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DANIELL-LOOMIS INTEGRALS

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ABSTRACT. In [**2**] and [**3**] for arbitrary nonnegative linear functionals on functions vector lattices an integral extension of Lebesgue power has been discussed. Here we generalize this extension process, prove convergence theorems using a suitable "local convergence in measure," discuss measurability and give characterizations by equality of upper and lower integrals. Riemann-μ, abstract Riemann-Loomis and Bourbaki integrals are subsumed.

0. Introduction. For a semi-ring Ω of sets from an arbitrary set X and $\mu : \Omega \to [0, \infty[$, only finitely additive, an analogue $R_1(\mu, \overline{\mathbf{R}})$ to the space $L^1(\mu, \overline{\mathbf{R}})$ of Lebesgue-μ-integrable functions was introduced by Loomis [**11**]; this has been extended to Banach space-valued functions by Dunford-Schwartz [**4**], and in more general form in [**6, 7**]. Analogues to the Daniell extension process, but without or with weaker continuity assumptions on the elementary integral, have been treated by Aumann [**1**], Loomis [**11**] and Gould [**5**].

The Daniell-Bourbaki integral extension has been generalized with the integral $\overline{I} : \overline{B} \to \mathbf{R}$ introduced in [**2**], starting with any nonnegative linear functional I on a vector lattice B of real-valued functions on X. If Ω is a δ-ring, μ σ-additive, $I = \int .d\mu$ on $B =$ step functions over Ω, then R_1, L^1 and $\overline{B}$ coincide modulo null functions [**3, 9**].

In Sections 2 and 3 we generalize the extension $I|B \to \overline{I}|\overline{B}$ to $I|B \to J|L$ by "localization," using an appropriate local convergence in measure, which is very useful to obtain convergence theorems in a form analogous to the classical ones (some of which are not true for $\overline{B}$). In Section 4 we give various descriptions of the set L of integrable functions, in particular a Darboux-type characterization on L is proved. Always $R_1 \subset L$ (not true for $\overline{B}$), in general $\overline{B}$ has infinite codimension in L, even modulo null functions.

We recall that the abstract space of integrable functions L is constructed similar to the Daniell L^1 and which coincides with L^1 in the

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classical case, but contrary to the L^1 case, no continuity conditions on the starting elementary integral I/B, e.g., of Daniell type or starke integral norm of [**13**], are needed; so that, our results subsume most known situations of integration with respect to finitely additive measures.

Finally, in Section 5, all this is specialized to proper and abstract Riemann, Loomis, Daniell and Bourbaki integrals.

1. Notations and earlier results. As in [**1, 13**], we extend the usual $+$ in $\overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, \infty\}$ to $\overline{\mathbf{R}} \times \overline{\mathbf{R}}$ by

$$(1)\qquad \begin{gathered} a + b := 0, \qquad a \dot{+} b := \infty \quad \text{if} \quad a = -b \in \{-\infty, \infty\}; \\ a - b := a + (-b), \text{ etc.} \end{gathered}$$

Life would be easier using only the associative "$\dot{+}$," but convergence theorems with "$+$" are stronger.

We denote $a \vee b := \max(a, b)$, $a \wedge b = \min(a, b)$, $a \cap b := (a \wedge b) \vee (-b)$ if $b \geq 0$, $a^+ := a \vee 0$, $a^- := (-a)^+$.

For $a, b, c, d, e \in \overline{\mathbf{R}}$, $t, s \in \overline{\mathbf{R}}_+ := [0, \infty]$, one has

$$(2)\qquad \begin{gathered} |a \cap t - b \cap t| \leq 2(|a - b| \wedge t), \\ |(a + b) - (c + d)| \leq |a - e| + |b - d|, \\ |a \cap t - a \cap s| \leq |t - s|. \end{gathered}$$

In the following, X is an arbitrary nonempty set and we always assume B is a vector lattice $\subset \mathbf{R}^X$, $I : B \to \mathbf{R}$ is linear, $I(f) \geq 0$ if $0 \leq f \in B$.

