ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 1, Spring 1998

INCOMPLETE LIPSCHITZ-HANKEL INTEGRALS OF ANGER AND WEBER FUNCTIONS

ALLEN R. MILLER

ABSTRACT. Representations in terms of Kampé de Fériet functions are derived for certain generalized hypergeometric integrals. These are then used to obtain representations for incomplete Lipschitz-Hankel and related integrals of Anger and Weber functions. Reduction formulas for the associated Kampé de Fériet functions may then be employed to reduce special cases of the resulting Lipschitz-Hankel integrals to simpler generalized hypergeometric functions. In addition, four new reduction formulas for the Kampé de Fériet functions associated with incomplete Lipschitz-Hankel integrals of Anger-Weber functions are given.

1. Introduction. An integral having the form

$$C_{e_{\mu,\nu}}(a,z) \equiv \int_0^z \exp(at)t^\mu C_\nu(t)\,dt$$

where $C_\nu(t)$ is a cylindrical function or an associated Bessel function is called a Lipschitz-Hankel integral if $z = \infty$, and if $|z| < \infty$, then it is called an incomplete Lipschitz-Hankel integral.

In a series of recent papers, see [**3**] through [**9**], the author has given representations for incomplete Lipschitz-Hankel and related integrals of Bessel functions $J_\nu(t)$, modified Bessel functions $I_\nu(t)$, Hankel functions $H_\nu^{(1)}(t)$ and $H_\nu^{(2)}(t)$, Macdonald functions $K_\nu(t)$, Neumann functions $Y_\nu(t)$, Struve functions $\mathbf{H}_\nu(t)$ and modified Struve functions $\mathbf{L}_\nu(t)$.

This program shall now be completed in the present paper by deriving representations in terms of Kampé de Fériet functions for incomplete Lipschitz-Hankel and related integrals of Anger functions $\mathbf{J}_\nu(t)$ and Weber functions $\mathbf{E}_\nu(t)$. To this end, we define the following functions

Received by the editors on October 17, 1994, and in revised form on August 16, 1995.

in two complex variables:

$$\mathbf{J}_{e_{\mu,\nu}}(a,z) \equiv \int_0^z \exp(at)t^{\mu}\mathbf{J}_{\nu}(t)\,dt \tag{1.1}$$

$$\mathbf{E}_{e_{\mu,\nu}}(a,z) \equiv \int_0^z \exp(at)t^{\mu}\mathbf{E}_{\nu}(t)\,dt \tag{1.2}$$

and the related functions

$$\mathbf{J}_{s_{\mu,\nu}}(a,z) \equiv \int_0^z \sin(at)t^{\mu}\mathbf{J}_{\nu}(t)\,dt \tag{1.3}$$

$$\mathbf{J}_{c_{\mu,\nu}}(a,z) \equiv \int_0^z \cos(at)t^{\mu}\mathbf{J}_{\nu}(t)\,dt \tag{1.4}$$

$$\mathbf{E}_{s_{\mu,\nu}}(a,z) \equiv \int_0^z \sin(at)t^{\mu}\mathbf{E}_{\nu}(t)\,dt \tag{1.5}$$

$$\mathbf{E}_{c_{\mu,\nu}}(a,z) \equiv \int_0^z \cos(at)t^{\mu}\mathbf{E}_{\nu}(t)\,dt. \tag{1.6}$$

Here μ and ν may be complex numbers, and the presence of the exponential, sine and cosine functions in the integrand are indicated respectively by the subscripts e, s and c.

2. Preliminary results. In order to facilitate calculation of the integrals in equations (1.1)–(1.6), it will be convenient to obtain a representation for the hypergeometric integral

$$J(a,b,z) \equiv \int_0^z t^{\mu-1}{}_pF_q[(\alpha_p);(\beta_q);at]{}_rF_s[(\xi_r);(\eta_s);bt^2/4]\,dt,$$

where for convergence at the lower limit of integration, $\operatorname{Re}\mu > 0$.

