det(a_1, \ldots, a_n, b)$ holds for any vector $b \in \mathbf{R}^{n+1}$. (.,...,.) denotes the parallelepipedal product and "·" the usual scalar product.

Definition. $u \in L^\infty(\Omega) \cap H^{1,n}(\Omega) := (L^\infty(\Omega))^{n+1} \cap (H^{1,n}(\Omega))^{n+1}$ is called a *weak solution* to the Dirichlet problem (1), if $u - u_0 \in H_0^{1,n}(\Omega)$ and if the relation

$$-\sum_{i=1}^n \int_\Omega (|\nabla u|^{n-2} u_{x_i}) \cdot \varphi_{x_i}\, dx = n^{n/2} H \int_\Omega (u_{x_1}, \ldots, u_{x_n}, \varphi)\, dx \tag{2}$$

holds for any vector function $\varphi \in L^\infty(\Omega) \cap H_0^{1,n}(\Omega)$.

Under the additional assumption $|H| \cdot \sup_{\partial\Omega} |u_0| < 1$, F. Duzaar and M. Fuchs [**1**] show existence and, if the data are sufficiently smooth, $C^{1,\alpha}$-regularity of "small" weak solutions, i.e., they satisfy

$$\sup_\Omega |u| \le \sup_{\partial\Omega} |u_0|. \tag{3}$$

We remark that all the authors mentioned above also treat variable curvature functions $H(u)$. In the present paper we prove the following uniqueness result for "small" solutions. Property (3) ensures its application, if $\sup_{\partial\Omega} |u_0|$ is small enough.

Theorem. *Let $u, v \in C^0(\overline{\Omega}) \cap H^{1,n}(\Omega)$ be weak solutions to the Dirichlet problem* (1)*. Assume that u and v obey the smallness condition*

$$|H| \cdot \max\left(\sup_\Omega |u|, \sup_\Omega |v|\right) < \frac{1}{n}. \tag{4}$$

Then u and v coincide, i.e., $u = v$ in Ω.

For $n = 2$, this result is already contained in [**3**]. Moreover, for $n = 2$, W. Jäger [**7**] gives a stronger and presumably optimal result: in (4) 1/2 may be replaced by 1; furthermore, continuous dependence in the C^0-norm of the solutions on the boundary data is shown. The proof of Jäger's result requires the choice of a refined auxiliary function and a corresponding auxiliary differential equation. This technique fails for $n \geq 3$ because of the quasilinear, degenerate and coupled character of the principal part of equation (1).

The proof of the theorem is achieved by multiplying the difference of the differential equations for u and v (scalarly) by $u-v$. The estimation of $(u_{x_1} \times \cdots \times u_{x_n} - v_{x_1} \times \cdots \times v_{x_n}) \cdot (u - v)$ requires some effort. In particular, an optimal bound for the permanent is needed, see Lemma 1 below.

Throughout this paper permutations will play an essential role. $\mathcal{S}_n$ denotes the symmetric group of $\{1, \dots, n\}$, i.e., the set of all permutations τ of $\{1, \dots, n\}$.

Lemma 1 (Estimate of the permanent). *Let $n \in \mathbf{N}$, $A = (a_{ij})_{i,j=1,\dots,n}$ be an arbitrary $n \times n$-matrix with columns $a_1, \dots, a_n$. Then there holds*:

$$\left| \sum_{\tau \in \mathcal{S}_n} a_{\tau(1),1} \cdot \ldots \cdot a_{\tau(n),n} \right| \leq \frac{n!}{n^{n/2}} |a_1| \cdot \ldots \cdot |a_n|. \tag{5}$$

We recall that $|\cdot|$ denotes the Euclidean norm of vectors in $\mathbf{R}^n$. The constant $n!/n^{n/2}$ is optimal.

It is shown, e.g., in [**6, 8**], see also the references therein, that the maximum absolute value of a *symmetric* multilinear form on unit vectors can be attained with all vectors equal. Together with the geometric-arithmetic mean inequality, this gives Lemma 1. For the reader's convenience, an elementary proof of Lemma 1 is given in the appendix.