For such $I|B$ we need the following results of [**2**], in somewhat modified notation:

(3)

$$\begin{gathered} B^\tau := \sup\{M, \varnothing \neq M \subset B\}, \text{ where the sup is pointwise on } X, \\ I_*(f) := \sup\{I(h); h \in B, h \leq f\} \text{ for } f \in \overline{\mathbf{R}}^X, \text{ with } \sup \varnothing = -\infty, \\ B_\tau := \{g \in B^\tau; I_*(f + g) = I_*(f) + I_*(g) \text{ for all } f \in B^\tau\}, \\ \overline{I}(f) := \inf\{I_*(g); f \leq g \in B_\tau\}, \underline{I}(f) := -\overline{I}(-f) \text{ for } f \in \overline{\mathbf{R}}^X. \end{gathered}$$

The elements of $\overline{B} := \{f \in \overline{\mathbf{R}}^X; \underline{I}(f) = \overline{I}(f) \in \mathbf{R}\}$ are called *I-summable functions.*

The B^τ and B_τ are $+$ and $\vee$ closed, B^τ is also $\wedge$ closed. $\overline{I}$ is $\dot{+}$ subadditive on $\overline{\mathbf{R}}^X$, $\overline{I}$ and I_* are $\mathbf{R}_+$-homogeneous and monotone on $\overline{\mathbf{R}}^X$.

For any $f \in \overline{\mathbf{R}}^X$ one has

$$I_*(f) \leq \underline{I}(f) \leq \overline{I}(f) \leq -I_*(-f) := I^*(f).$$

$\overline{B}$ is closed under $+, \dot{+}, \alpha, \wedge, \vee, |\ |$; $\overline{B}$ is the closure of B in $\overline{\mathbf{R}}^X$ with respect to the integral seminorm $\overline{I}$, $\overline{I}|\overline{B}$ is the unique $\overline{I}$-continuous extension of $I|B$ to $\overline{B}$ and is "linear" on $\overline{B}$, Aumann [**1**].

2. Integral extension with local convergence. For arbitrary net $(f_i)_{i\in S}$ with $f_i \in \overline{\mathbf{R}}^X$, $i \in S =$ directed set, we use a special type of convergence that will play an important role in what follows.

Definition 1. $f_i \to f(\overline{I})$ means $\overline{I}(|f_i - f| \wedge h) \to 0$ for each fixed $0 \leq h \in B$, where $f \in \overline{\mathbf{R}}^X$ and, e.g., $\infty - \infty = 0$ by (1). $(f_i) \subset \overline{\mathbf{R}}^X$ is called an $\overline{I}$-*Cauchy net* if $\overline{I}(|f_i - f_j|) \to 0$.

Theorem 1 of [**8**] yields

(4) If f_i, $f \in \overline{\mathbf{R}}^X$ with $|f_i - f| \leq g \in \overline{B}$, then $f_i \to f(\overline{I})$ if and only if $\overline{I}(|f_i - f|) \to 0$.

The following result is to prepare the basic definition.

Lemma 1. *If $(f_i) \subset \overline{B}$ is an $\overline{I}$-Cauchy net with $f_i \to 0(I^-)$, then $\overline{I}(|f|) \to 0$.*

Proof. To $\varepsilon > 0$ choose $j \in S$ with $\overline{I}(|f_i - f_j|) < \varepsilon$ if $i \geq j$. By Definition 1 there is $0 \leq h \in B$ with $\overline{I}(||f_i| - h|) < \varepsilon$; now with (2) we have $|f_i| \leq |f_i - f_j| + |f_i \cap |f_j| - f_j \cap h| + |f_j \cap h - f_i \cap h| + |f_i \cap h| \leq |f_i - f_j| + ||f_j| - h| + 2|f_j - f_i| + |f_i \cap h|$, so that $\overline{I}(|f_i|) \leq 5\varepsilon$ if $i \geq$ some $k \in S$. $\square$

This shows that if $f \in \overline{\mathbf{R}}^X$, (f_i) and $(g_i) \subset B$ are $\overline{I}$-Cauchy nets with $f_i \to f(\overline{I})$ and $g_i \to f(\overline{I})$. Then the limits exist and

$\lim I(f_i) = \lim I(g_i) \in \mathbf{R}$.