Now, writing each of the generalized hypergeometric functions as a series sum, interchanging the resulting double sum and integral and then integrating term by term, we obtain

$$J(a,b,z) = z^{\mu}\sum_{m,n=0}^{\infty}\frac{(\alpha_p)_n}{(\beta_q)_n}\frac{(\xi_r)_m}{(\eta_s)_m}\frac{(az)^n}{n!}\frac{(bz^2/4)^m}{m!}\frac{1}{\mu+n+2m},$$

where, for convenience, $(a_i)_m \equiv (a_1)_m(a_2)_m\cdots(a_i)_m$. Now, since [**5**, p. 478],

$$\frac{1}{\mu+n+2m}=\frac{1}{\mu}\frac{(\mu/2)m}{((2+\mu)/2)m}\frac{(\mu+2m)_n}{(1+\mu+2m)_n},$$

it is easily seen that

$$J(a,b,z)=\frac{z^{\mu}}{\mu}\sum_{m=0}^{\infty}\frac{(\mu/2)_m}{((2+\mu)/2)_m}\frac{(\xi_r)_m}{(\eta_s)_m}\frac{(bz^2/4)^m}{m!} \cdot {}_{p+1}F_{q+1}\left[\begin{matrix}\mu+2m,(\alpha_p) & ; \\ & & az \\ 1+\mu+2m,(\beta_q) & ;\end{matrix}\right] \tag{2.1}$$

where $\operatorname{Re}\mu>0$.

Next, splitting ${}_{p+1}F_{q+1}$ into even and odd terms using MacRobert's identity [**13**, p. 200], we arrive at, after some computation, the following result for $\operatorname{Re}\mu>0$

$$\int_0^z t^{\mu-1}{}_pF_q[(\alpha_p);(\beta_q);at]_rF_s[(\xi_r);(\eta_s);bt^2/4]\,dt \tag{2.2}$$
$$=\frac{z^{\mu}}{\mu}F\begin{matrix}1:r;2p\\1:s;2q+1\end{matrix}\left[\begin{matrix}\frac{\mu}{2}: & (\xi_r); & (\frac{\alpha_p}{2}),(\frac{1+\alpha_p}{2}); \\ \frac{2+\mu}{2}: & (\eta_s); & \frac{1}{2},(\frac{\beta_q}{2}),(\frac{1+\beta_q}{2});\end{matrix}\ \frac{bz^2}{4},\frac{a^2z^2}{4^{q-p+1}}\right]$$
$$+\frac{az^{1+\mu}}{1+\mu}\frac{\alpha_1\cdots\alpha_p}{\beta_1\cdots\beta_q}$$
$$\cdot F\begin{matrix}1:r;2p\\1:s;2q+1\end{matrix}\left[\begin{matrix}\frac{1+\mu}{2}: & (\xi_r); & (\frac{1+\alpha_p}{2}),(\frac{2+\alpha_p}{2}); \\ \frac{3+\mu}{2}: & (\eta_s); & \frac{3}{2},(\frac{1+\beta_q}{2}),(\frac{2+\beta_q}{2});\end{matrix}\ \frac{bz^2}{4},\frac{a^2z^2}{4^{q-p+1}}\right]$$

where $\alpha_1\cdots\alpha_p=1$ if $p=0$. In equation (2.2) if we set $p=q=0$ and replace μ by $1+\mu$ we deduce the special case:

$$\int_0^z \exp(at)t^{\mu}{}_rF_s[(\xi_r);(\eta_s);bt^2/4]\,dt \tag{2.3}$$
$$=\frac{z^{1+\mu}}{1+\mu}F\begin{matrix}1:r;0\\1:s;1\end{matrix}\left[\begin{matrix}\frac{1+\mu}{2}: & (\xi_r); & -; \\ \frac{3+\mu}{2}: & (\eta_s); & \frac{1}{2};\end{matrix}\ \frac{bz^2}{4},\frac{a^2z^2}{4}\right]$$
$$+\frac{az^{2+\mu}}{2+\mu}F\begin{matrix}1:r;0\\1:s;1\end{matrix}\left[\begin{matrix}\frac{2+\mu}{2}: & (\xi_r); & -; \\ \frac{4+\mu}{2}: & (\eta_s); & \frac{3}{2};\end{matrix}\ \frac{bz^2}{4},\frac{a^2z^2}{4}\right]$$

where $\operatorname{Re}\mu>-1$.