Lemma 2. *Let $u, v \in (C^0(\overline{\Omega}) \cap H^{1,n}(\Omega))^{n+1}$ be arbitrary vector functions which satisfy $u = v$ on $\partial\Omega$. Then we have*:

$$(6)\quad \left|\int_\Omega (u_{x_1}\times\cdots\times u_{x_n} - v_{x_1}\times\cdots\times v_{x_n})\cdot(u-v)\,dx\right| \\ \leq \frac{n}{2n^{n/2}}\max\Big(\sup_\Omega |u|,\sup_\Omega |v|\Big) \\ \int_\Omega \{|\nabla u|^{n-2}+|\nabla v|^{n-2}\}|\nabla(u-v)|^2\,dx.$$

Proof. By density we may assume $u, v \in C^2(\overline{\Omega})$. In the following calculations, all variables $x_1, \dots, x_n$ should play an equivalent role. This is achieved with the help of permutations.

$$\begin{aligned}
&\int_\Omega (u_{x_1}\times\cdots\times u_{x_n} - v_{x_1}\times\cdots\times v_{x_n})\cdot(u-v)\,dx\\
&= \frac{1}{n!}\sum_{\tau\in\mathcal{S}_n}\operatorname{sgn}\tau\int_\Omega \big\{(u-v)_{x_{\tau(1)}}\times u_{x_{\tau(2)}}\times\cdots\times u_{x_{\tau(n)}}\\
&\qquad\qquad + v_{x_{\tau(1)}}\times(u-v)_{x_{\tau(2)}}\times\cdots\times u_{x_{\tau(n)}}\\
&\qquad\qquad \vdots\\
&\qquad\qquad + v_{x_{\tau(1)}}\times\cdots\times v_{x_{\tau(n-1)}}\cdot(u-v)_{x_{\tau(n)}}\big\}\cdot(u-v)\,dx\\
&= \frac{1}{n!}\sum_{\tau\in\mathcal{S}_n}\operatorname{sgn}\tau\int_\Omega ((u-v)_{x_{\tau(1)}}, u_{x_{\tau(2)}},\dots,u_{x_{\tau(n)}}, u-v)\,dx\\
&\quad + \frac{1}{n!}\sum_{\tau\in\mathcal{S}_n}\operatorname{sgn}\tau\int_\Omega (v_{x_{\tau(1)}},\dots,v_{x_{\tau(n-1)}},(u-v)_{x_{\tau(n)}}, u-v)\,dx\\
&\quad + \frac{1}{n!}\sum_{\tau\in\mathcal{S}_n}\operatorname{sgn}\tau\int_\Omega \big\{(v_{x_{\tau(1)}},(u-v)_{x_{\tau(2)}},u_{x_{\tau(3)}},\dots,u_{x_{\tau(n)}}, u-v)\\
&\qquad\qquad \vdots\\
&\qquad\qquad + (v_{x_{\tau(1)}},\dots,v_{x_{\tau(n-2)}},(u-v)_{x_{\tau(n-1)}},u_{x_{\tau(n)}},u-v)\big\}\,dx.
\end{aligned}$$

The first summand is integrated by parts with respect to $x_{\tau(n)}$. This yields

(7)

$$-\frac{1}{n!}\sum_{\tau\in\mathcal{S}_n}\operatorname{sgn}\tau\int_\Omega\big\{((u-v)_{x_{\tau(1)}x_{\tau(n)}},u_{x_{\tau(2)}},\dots,u_{x_{\tau(n-1)}},u,u-v)$$

$$\vdots$$

$$+((u-v)_{x_{\tau(1)}},u_{x_{\tau(2)}},\dots,u_{x_{\tau(n-2)}},u_{x_{\tau(n-1)}x_{\tau(n)}},u,u-v)$$

$$+((u-v)_{x_{\tau(1)}},u_{x_{\tau(2)}},\dots,u_{x_{\tau(n-1)}},u,(u-v)_{x_{\tau(n)}})\big\}\,dx.$$