Definition 2. The set $L := L(B, I)$ of *I-integrable functions* is defined as the set of all those $f \in \overline{\mathbf{R}}^X$ for which there exists an $\overline{I}$-Cauchy net $(h_i) \subset B$ with $h_i \to f(\overline{I})$. Then $J(f) := \lim I(h_i)$, (h_i) is called a defining net for f.

A function $f \in \overline{\mathbf{R}}^X$ is called *L-null* if $f \in L$ and $J(|f|) = 0$.

By the above the $J(f)$ is well defined, independent of the particular choice of the (h_i).

Simple consequences of Definition 2 are, see [**2**],

If $f \in \overline{\mathbf{R}}^X$, $g \in L$ and $f(x) = g(x)$ where $|g(x)| < \infty$, then $f \in L$ and $J(f) = J(g)$; especially $g_e \in L$, where $g_e(x) := g(x)$ if $g(x) \in \mathbf{R}$, else $g_e(x) := 0$ and $g - g_e$ is an L-null function.

L contains B and is closed with respect to $+, \dot{+}, \alpha, \alpha \in \mathbf{R}, \wedge, \vee, |\ |$. Also, $J(\alpha f) = \alpha J(f)$, $J(f+g) = J(f \dot{+} g) = J(f) + J(g)$, $|J(f)| \leq J(|f|)$ if $f, g \in L$, and $J(f) \leq J(g)$ if $f \leq g$.

Next $\|f\| := J(|f|)$ defines a seminorm on L, and it is easy to prove that B is $\|\ \|$-dense in L, i.e., for any $\overline{I}$-Cauchy net $(h_i) \subset B$ with $h_i \to f(\overline{I})$ one has $\|h_i - f\| \to 0$, (so, note that sequences suffice in Definition 2).

Lemma 2. *$\overline{B} \subset L(B, I)$ and $\overline{I}(f) = J(f)$ for any $f \in \overline{B}$.*

Proof. Since B is $\overline{I}$-dense in $\overline{B}$ by Definition 1, there exists a defining sequence (h_n) for any $f \in \overline{B}$ by (2) so that $f \in L$. Now $|\overline{I}(f) - I(h_n)| = |\overline{I}(f - h_n)| \leq \overline{I}(|f - h_n|) \to 0$ gives $J(f) = \lim I(h_n) = \overline{I}(f)$. □

3. Convergence theorems.

Lemma 3. *If $0 \leq f \in L(B, I)$ and $0 \leq g \in \overline{B}$, then $f \wedge g \in \overline{B}$ and $\overline{I}(f \wedge g) \leq J(f)$.*

Proof. By Definition 2, there are $h_n \in B$ with $0 \leq h_n \to f(\overline{I})$, $(h_n)\overline{I}$-

Cauchy. Now with (2) one gets $h_n \cap g \to f \cap g(\overline{I})$. It follows from (4) that $\overline{I}(|h_n \cap g - f \cap g|) \to 0$; by Definition 1, $h_n \cap g = h_n \wedge g \in \overline{B}$, so also $f \wedge g = f \cap g \in \overline{B}$. Finally, Lemma 2 gives $\overline{I}(f \wedge g) = J(f \wedge g) \leq J(f)$. $\square$

Corollary 1. *If $f \in L(B, I)$ and $|f| \leq g \in \overline{B}$, then $f \in \overline{B}$.*

As a substitute for the general missing completeness of L, one has the following

Theorem 1. *If $f_i \in L(B, I)$, $f \in \overline{\mathbf{R}}^X$, (f_i) is a $\| \|$-Cauchy $= J$-Cauchy net with $f_i \to f(\overline{I})$, then $f \in L(B, I)$, $J(|f_i - f|) \to 0$ and $J(f_i) \to J(f)$.*

Proof. By Definition 2, given any $i \in S$ and $\varepsilon > 0$, there exist $h_{i,\varepsilon} \in B$ with $J(|f_i - h_{i,\varepsilon}|) < \varepsilon$.

Since $|h_{i,\varepsilon} - f| \leq |h_{i,\varepsilon} - f_i| + |f_i - f|$, $|a + b| \wedge t \leq |a| \wedge t + |b| \wedge t$, $a, b \in \overline{\mathbf{R}}$, $t \in \overline{\mathbf{R}}_+$, and Lemma 3 give $h_{i,\varepsilon} \to f(\overline{I})$, where $(h_{i,\varepsilon})$ is an $\overline{I}$-Cauchy net with index set Sx$]0, \infty[$.