However, if we replace μ by $\mu+1$ and set $p=q=0$ in equation (2.1) by using Kummer's first transformation

$$ {}_1F_1[a;c;z] = \exp(z)\,{}_1F_1[c-a;c;-z] $$

we arrive at

$$ \int_0^z \exp(at)t^{\mu}{}_rF_s[(\xi_r);(\eta_s);bt^2/4]\,dt \\ = \frac{z^{1+\mu}}{1+\mu}\exp(az)\sum_{m=0}^{\infty}\frac{\left(\frac{1+\mu}{2}\right)_m}{\left(\frac{3+\mu}{2}\right)_m}\frac{(\xi_r)_m}{(\eta_s)_m}\frac{(bz^2/4)^m}{m!} \\ \cdot {}_1F_1[1;2+\mu+2m;-az]. $$

Now, splitting ${}_1F_1$ into even and odd terms using MacRobert's identity, we deduce after some computation a second representation for this integral:

$$ (2.4)\quad \int_0^z \exp(at)t^{\mu}{}_rF_s[(\xi_r);(\eta_s);bt^2/4]\,dt \\ = \frac{z^{1+\mu}}{1+\mu}\exp(az) \\ \cdot\left\{F\begin{matrix}0:2+r;1\\2:s\;\;\;;0\end{matrix}\left[\begin{matrix}\text{——}: & \frac{1+\mu}{2},\frac{2+\mu}{2},(\xi_r); & 1; & \frac{bz^2}{4},\frac{a^2z^2}{4}\\ \frac{2+\mu}{3},\frac{3+\mu}{2}: & (\eta_s); & -; & \end{matrix}\right]\right. \\ \left.-\frac{az}{2+\mu}F\begin{matrix}0:2+r;1\\2:s\;\;\;;0\end{matrix}\left[\begin{matrix}\text{——}: & \frac{1+\mu}{2},\frac{2+\mu}{2},(\xi_r); & 1; & \frac{bz^2}{4},\frac{a^2z^2}{4}\\ \frac{3+\mu}{2},\frac{4+\mu}{2}: & (\eta_s); & -; & \end{matrix}\right]\right\} $$

where $\operatorname{Re}\mu > -1$.

3. Representations for $\mathbf{J}_{e_{\mu,\nu}}(a,z)$ and $\mathbf{E}_{e_{\mu,\nu}}(a,z)$. We define the Anger-Weber functions $\mathbf{C}_\nu(t)$, see, e.g., [**2**, p. 218], in the following concise manner:

$$ \mathbf{C}_\nu(t) \equiv \alpha_\nu t\,{}_1F_2\,[1;\quad (3-\nu)/2,\quad (3+\nu)/2;\quad -t^2/4] \\ +\beta_\nu\,{}_1F_2\,[1;\quad (2-\nu)/2,\quad (2+\nu)/2;\quad -t^2/4] $$

where

$$\alpha_\nu \equiv \begin{cases} \dfrac{1}{\pi}\dfrac{\sin\nu\pi}{1-\nu^2} & \text{if } \mathbf{C}_\nu(t)=\mathbf{J}_\nu(t) \\ \dfrac{-1}{\pi}\dfrac{1+\cos\nu\pi}{1-\nu^2} & \text{if } \mathbf{C}_\nu(t)=\mathbf{E}_\nu(t) \end{cases}$$

$$\beta_\nu \equiv \begin{cases} \dfrac{\sin\nu\pi}{\nu\pi} & \text{if } \mathbf{C}_\nu(t)=\mathbf{J}_\nu(t) \\ \dfrac{1-\cos\nu\pi}{\nu\pi} & \text{if } \mathbf{C}_\nu(t)=\mathbf{E}_\nu(t). \end{cases}$$

Thus, by using equations (2.3) and (2.4), respectively, we obtain for $\operatorname{Re}\mu > -1$,