Let $i,j\in\{1,\dots,n\}$, $i\neq j$, $\tau_0\in\mathcal{S}_n$ be fixed. Then there is precisely one permutation $\tau_1\in\mathcal{S}_n\backslash\{\tau_0\}$ with $\tau_1(k)=\tau_0(k)$ for any $k\in\{1,\dots,n\}\backslash\{i,j\}$. There holds $\operatorname{sgn}\tau_1=-\operatorname{sgn}\tau_0$. That means that all terms containing second derivatives occur in pairs but with opposite sign. Thus, they cancel out. Hence, (7) is equal to

$$-\frac{1}{n!}\sum_{\tau\in\mathcal{S}_n}\operatorname{sgn}\tau\int_\Omega((u-v)_{x_{\tau(1)}},u_{x_{\tau(2)}},\dots,u_{x_{\tau(n-1)}},u,(u-v)_{x_{\tau(n)}})\,dx.$$

The other terms in the sum above are treated in the same way. The second summand is integrated with respect to $x_{\tau(1)}$. The third summand is split into two identical terms with weight $1/2$. The first of these terms is integrated with respect to $x_{\tau(1)}$, the second one with respect to $x_{\tau(n)}$. We obtain

$$\int_\Omega(u_{x_1}\times\cdots\times u_{x_n}-v_{x_1}\times\cdots\times v_{x_n})\cdot(u-v)\,dx$$

$$=-\frac{1}{n!}\sum_{\tau\in\mathcal{S}_n}\operatorname{sgn}\tau\int_\Omega\big\{((u-v)_{x_{\tau(1)}},u_{x_{\tau(2)}},\dots,u_{x_{\tau(n-1)}},u,(u-v)_{x_{\tau(n)}})$$

$$+(v,v_{x_{\tau(2)}},\dots,v_{x_{\tau(n-1)}},(u-v)_{x_{\tau(n)}},(u-v)_{x_{\tau(1)}})$$

$$+(1/2)(v,(u-v)_{x_{\tau(2)}},u_{x_{\tau(3)}},\dots,u_{x_{\tau(n)}},(u-v)_{x_{\tau(1)}})$$

$$\vdots$$

$$+(1/2)(v,v_{x_{\tau(2)}},\dots,v_{x_{\tau(n-2)}},(u-v)_{x_{\tau(n-1)}},u_{x_{\tau(n)}},(u-v)_{x_{\tau(1)}})$$

$$+(1/2)(v_{x_{\tau(1)}},(u-v)_{x_{\tau(2)}},u_{x_{\tau(3)}},\dots,u_{x_{\tau(n-1)}},u,(u-v)_{x_{\tau(n)}})$$

$$\vdots$$

$$+(1/2)(v_{x_{\tau(1)}},\dots,v_{x_{\tau(n-2)}},(u-v)_{x_{\tau(n-1)}},u,(u-v)_{x_{\tau(n)}})\big\}\,dx.$$

By means of Hadamard's estimate, $|\det(b_1,\dots,b_{n+1})| \le |b_1|\cdot\ldots\cdot|b_{n+1}|$, change of index of summation and Lemma 1, we conclude