Therefore, $f \in L$ and $J(f) = \lim I(h_{i,\varepsilon}) = \lim J(f_i)$. This applied to $|f_i - f|$ gives $\|f_i - f\| \to 0$. $\square$

In contrast to Theorem 1, by Example 2 below, the space $\overline{B}$ is not closed in this sense; by the same example also the usual monotone convergence theorem is false for $\overline{B}$ with "$\to (\overline{I})$." For L, however, one has

Corollary 2 (Monotone convergence theorem). *If $f_i \in L(B, I)$, $f \in \overline{\mathbf{R}}^X$, (f_i) is an increasing net such that $f_i \to f(\overline{I})$ and $\sup J(f_i) < \infty$, then $f \in L(B, I)$ and $J(f_i) \to J(f)$.*

Proof. By Theorem 1 we have only to show that (f_i) is J-Cauchy. Observe that $J(f_i) \leq J(f_j)$ if $i \leq j$ implies $J(f_i) \to \sup J(f_i)$, so indeed $J(|f_i - f_j|) = J(f_j) - J(f_i) \to 0$. $\square$

Theorem 2 (Dominated convergence theorem). *If* $f_i \in L(B,I)$, $f \in \overline{\mathbf{R}}^X$, (f_i) *is a net with* $f_i \to f(\overline{I})$ *and* $|f_i| \le$ *some* $g \in L(B,I)$ *for* $i \in S$, *then* $f \in L(B,I)$ *and* $\|f_i - f\| \to 0$, $J(f_i) \to J(f)$.

Proof. By Theorem 1, it suffices to show that (f_i) is J-Cauchy. If not, there are $\varepsilon_0 > 0$ and $k \in S$ indices $i_k, j_k \ge k$ with $J(g_k) \ge 2\varepsilon_0$ for $k \in S$, with $g_k := |f_{i_k} - f_{j_k}| \in L$, $g_k \to 0(\overline{I})$, $g_k \le \varphi := 2g \in L$.

By Definition 2, there is a $0 \le h \in B$ with $\|\varphi - h\| < \varepsilon_0$, hence $J(\varphi - \varphi \wedge h) \le J(|\varphi - h|)| < \varepsilon_0$.

Now (2) implies $g_k \le g_k \wedge h + (\varphi - \varphi \wedge h)$, with Lemmas 2 and 3 one gets $2\varepsilon_0 \le J(g_k) \le J(g_k \wedge h) + J(\varphi - \varphi \wedge h) < \overline{I}(g_k \wedge h) + \varepsilon_0 < 2\varepsilon_0$ for some $k \in S$, the desired contradiction. □

Even for sequences, with pointwise convergence the above results are of course false in the general (finitely additive) situation treated here; the applicability rests on verifying the assumption "$f_i \to f(\overline{I})$"; nevertheless, in the measure space situation pointwise convergence implies "$\to (\overline{I})$" (see Section 5.3).

4. Measurable and integrable functions.

Definition 3 (Stone). A function $f \in \overline{\mathbf{R}}^X$ is called *I-measurable* if $f \cap h \in L(B,I)$ for all $0 \le h \in B$.

$M_\cap := M_\cap(B,I)$ denotes the set of all the I-measurable functions.

Obviously $B \subset \overline{B} \subset L \subset M_\cap$. Also, by Corollary 1, $f \cap h \in \overline{B}$ in Definition 3.

Lemma 4. *$M_\cap(B,I)$ is closed with respect to* $\wedge, \vee, |\;|, \alpha, \alpha \in \mathbf{R}, \to (\overline{I})$. *If* $f, g \in M_\cap$, $l \in L$, *then* $f \cap |g|$, $f + l \in M_\cap f \in M_\cap$ *if and only if* f^+ *and* f^- *belong to* $M_\cap$.