$$\text{(3.1)} \quad \mathbf{C}_{e_{\mu,\nu}}(a,z)$$
$$= \alpha_\nu \left\{ \frac{z^{2+\mu}}{2+\mu} F\,\begin{matrix}1:1;0\\1:2;1\end{matrix}\left[\begin{matrix}\frac{2+\mu}{2}: & 1; & -; \\ \frac{4+\mu}{2}: & \frac{3-\nu}{2},\frac{3+\nu}{2}; & \frac{1}{2};\end{matrix}\ \frac{-z^2}{4},\frac{a^2z^2}{4}\right]\right.$$
$$\left. + \frac{az^{3+\mu}}{3+\mu} F\,\begin{matrix}1:1;0\\1:2;1\end{matrix}\left[\begin{matrix}\frac{3+\mu}{2}: & 1; & -; \\ \frac{5+\mu}{2}: & \frac{3-\nu}{2},\frac{3+\nu}{2}; & \frac{3}{2};\end{matrix}\ \frac{-z^2}{4},\frac{a^2z^2}{4}\right]\right\}$$
$$+ \beta_\nu \left\{ \frac{z^{1+\mu}}{1+\mu} F\,\begin{matrix}1:1;0\\1:2;1\end{matrix}\left[\begin{matrix}\frac{1+\mu}{2}: & 1; & -; \\ \frac{3+\mu}{2}: & \frac{2-\nu}{2},\frac{2+\nu}{2}; & \frac{1}{2};\end{matrix}\ \frac{-z^2}{4},\frac{a^2z^2}{4}\right]\right.$$
$$\left. + \frac{az^{2+\mu}}{2+\mu} F\,\begin{matrix}1:1;0\\1:2;1\end{matrix}\left[\begin{matrix}\frac{2+\mu}{2}: & 1; & -; \\ \frac{4+\mu}{2}: & \frac{2-\nu}{2},\frac{2+\nu}{2}; & \frac{3}{2};\end{matrix}\ \frac{-z^2}{4},\frac{a^2z^2}{4}\right]\right\}$$

and

$$\text{(3.2)} \quad \mathbf{C}_{e_{\mu,\nu}}(a,z) = \alpha_\nu \frac{z^{2+\mu}}{2+\mu}$$
$$\cdot \exp(az) \left\{ F\,\begin{matrix}0:3;1\\2:2;0\end{matrix}\left[\begin{matrix}\text{———}: & \frac{2+\mu}{2},\frac{3+\mu}{2},1; & 1; \\ \frac{3+\mu}{2},\frac{4+\mu}{2}: & \frac{3-\nu}{2},\frac{3+\nu}{2}; & -;\end{matrix}\ \frac{-z^2}{4},\frac{a^2z^2}{4}\right]\right.$$
$$\left. - \frac{az}{3+\mu} F\,\begin{matrix}0:3;1\\2:2;0\end{matrix}\left[\begin{matrix}\text{———}: & \frac{2+\mu}{2},\frac{3+\mu}{2},1; & 1; \\ \frac{4+\mu}{2},\frac{5+\mu}{2}: & \frac{3-\nu}{2},\frac{3+\nu}{2}; & -;\end{matrix}\ \frac{-z^2}{4},\frac{a^2z^2}{4}\right]\right\}$$
$$+ \beta_\nu \frac{z^{1+\mu}}{1+\mu}$$
$$\cdot \exp(az) \left\{ F\,\begin{matrix}0:3;1\\2:2;0\end{matrix}\left[\begin{matrix}\text{———}: & \frac{1+\mu}{2},\frac{2+\mu}{2},1; & 1; \\ \frac{2+\mu}{2},\frac{3+\mu}{2}: & \frac{2-\nu}{2},\frac{2+\nu}{2}; & -;\end{matrix}\ \frac{-z^2}{4},\frac{a^2z^2}{4}\right]\right.$$
$$\left. - \frac{az}{2+\mu} F\,\begin{matrix}0:3;1\\2:2;0\end{matrix}\left[\begin{matrix}\text{———}: & \frac{1+\mu}{2},\frac{2+\mu}{2},1; & 1; \\ \frac{3+\mu}{2},\frac{4+\mu}{2}: & \frac{2-\nu}{2},\frac{2+\nu}{2}; & -;\end{matrix}\ \frac{-z^2}{4},\frac{a^2z^2}{4}\right]\right\}.$$

4. Representations for $\mathbf{C}_{s_{\mu,\nu}}(a,z)$ and $\mathbf{C}_{c_{\mu,\nu}}(a,z)$. To compute representations for $\mathbf{C}_{s_{\mu,\nu}}(a,z)$ and $\mathbf{C}_{c_{\mu,\nu}}(a,z)$, observe that

$$\mathbf{C}_{s_{\mu,\nu}}(a,z) = \frac{1}{2i}[\mathbf{C}_{e_{\mu,\nu}}(ia,z) - \mathbf{C}_{e_{\mu,\nu}}(-ia,z)]$$
$$\mathbf{C}_{c_{\mu,\nu}}(a,z) = \frac{1}{2}[\mathbf{C}_{e_{\mu,\nu}}(ia,z) + \mathbf{C}_{e_{\mu,\nu}}(-ia,z)].$$