$$
\begin{aligned}
&\left|\int_\Omega (u_{x_1}\times\cdots\times u_{x_n} - v_{x_1}\times\cdots\times v_{x_n})\cdot(u-v)\,dx\right| \\
&\quad\le \frac{1}{n!}\max\Big(\sup_\Omega|u|,\sup_\Omega|v|\Big) \\
&\quad\cdot\sum_{\tau\in\mathcal{S}_n}\int_\Omega\Big\{|(u-v)_{x_{\tau(1)}}|\cdot|(u-v)_{x_{\tau(2)}}|\cdot|u_{x_{\tau(3)}}|\cdot\ldots\cdot|u_{x_{\tau(n)}}| \\
&\qquad+|(u-v)_{x_{\tau(1)}}|\cdot|(u-v)_{x_{\tau(2)}}|\cdot|v_{x_{\tau(3)}}|\cdot\ldots\cdot|v_{x_{\tau(n)}}| \\
&\qquad+\frac12|(u-v)_{x_{\tau(1)}}|\cdot|(u-v)_{x_{\tau(2)}}|\cdot|u_{x_{\tau(3)}}|\cdot\ldots\cdot|u_{x_{\tau(n-1)}}|\cdot|u_{x_{\tau(n)}}| \\
&\qquad\vdots \\
&\qquad+\frac12|(u-v)_{x_{\tau(1)}}|\cdot|(u-v)_{x_{\tau(2)}}|\cdot|v_{x_{\tau(3)}}|\cdot\ldots\cdot|v_{x_{\tau(n-1)}}|\cdot|u_{x_{\tau(n)}}| \\
&\qquad+\frac12|(u-v)_{x_{\tau(1)}}|\cdot|(u-v)_{x_{\tau(2)}}|\cdot|u_{x_{\tau(3)}}|\cdot\ldots\cdot|u_{x_{\tau(n-1)}}|\cdot|v_{x_{\tau(n)}}| \\
&\qquad\vdots \\
&\qquad+\frac12|(u-v)_{x_{\tau(1)}}|\cdot|(u-v)_{x_{\tau(2)}}|\cdot|v_{x_{\tau(3)}}|\cdot\ldots\cdot|v_{x_{\tau(n-1)}}|\cdot|v_{x_{\tau(n)}}|\Big\}\,dx \\
&\quad\le\frac{1}{n^{n/2}}\max\Big(\sup_\Omega|u|,\sup_\Omega|v|\Big) \\
&\qquad\cdot\int_\Omega|\nabla(u-v)|^2\Big\{\frac32|\nabla u|^{n-2}+\frac32|\nabla v|^{n-2} \\
&\qquad\qquad+\frac12\sum_{j=1}^{n-3}(|\nabla v|^j|\nabla u|^{n-2-j}+|\nabla u|^j|\nabla v|^{n-2-j})\Big\}\,dx \\
&\quad\le\frac{1}{n^{n/2}}\max\Big(\sup_\Omega|u|,\sup_\Omega|v|\Big) \\
&\qquad\cdot\int_\Omega|\nabla(u-v)|^2\Big\{\frac32|\nabla u|^{n-2}+\frac32|\nabla v|^{n-2} \\
&\qquad\qquad+\frac12\sum_{j=1}^{n-3}\Big(\frac{j}{n-2}|\nabla v|^{n-2}+\frac{n-2-j}{n-2}|\nabla u|^{n-2}
\end{aligned}
$$

$$+\frac{j}{n-2}|\nabla u|^{n-2}+\frac{n-2-j}{n-2}|\nabla v|^{n-2}\Big)\Big\}\,dx$$
$$=\frac{n}{2n^{n/2}}\max\Big(\sup_\Omega|u|,\sup_\Omega|v|\Big)$$
$$\cdot\int_\Omega|\nabla(u-v)|^2\{|\nabla u|^{n-2}+|\nabla v|^{n-2}\}\,dx. \qquad \square$$

Now it is not difficult to furnish the proof of the theorem. Let u, v be solutions to the H-system as described there. We multiply the difference of the equations for u and v (scalarly) by $(u-v)$:

$$-\sum_{i=1}^n\int_\Omega(|\nabla u|^{n-2}u_{x_i}-|\nabla v|^{n-2}v_{x_i})\cdot(u-v)_{x_i}\,dx$$
$$=n^{n/2}H\int_\Omega(u_{x_1}\times\cdots\times u_{x_n}-v_{x_1}\times\cdots\times v_{x_n})\cdot(u-v)\,dx.$$

For the lefthand side, we have

$$\begin{aligned}
-\sum_{i=1}^n\int_\Omega&(|\nabla u|^{n-2}u_{x_i}-|\nabla v|^{n-2}v_{x_i})\cdot(u-v)_{x_i}\,dx\\
&=-\frac12\int_\Omega(|\nabla u|^{n-2}+|\nabla v|^{n-2})|\nabla(u-v)|^2\,dx\\
&\quad-\frac12\int_\Omega(|\nabla u|^n+|\nabla v|^n)\,dx\\
&\quad+\frac12\int_\Omega(|\nabla u|^{n-2}\cdot|\nabla v|^2+|\nabla u|^2\cdot|\nabla v|^{n-2})\,dx\\
&\le-\frac12\int_\Omega(|\nabla u|^{n-2}+|\nabla v|^{n-2})|\nabla(u-v)|^2\,dx\\
&\quad-\frac12\int_\Omega(|\nabla u|^n+|\nabla v|^n)\,dx\\
&\quad+\frac12\int_\Omega\bigg(\frac{n-2}{n}|\nabla u|^n+\frac2n|\nabla v|^n\\
&\qquad\qquad+\frac2n|\nabla u|^n+\frac{n-2}{n}|\nabla v|^n\bigg)\,dx\\
&=-\frac12\int_\Omega(|\nabla u|^{n-2}+|\nabla v|^{n-2})|\nabla(u-v)|^2\,dx.
\end{aligned}$$