We show only the closedness with respect to $\to (\overline{I})$: If $f_i \in M_\cap$, $f \in \overline{\mathbf{R}}^X$ and $f_i \to f(\overline{I})$, with (2) one gets, for $0 \le h \in B$, $\overline{I}(|f_i \cap h - f \cap h|) \le 2\overline{I}(|f_i - f| \wedge h) \to 0$. Hence, $f_i \cap h \in B$ by Corollary 1 and since L is a lattice. Therefore, $f \cap h \in \overline{B} \subset L$, since $\overline{B}$ is $\overline{I}$-closed. □

In general, however, $M_\cap$ is not $+$-closed by Example 3 below.

Using $f \cap h_n$ with $J(|g-h_n|) \to 0$, (2) and the dominated convergence theorem we obtain

(5) f is I-measurable, $|f| \le$ some I-integrable g implies f is I-integrable.

Definition 4. For any $f \in \overline{\mathbf{R}}^X$ we define *lower* and *upper* (*Darboux*) *integrals* by

(6)
$$J_*(f) := \sup\{J(g); g \le f, g \in L(B,I)\} \quad \text{and} \quad J^*(f) := -J_*(-f).$$

A functional $q : \overline{\mathbf{R}}_+^X \to \overline{\mathbf{R}}_+$ is called an *integral metric* (see, for example, [**1, 13**]) if $q(0) = 0$ and $q(f) \le q(g) + q(h)$ if $f \le g + h$, $f, g, h \in \overline{\mathbf{R}}_+^X$.

For $f, g, h, k \in \overline{\mathbf{R}}^X$ with $0 \le h \le k$, one gets

$$|q(|f|) - q(|g|)| \le q(|f-g|) \quad \text{and} \quad q(h) \le q(k). \tag{7}$$

For any $T : D \to \overline{\mathbf{R}}$ with $D = \overline{\mathbf{R}}^X$ or $\overline{\mathbf{R}}_+^X$ we define the *localization*

$$T_B(f) := \sup\{T(f \wedge h); 0 \le h \in B\} \quad \text{for all } f \in D. \tag{8}$$

This is a simplified version of Schäfke's definition [**13**, p. 120].

If $T = q =$ integral metric, q_B is also an integral metric.

From these definitions and our results above, one gets

$$I_* \le \underline{I}_B = \underline{I} \le J_{*B} = J_* \le J_B^* = \overline{I}_B \le J^* \le \overline{I} \le I^* \quad \text{on } \overline{\mathbf{R}}^X. \tag{9}$$

Only under additional assumptions the "$\le$" can be improved, e.g., $J_* = J^*$ on L, $J_* = J_B^*$ on $M_\cap$, $\overline{I}_B = \overline{I}$ if $\overline{I} < \infty$, we omit the details.

As in the proof of Theorem 1.5 of Schäfke [**13**], one shows

(10) $L(B,I) = \overline{I}_B$-closure of B in $\overline{\mathbf{R}}^X$ and $J|L$ is the unique $\overline{I}_B$-continuous extension on $I|B$. Especially is $L(B,I)\overline{I}_B$-closed.

With the above Darboux integrals, (5) can be generalized:

Theorem 3. *$f \in L(B,I)$ if and only if $f \in M_\cap(B,I)$ and $J_*(|f|) < \infty$.*

Then the same is true with J_* replaced by J^*, or with $J_*(|f|) < \infty$; $J_*(f) < \infty$ however is not sufficient.

Proof. Since $J_*(f) \in \mathbf{R}$, there are $g_n \in L$ with $g_n \le g_{n+1} \le f$ and $J(g_n) \to J_*(f)$.

Now for any $a, b \in \overline{\mathbf{R}}$, $0 \le t < \infty$, one has $(a-b) \wedge t \le a \wedge (b+t) \dot{-} b$, yielding $|f - g_n| \wedge h \le f \wedge (g_n + h) \dot{-} g_n =: l_n$ if $0 \le h \in B$.

Because $g_1 \wedge (g_n + h) \le f \wedge (g_n + h) \le g_n + h$, Lemma 4 and (5) yield $f \wedge (g_n + h) \in L$.

Therefore, $l_n \in L$ and $J(l_n) = J(f \wedge (g_n + h) \dot{-} g_n) \le J_*(f) - J(g_n) \to 0$ with Definition 2.