Thus, from equation (3.1), we have

$$\begin{aligned}(4.1)\quad \mathbf{C}_{s_{\mu,\nu}}(a,z) = {} & \alpha_\nu \frac{az^{3+\mu}}{3+\mu} \\ & \cdot F\,{}^{1:1;0}_{1:2;1}\left[\begin{matrix}\frac{3+\mu}{2}: & 1; & -; \\ \frac{5+\mu}{2}: & \frac{3-\nu}{2}, \frac{3+\nu}{2}; & \frac{3}{2};\end{matrix}\ \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right] \\ & + \beta_\nu \frac{az^{2+\mu}}{2+\mu} \\ & \cdot F\,{}^{1:1;0}_{1:2;1}\left[\begin{matrix}\frac{2+\mu}{2}: & 1; & -; \\ \frac{4+\mu}{2}: & \frac{2-\nu}{2}, \frac{2+\nu}{2}; & \frac{3}{2};\end{matrix}\ \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right], \\ & \operatorname{Re}\mu > -2\end{aligned}$$

$$\begin{aligned}(4.2)\quad \mathbf{C}_{c_{\mu,\nu}}(a,z) = {} & \alpha_\nu \frac{z^{2+\mu}}{2+\mu} \\ & \cdot F\,{}^{1:1;0}_{1:2;1}\left[\begin{matrix}\frac{2+\mu}{2}: & 1; & -; \\ \frac{4+\mu}{2}: & \frac{3-\nu}{2}, \frac{3+\nu}{2}; & \frac{1}{2};\end{matrix}\ \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right] \\ & + \beta_\nu \frac{z^{1+\mu}}{1+\mu} \\ & \cdot F\,{}^{1:1;0}_{1:2;1}\left[\begin{matrix}\frac{1+\mu}{2}: & 1; & -; \\ \frac{3+\mu}{2}: & \frac{2-\nu}{2}, \frac{2+\nu}{2}; & \frac{1}{2};\end{matrix}\ \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right], \\ & \operatorname{Re}\mu > -1,\end{aligned}$$

and from equation (3.2) we obtain,

$$\begin{aligned}(4.3)\quad \mathbf{C}_{s_{\mu,\nu}}(a,z) = {} & \sin(az)\left\{\alpha_\nu \frac{z^{2+\mu}}{2+\mu}\right. \\ & \cdot F\,{}^{0:3;1}_{2:2;0}\left[\begin{matrix}\text{———}: & \frac{2+\mu}{2}, \frac{3+\mu}{2}, 1; & 1; \\ \frac{3+\mu}{2}, \frac{4+\mu}{2}: & \frac{3-\nu}{2}, \frac{3+\nu}{2}; & -;\end{matrix}\ \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right]\end{aligned}$$

$$+ \beta_\nu \frac{z^{1+\mu}}{1+\mu}$$
$$\cdot F^{0:3;1}_{2:2;0}\left[\begin{matrix} \text{———}: & \frac{1+\mu}{2}, \frac{2+\mu}{2}, 1; & 1; \\ \frac{2+\mu}{2}, \frac{3+\mu}{2}: & \frac{2-\nu}{2}, \frac{2+\nu}{2}; & -; \end{matrix} \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right]\Bigg\}$$
$$- a\cos(az)\Bigg\{\alpha_\nu \frac{z^{3+\mu}}{(2+\mu)(3+\mu)}$$
$$\cdot F^{0:3;1}_{2:2;0}\left[\begin{matrix} \text{———}: & \frac{2+\mu}{2}, \frac{3+\mu}{2}, 1; & 1; \\ \frac{4+\mu}{2}, \frac{5+\mu}{2}: & \frac{3-\nu}{2}, \frac{3+\nu}{2}; & -; \end{matrix} \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right]$$
$$+ \beta_\nu \frac{z^{2+\mu}}{(1+\mu)(2+\mu)}$$
$$\cdot F^{0:3;1}_{2:2;0}\left[\begin{matrix} \text{———}: & \frac{1+\mu}{2}, \frac{2+\mu}{2}, 1; & 1; \\ \frac{3+\mu}{2}, \frac{4+\mu}{2}: & \frac{2-\nu}{2}, \frac{2+\nu}{2}; & -; \end{matrix} \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right]\Bigg\},$$
$$\operatorname{Re}\mu > -2$$