With the help of Lemma 2, the righthand side is estimated as follows:

$$n^{n/2}H\int_\Omega (u_{x_1}\times\cdots\times u_{x_n}-v_{x_1}\times\cdots\times v_{x_n})\cdot(u-v)\,dx$$
$$\geq -|H|\cdot\frac{n}{2}\cdot\max\Big(\sup_\Omega|u|,\sup_\Omega|v|\Big)$$
$$\cdot\int_\Omega(|\nabla u|^{n-2}+|\nabla v|^{n-2})|\nabla(u-v)|^2\,dx.$$

Combining these estimates, we arrive at

$$0\geq\Big(1-n|H|\max\Big(\sup_\Omega|u|,\sup_\Omega|v|\Big)\Big)$$
$$\cdot\int_\Omega(|\nabla u|^{n-2}+|\nabla v|^{n-2})|\nabla(u-v)|^2\,dx.$$

This inequality proves Theorem 1. □

Appendix

As the estimate of the permanent may not be so widely known, we give a proof of Lemma 1, which relies only on some elementary knowledge about permutations and the very simple inequality $2ab\leq a^2+b^2$. This inequality is used repeatedly without mention.

The case $n\in\{1,2\}$ is trivial, so we assume $n\geq 3$. By homogeneity, it suffices to consider $f((a_{ij})_{i,j=1,\dots,n}):=\sum_{\tau\in\mathcal{S}_n}a_{\tau(1),1}\cdot\ldots\cdot a_{\tau(n),n}$ on

$$P:=\Big\{(a_{ij}):\sum_{i=1}^n a_{ij}^2=1,\ j=1,\dots,n\Big\}. \tag{8}$$

We have to calculate $\lambda:=\max\{f((a_{ij})):(a_{ij})\in P\}$. Inserting $a_{ij}=1/\sqrt{n}$ we immediately find $\lambda\geq n!/n^{n/2}$. So it remains to prove $\lambda\leq n!/n^{n/2}$.

Let $(a_{ij})_{i,j=1,\dots,n}\in P$ be a matrix on which f attains its maximum.

We have

$$\begin{aligned}
\lambda^2 &= \left(\sum_{\tau \in \mathcal{S}_n} a_{\tau(1),1} \cdot \ldots \cdot a_{\tau(n),n} \right)^2 \\
&= \left[\sum_{k=1}^{n} a_{k,1} \left(\sum_{\substack{\tau \in \mathcal{S}_n \\ \tau(1)=k}} a_{\tau(2),2} \cdot \ldots \cdot a_{\tau(n),n} \right) \right]^2 \\
&= \sum_{k=1}^{n} a_{k,1}^2 \left(\sum_{\substack{\tau \in \mathcal{S}_n \\ \tau(1)=k}} a_{\tau(2),2} \cdot \ldots \cdot a_{\tau(n),n} \right)^2 \\
&\quad + \sum_{\substack{k,l=1 \\ k \neq l}}^{n} a_{k,1} a_{l,1} \left(\sum_{\substack{\tau \in \mathcal{S}_n \\ \tau(1)=k}} a_{\tau(2),2} \cdot \ldots \cdot a_{\tau(n),n} \right) \\
&\quad \cdot \left(\sum_{\substack{\tau \in \mathcal{S}_n \\ \tau(1)=l}} a_{\tau(2),2} \cdot \ldots \cdot a_{\tau(n),n} \right) \\
&\leq \sum_{k=1}^{n} a_{k,1}^2 \left(\sum_{\substack{\tau \in \mathcal{S}_n \\ \tau(1)=k}} a_{\tau(2),2} \cdot \ldots \cdot a_{\tau(n),n)} \right)^2 \\
&\quad + \frac{1}{2} \sum_{\substack{k,l=1 \\ k \neq l}}^{n} a_{l,1}^2 \left(\sum_{\substack{\tau \in \mathcal{S}_n \\ \tau(1)=k}} a_{\tau(2),2} \cdot \ldots \cdot a_{\tau(n),n} \right)^2 \\
&\quad + \frac{1}{2} \sum_{\substack{k,l=1 \\ k \neq l}}^{n} a_{k,1}^2 \left(\sum_{\substack{\tau \in \mathcal{S}_n \\ \tau(1)=l}} a_{\tau(2),2} \cdot \ldots \cdot a_{\tau(n),n} \right)^2 .
\end{aligned}$$