Furthermore, $|f - g_n| \wedge h \le l_n \wedge h \in \overline{B}$ by Corollary 1, hence $\overline{I}(|f - g_n| \wedge h) \le \overline{I}(l_n \wedge h) \le J(l_n)$ by Lemma 3. This implies $\overline{I}_B(|f - g_n|) \le J_*(f) - J(g_n) \to 0$, so that $f \in L$ with (10). □

I-integrability can be characterized as in the classical cases, without any measurability assumptions:

Theorem 4. *For any $f \in \overline{\mathbf{R}}^X$ the following conditions are equivalent*:

i) $f \in L(B,I)$

ii) $J_*(f) = J^*(f) \in \mathbf{R}$

iii) $J_*(f) = (J^*)_B(f) \in \mathbf{R}$.

Proof. In view of (9) and the remarks after it, it is enough to show iii) $\Rightarrow$ i). For this, by Theorem 3, we only have to show that $f \cap h \in L$ if $0 \le h \in B$.

Now for $\varepsilon > 0$ there are $g \in L$, $k \in B$ with $g \le h$, $J_*(f) - \varepsilon < J(g)$, $J(g - g \wedge k) \le J(|g - k|) < \varepsilon$.

Set $0 \le l := h + |k|$, then $-J(g \wedge l) \le -J(g \wedge k) < -J(g) + \varepsilon$.

There also is $p \in L$ with $f \wedge l \leq p$ and $J(p) < J_*(f) + \varepsilon$.

Since $f \cap h = (f \wedge l) \cap h$, one has $|f \cap h - (g \wedge h) \cap h| \leq 2(f \wedge l - g \wedge l) \leq 2(p - g \wedge l) =: q \in L$, or, with $J_* = J^*$ on L, $\overline{I}_B(|f \cap h - (g \wedge l) \cap h|) \leq \overline{I}_B(q) = J(q) = 2[J(p) - J(g \wedge l)] < 2[J_*(f) + \varepsilon - J(g) + \varepsilon] < 2[J_*(f) + 2\varepsilon - J_*(g) + \varepsilon] = 6\varepsilon$.

This proves that $f \cap h \in L$ by (10). $\square$

Using the above results, we proceed to add further properties of these extensions of I/B.

Almost by definition, our I-measurable functions contain the integrable functions of [**2**, p. 253].

The inclusion $B^\tau \cap \overline{B} \subset B_\tau$ of [**3**, p. 261] here generalizes to

$$B^\tau \cap L(B, I) \subset \{g \in B_\tau; I_*(g) < \infty\} \subset \overline{B}. \tag{11}$$

Similarly as in the case of Riemann-μ-, Lebesgue- and $\overline{B}$-integrals [**7**, p. 262, **8**, Theorem 3, p. 86] one shows that with Stone's axiom, i.e., $h \wedge 1 \in B$ if $0 \leq h \in B$, the following result holds:

(12) If $0 \leq h \in B$, then $h \wedge 1 \in L(B, I)$, and $J(h \wedge 1) \to I(h)$ as $n \to \infty$.

Thus it remains to show that $L(B, I)$ is closed with respect to improper integration, i.e.,

(13) $f \in L$ if and only if $f \cap n \in L$ for $n = 1, 2, \ldots$ and $\sup_n J(|f \cap n|) < \infty$.

Without (12) or with $\sup_n |J(f \cap n)| < \infty$, (13) is false for Example 1 below.

Concerning iteration of the extension process $I|B \to J|L$, one can show

(14) $L(B, I) = R_1(\tilde{B}, \tilde{I})$, with coinciding integrals,

where $\tilde{B} := L(B, I) \cap \mathbf{R}^X$, $\tilde{I} := J|\tilde{B}$, and R_1 is defined in Section 5.2 so that every L-space (integral) is an R_1-space (integral) of abstract Riemann integrable functions.

(15) Since always $R_1 \subset L$ by (17) below, one has at least $L(B, I) \subset L(\tilde{B}, \tilde{I})$ (an analogous inclusion for $\overline{B}$ false).

Let us finally remark that most of the above can be extended to Banach space-valued functions, using $a \cap t := \|a\|^{-1}(\|a\| \wedge t)a$ of [**7**, p. 327] and a controlling $I_0 : B_0 \to \mathbf{R}_+$, $B_0 \subset \mathbf{R}_+^X$, with $\|I(h)\| \le I_0(|h|)$, as in Schäfke [**13**].