By interchanging k and l in the third summand, we see that the second and third summand are equal. With the help of (8), there follows:

$$
\begin{aligned}
\lambda^2 \le & \sum_{k=1}^{n} a_{k,1}^2 \Bigg(\sum_{\substack{\tau\in\mathcal{S}_n \\ \tau(1)=k}} a_{\tau(2),2}\cdot\ldots\cdot a_{\tau(n),n} \Bigg)^2 \\
& + \sum_{k=1}^{n} \Bigg(\sum_{\substack{l=1 \\ l\neq k}}^{n} a_{l,1}^2 \Bigg) \Bigg(\sum_{\substack{\tau\in\mathcal{S}_n \\ \tau(1)=k}} a_{\tau(2),2}\cdot\ldots\cdot a_{\tau(n),n} \Bigg)^2 \qquad (9)\\
\Longrightarrow \lambda^2 \le & \sum_{k=1}^{n} \Bigg(\sum_{\substack{\tau\in\mathcal{S}_n \\ \tau(1)=k}} a_{\tau(2),2}\cdot\ldots\cdot a_{\tau(n),n} \Bigg)^2 .
\end{aligned}
$$

Thus the first column of (a_{ij}) has been eliminated from the representation of λ. We want to proceed in a similar way. In each step, one column has to be eliminated with the help of condition (8).

We now demonstrate how to eliminate the $(r+1)$th column of (a_{ij}) from the expression

$$
\lambda_r^2 := \sum_{1\le k_1<\cdots<k_r\le n} \Bigg(\sum_{\substack{\tau\in\mathcal{S}_n \\ \tau(1)=k_1 \\ \vdots \\ \tau(r)=k_r}} a_{\tau(r+1),r+1}\cdot\ldots\cdot a_{\tau(n),n} \Bigg)^2 ,
$$

$$
r = 1,\ldots,n-1.
$$

Obviously there holds

$$
\lambda_{n-1}^2 = \sum_{k=1}^{n} a_{k,n}^2 = 1. \qquad (10)
$$

For $r \in \{1, \dots, n-2\}$, we have

$$\lambda_r^2 = \frac{1}{r!} \sum_{\substack{k_1,\dots,k_r=1 \\ \text{mutually} \\ \text{distinct}}}^{n} \left[\sum_{\substack{l=1 \\ l\neq k_1 \\ \vdots \\ l\neq k_r}}^{n} a_{l,r+1} \left(\sum_{\substack{\tau\in\mathcal{S}_n \\ \tau(1)=k_1 \\ \vdots \\ \tau(r)=k_r \\ \tau(r+1)=l}} a_{\tau(r+2),r+2} \cdot \dots \cdot a_{\tau(n),n} \right) \right]^2$$

$$= \frac{1}{r!} \sum_{\substack{k_1,\dots,k_r,l=1 \\ \text{mutually} \\ \text{distinct}}}^{n} a_{l,r+1}^2 \left(\sum_{\substack{\tau\in\mathcal{S}_n \\ \tau(1)=k_1 \\ \vdots \\ \tau(r)=k_r \\ \tau(r+1)=l}} a_{\tau(r+2),r+2} \cdot \dots \cdot a_{\tau(n),n} \right)^2$$

$$+ \frac{1}{r!} \sum_{\substack{k_1,\dots,k_r,l,j=1 \\ \text{mutually} \\ \text{distinct}}}^{n} a_{l,r+1} \cdot a_{j,r+1}$$

$$\cdot \left(\sum_{\substack{\tau\in\mathcal{S}_n \\ \tau(1)=k_1 \\ \vdots \\ \tau(r)=k_r \\ \tau(r+1)=l}} a_{\tau(r+2),r+2} \cdot \dots \cdot a_{\tau(n),n} \right)$$

$$\cdot \left(\sum_{\substack{\tau\in\mathcal{S}_n \\ \tau(1)=k_1 \\ \vdots \\ \tau(r)=k_r \\ \tau(r+1)=j}} a_{\tau(r+2),r+2} \cdot \dots \cdot a_{\tau(n),n} \right).$$