5. Applications and examples.

1. If Ω is a semi-ring of sets in X and $\mu : \Omega \to [0, \infty[$ is additive, $B = B_\Omega :=$ real valued step functions over Ω and $I = I_\mu := \int \cdot d_\mu$ are admissible.

Then the *proper Riemann μ-integrable functions* $R^1_{\text{prop}}(\mu, \mathbf{R})$ are defined as a I^*_μ-closure of B_Ω in $\mathbf{R}^X$, with integral metric $I^*_\mu(f) := \inf\{I_\mu(h); f \le h \in B_\Omega\}$.

For $\Omega =$ intervals $\subset \mathbf{R}^n$ and $\mu =$ Lebesgue measure μ_L, $R^1_{\text{prop}} =$ classical Riemann integrable functions [**1**].

The space of *abstract Riemann μ-integrable functions* $R_1(\mu, \overline{\mathbf{R}})$ is defined as in Definition 2, but with $h_i \to f$ "μ-locally" [**7**, pp. 199, 70]. By Lemma 9 of [**8**], this convergence is equivalent to $h_i \to f(I^*_\mu)$, where in Definition 1 the $\overline{I}$ is replaced by I^*_μ. Obviously $\overline{I}_\mu \le I^*_\mu$ and $f_i \to f(I^*_\mu)$ implies $f_i \to f(\overline{I}_\mu)$. So with $L(X, \Omega, \mu, \mathbf{R})$ of Dunford-Schwartz [**4**, p. 112], one has

(16) $R^1_{\text{prop}}(\mu, \mathbf{R}) \subset L(X, \Omega, \mu, \mathbf{R}) \subset R_1(\mu, \overline{\mathbf{R}}) = (I^*_\mu)_B$-closure of $B_\Omega \subset L(B_\Omega, I_\mu)$, with coinciding integrals; all $\subset$ are in general strict, see Example 4 below.

In Gould's paper [**5**], Stone's axiom is assumed, so by [**6**] his results are already subsumed by the R_1-theory, see [**7**, pp. 57, 268].

2. For $I|B$ a "one-sided closure" U of B has been introduced by Loomis in [**11**]. This can be seen as the $(I^*)_B$-closure of B in $\overline{\mathbf{R}}^X$, so, in view of Definition 2, we define $R_1(B, I)$ as the I^*-closure of B in $\overline{\mathbf{R}}^X$. One has convergence theorems similar as in the $R_1(\mu, \overline{\mathbf{R}})$ case.

As in 1 (see also [**3**]), with $B = B_\Omega$, $I = I_\mu$ only, one gets

(17) $R_1(B, I) \subset \overline{B} + R_1$-null functions $\subset L(B, I)$ with coinciding integrals.

By the counterexamples in [**8, 9**], these are the only relations between $R_1, \overline{B}, L$ and their null functions.

3. If $I|B$ satisfies Daniell's continuity condition, i.e., $I(h_n) \to 0$, if $0 \leq h_{n+1} \leq h_n \in B$, $n \in \mathbf{N}$, with $h_n(x) \to 0$ for each $x \in X$, then $\overline{B} \subset L^1 + \overline{B}$-null functions, by [**9**].

With $R_1 \subset L_1$ and (18) below, these are the only relations here, see [**9**]. If, however, $B = B_\Omega$ with $\Omega = \delta$-ring and $I = I_\mu$ with μ σ-additive, then $R_1(\mu, \overline{\mathbf{R}}) = L_1(\mu, \overline{\mathbf{R}}) \subset L(B_\Omega, I_\mu)$, and $f_n \to f$ μ-almost everywhere implies $f_n \to f$ (I_μ^*) for μ-measurable f_n, by [**7**, p. 265], so the classical Lebesgue convergence theorems are subsumed by the R_1 and therefore $L(B, I)$ theory.