We split the second sum with weights $(n-r)/n$ and r/n. In these sums each of the first two factors is combined with each of the last two factors. To these couples of factors the inequality $ab \leq (a^2+b^2)/2$ is applied.

$$\lambda_r^2 \leq \frac{1}{r!} \sum_{\substack{k_1,\dots,k_r,l=1\\ \text{mutually}\\ \text{distinct}}}^{n} a_{l,r+1}^2 \Bigg(\sum_{\substack{\tau\in\mathcal{S}_n\\ \tau(1)=k_1\\ \vdots\\ \tau(r)=k_r\\ \tau(r+1)=l}} a_{\tau(r+2),r+2}\cdot\ldots\cdot a_{\tau(n),n} \Bigg)^2$$

$$+ \frac{1}{r!}\cdot\frac{n-r}{2n} \sum_{\substack{k_1,\dots,k_r,l,j=1\\ \text{mutually}\\ \text{distinct}}}^{n} a_{j,r+1}^2$$

$$\cdot \Bigg(\sum_{\substack{\tau\in\mathcal{S}_n\\ \tau(1)=k_1\\ \vdots\\ \tau(r)=k_r\\ \tau(r+1)=l}} a_{\tau(r+2),r+2}\cdot\ldots\cdot a_{\tau(n),n} \Bigg)^2$$

$$+ \frac{1}{r!}\cdot\frac{n-r}{2n} \sum_{\substack{k_1,\dots,k_r,l,j=1\\ \text{mutually}\\ \text{distinct}}}^{n} a_{l,r+1}^2$$

$$\cdot \Bigg(\sum_{\substack{\tau\in\mathcal{S}_n\\ \tau(1)=k_1\\ \vdots\\ \tau(r)=k_r\\ \tau(r+1)=j}} a_{\tau(r+2),r+2}\cdot\ldots\cdot a_{\tau(n),n} \Bigg)^2$$

$$+ \frac{1}{r!}\cdot\frac{r}{2n} \sum_{\substack{k_1,\dots,k_r,l,j=1\\ \text{mutually}\\ \text{distinct}}}^{n} a_{l,r+1}^2$$

$$\cdot \Bigg(\sum_{\substack{\tau\in\mathcal{S}_n\\ \tau(1)=k_1\\ \vdots\\ \tau(r)=k_r\\ \tau(r+1)=l}} a_{\tau(r+2),r+2}\cdot\ldots\cdot a_{\tau(n),n} \Bigg)^2$$

$$+\frac{1}{r!}\cdot\frac{r}{2n}\sum_{\substack{k_1,\dots,k_r,l,j=1\\ \text{mutually}\\ \text{distinct}}}^{n} a_{j,r+1}^2$$

$$\cdot\Bigg(\sum_{\substack{\tau\in\mathcal{S}_n\\ \tau(1)=k_1\\ \vdots\\ \tau(r)=k_r\\ \tau(r+1)=j}} a_{\tau(r+2),r+2}\cdot\dots\cdot a_{\tau(n),n}\Bigg)^2.$$