4. If $I|B$ satisfies Daniell's continuity condition and additionally

(18) $\lim h_n \in L(B, I)$ if $0 \leq h_n \leq h_{n+1} \in B$ and $h_n \leq k \in B$,

then $I^\sigma(|f - f_n|) \to 0$ implies $f_n \to f(\overline{I})$ in $\overline{\mathbf{R}}^X$; here $I\sigma(f) := \inf\{\sum_1^\infty I(h_n); h_n \in B, f \leq \sum_1^\infty h_n\}$ is the induced integral norm with which Daniell's $L^1 = I^\sigma$-closure of B, see, for example, [**1, 10**]. Now, since $\lim h_n \in \overline{B}$ by (11), we obtain

$L^1 + \overline{B}$-null functions $= \overline{B}, \overline{B} + L$-null functions $= L(B, I)$.

If even $\lim h_n \in R_1(B, I)$, e.g., $h_n \in B_\Omega$, corresponding to $\Omega = \delta$-ring in 3, then again $R_1(B, I) = L_1 = L^1$-null functions $\subset L(B, I)$.

5. If $I|B$ satisfies Bourbaki's condition, i.e., I/B is monotone-net-continuous, then $B^\tau = B_\tau$, $L^1 \subset \overline{B} =$ Bourbaki's $L^\tau \subset L(B, I) = L_\tau = L^\tau + \{L_\tau$-null functions$\}$, where L_τ is the localized version of L^τ.

Special cases include $B = C_0(X, \mathbf{R})$ with arbitrary topological X or $B = B_\Omega$ with $\Omega =$ intervals $\subset \mathbf{R}^n$, $\mu =$ Lebesgue measure, see [**2, 8** and **9**].

6. Local integral metrics and corresponding integrals have been introduced in Schäfke [**13**]; there also convergence theorems are proved. For $\overline{I}$ they are subsumed by the convergence theorems here; we do not need the restrictive condition 2 of [**13**], e.g., $C_0(X, \mathbf{R})$ does not satisfy condition 2.

For reference and the benefit of the reader, we collect some examples mostly given in [**7**] and [**8**].

Example 1. $X = \mathbf{N}$, $B = \{\{x_n\}_{n\in\mathbf{N}};\ \lim(x_n/n) \text{ exists} \in \mathbf{R}\}$, $I =$ this lim, $k = \{n^2\}$ $k \notin L$, though X is an I-null set, $I(\chi_X) = 0$; even $h \wedge 1 \in B$ if $0 \le h \in B$.

Example 2. $X = \mathbf{N}_0 \times J$, with $\mathbf{N}_0 = \{0, 1, 2, \dots\}$, $J = [0, 1[\subset \mathbf{R}$. $\Omega =$ ring containing all M of the form $\{n\} \in E$, $\{n\} \times (J - E)$, $F \times \{y\}$ or $(\mathbf{N}_0 - F) \times \{y\}$, with $0 \neq n \in \mathbf{N}_0$, E finite $\subset J$, $0 \notin F$ finite $\subset \mathbf{N}_0$, $y \in J$.

$\mu : \Omega \to \{0, 1\}$ is defined by $\mu(\{n\} \times (J - E)) = 1$, $\mu(M) = 0$ for all other $M \in \Omega$.

Let $T := \{0\} \times J$. One has $h_n := 0 \to \chi_T(\mu)$, i.e., $I(\chi_T) = 0$, but $\chi_T \notin \overline{B}$.

Example 3. X infinite, $\Omega = \{E \text{ or } X - E;\ E \text{ finite} \subset X\}$, $\mu = \delta_\infty$, $\delta_\infty(E) = 0, \delta_\infty(X - E) = 1$, $B = B_\Omega$, $I(f) = \int f\, d\delta_\infty$ on B (if X is uncountable, δ_∞ is σ-additive).

Let $X = \mathbf{N}$, $g(n) = n$ and $f = g + \chi_{2\mathbf{N}}$. One has $f \ge g \ge 0$, f and g I-measurable, but $f - g$ not I-measurable.

Example 4. $X = [0, 1]$, $\Omega = \{[a, b[; 0 \le a \le b \le 1\}$, $\mu =$ Lebesgue measure on Ω, $\mathbf{Q} =$ rationals $\subset X$, then $f_n := 0 \to \chi_{\mathbf{Q}}(\overline{I}_\mu)$ but not converging "μ-locally."

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