Interchanging l and j we see that the second and third, fourth and fifth summands coincide, respectively. In the last two summands, the summation with respect to j and l, respectively, is carried out. So these terms together are equal to $(r/n)(n-(r+1))$ times the first summand. Taking notice of $1+(r/n)\cdot(n-(r+1)) = (r+1)\cdot(n-r)/n$ and renaming l by k_{r+1}, we find

$$\lambda_r^2 \le \frac{1}{r!}\cdot\frac{n-r}{n}(r+1)\sum_{\substack{k_1,\dots,k_{r+1}=1\\ \text{mutually}\\ \text{distinct}}}^{n} a_{k_{r+1},r+1}^2$$

$$\cdot\Bigg(\sum_{\substack{\tau\in\mathcal{S}_n\\ \tau(1)=k_1\\ \vdots\\ \tau(r+1)=k_{r+1}}} a_{\tau(r+2),r+2}\cdot\dots\cdot a_{\tau(n),n}\Bigg)^2$$

$$+\frac{1}{r!}\cdot\frac{n-r}{n}\sum_{\substack{k_1,\dots,k_{r+1},j=1\\ \text{mutually}\\ \text{distinct}}}^{n} a_{j,r+1}^2$$

$$\cdot\Bigg(\sum_{\substack{\tau\in\mathcal{S}_n\\ \tau(1)=k_1\\ \vdots\\ \tau(r+1)=k_{r+1}}} a_{\tau(r+2),r+2}\cdot\dots\cdot a_{\tau(n),n)}\Bigg)^2.$$

In the first summand the indices k_{r+1} and k_i, $i=1,\dots,r+1$, are interchanged. This treatment doesn't affect the second factor in the first sum. So we come up with

$$\lambda_r^2 \leq \frac{1}{r!} \cdot \frac{n-r}{n} \sum_{\substack{k_1,\dots,k_{r+1}=1 \\ \text{mutually} \\ \text{distinct}}}^{n} (a_{k_1,r+1}^2 + \cdots + a_{k_{r+1},r+1}^2)$$

$$\cdot \Bigg(\sum_{\substack{\tau\in\mathcal{S}_n \\ \tau(1)=k_1 \\ \vdots \\ \tau(r+1)=k_{r+1}}} a_{\tau(r+2),r+2} \cdot \ldots \cdot a_{\tau(n),n} \Bigg)^2$$

$$+ \frac{1}{r!} \cdot \frac{n-r}{n} \sum_{\substack{k_1,\dots,k_{r+1}=1 \\ \text{mutually} \\ \text{distinct}}}^{n} \Bigg(\sum_{\substack{j=1 \\ j\neq k_1 \\ \vdots \\ j\neq k_{r+1}}}^{n} a_{j,r+1}^2 \Bigg)$$

$$\cdot \Bigg(\sum_{\substack{\tau\in\mathcal{S}_n \\ \tau(1)=k_1 \\ \vdots \\ \tau(r+1)=k_{r+1}}} a_{\tau(r+2),r+2} \cdot \ldots \cdot a_{\tau(n),n} \Bigg)^2$$

$$= \frac{r+1}{(r+1)!} \cdot \frac{n-r}{n}$$

$$\cdot \sum_{\substack{k_1,\dots,k_{r+1}=1 \\ \text{mutually} \\ \text{distinct}}}^{n} \Bigg(\sum_{\substack{\tau\in\mathcal{S}_n \\ \tau(1)=k_1 \\ \vdots \\ \tau(r+1)=k_{r+1}}} a_{\tau(r+2),r+2} \cdot \ldots \cdot a_{\tau(n),n} \Bigg)^2 \quad \text{by (8)}$$

$$= (r+1) \cdot \frac{n-r}{n}$$

$$\cdot \sum_{1\leq k_1<\cdots<k_{r+1}\leq n} \Bigg(\sum_{\substack{\tau\in\mathcal{S}_n \\ \tau(1)=k_1 \\ \vdots \\ \tau(r+1)=k_{r+1}}} a_{\tau(r+2),r+2} \cdot \ldots \cdot a_{\tau(n),n} \Bigg)^2$$

$$= (r+1) \cdot \frac{n-r}{n} \cdot \lambda_{r+1}^2.$$

To sum up, we have

$$\begin{aligned} \lambda^2 \le \lambda_1^2 \le \cdots &\le \left[\prod_{r=1}^{n-2} (r+1) \cdot \frac{n-r}{n} \right] \cdot \lambda_{n-1}^2 \\ &= \frac{((n-1)!)^2}{n^{n-2}} \quad \text{by (10)} \\ &= \left(\frac{n!}{n^{n/2}} \right)^2. \qquad \square \end{aligned}$$

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