(4.4)
$$\mathbf{C}_{c_{\mu,\nu}}(a,z) = \cos(az)\Bigg\{\alpha_\nu \frac{z^{2+\mu}}{2+\mu}$$
$$\cdot F^{0:3;1}_{2:2;0}\left[\begin{matrix} \text{———}: & \frac{2+\mu}{2}, \frac{3+\mu}{2}, 1; & 1; \\ \frac{3+\mu}{2}, \frac{4+\mu}{2}: & \frac{3-\nu}{2}, \frac{3+\nu}{2}; & -; \end{matrix} \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right]$$
$$+ \beta_\nu \frac{z^{1+\mu}}{1+\mu}$$
$$\cdot F^{0:3;1}_{2:2;0}\left[\begin{matrix} \text{———}: & \frac{1+\mu}{2}, \frac{2+\mu}{2}, 1; & 1; \\ \frac{2+\mu}{2}, \frac{3+\mu}{2}: & \frac{2-\nu}{2}, \frac{2+\nu}{2}; & -; \end{matrix} \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right]\Bigg\}$$
$$+ a\sin(az)\Bigg\{\alpha_\nu \frac{z^{3+\mu}}{(2+\mu)(3+\mu)}$$
$$\cdot F^{0:3;1}_{2:2;0}\left[\begin{matrix} \text{———}: & \frac{2+\mu}{2}, \frac{3+\mu}{2}, 1; & 1; \\ \frac{4+\mu}{2}, \frac{5+\mu}{2}: & \frac{3-\nu}{2}, \frac{3+\nu}{2}; & -; \end{matrix} \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right]$$
$$+ \beta_\nu \frac{z^{2+\mu}}{(1+\mu)(2+\mu)}$$
$$\cdot F^{0:3;1}_{2:2;0}\left[\begin{matrix} \text{———}: & \frac{1+\mu}{2}, \frac{2+\mu}{2}, 1; & 1; \\ \frac{3+\mu}{2}, \frac{4+\mu}{2}: & \frac{2-\nu}{2}, \frac{2+\nu}{2}; & -; \end{matrix} \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right]\Bigg\},$$
$$\operatorname{Re}\mu > -1.$$

We note that equations (4.1) and (4.2) may be obtained directly in terms of $F^{1:1;0}_{1:2;1}$ by expressing the sine and cosine in terms of ${}_0F_1$ and then using [**1**, p. 173].

5. Reduction of the integrals. Many reductions to generalized hypergeometric functions are presently known for the Kampé de Fériet function $F^{0:2;1}_{2:1;0}[x,y]$, see, for example, [**10, 11**] and [**12**]. Since this particular hypergeometric function in two variables is especially important and occurs often in representations for incomplete integrals of cylindrical functions, it has been denoted for conciseness by

$$Q\left[\begin{matrix}\alpha,\beta,\gamma;\\ \mu,\nu,\delta;\end{matrix} x,y\right] \equiv F\begin{matrix}0:2;1\\2:1;0\end{matrix}\left[\begin{matrix}\text{——}: & \alpha,\beta; & \gamma;\\ \mu,\nu: & \delta; & -;\end{matrix} x,y\right]. \tag{5.1}$$

Since we know, for example, that

$$\begin{aligned}F\begin{matrix}p:2;1\\q:1;0\end{matrix}&\left[\begin{matrix}(a_p): & \alpha,1; & 1;\\ (b_q): & \beta; & -;\end{matrix} x,x\right]\\ &= \frac{1-\beta}{1+\alpha-\beta}\,{}_{p+1}F_q\left[\begin{matrix}(a_p),1;\\(b_q);\end{matrix} x\right]\\ &\quad+\frac{\alpha}{1+\alpha-\beta}\,{}_{p+2}F_{q+1}\left[\begin{matrix}(a_p),1,1+\alpha;\\(b_q),\beta;\end{matrix} x\right],\end{aligned}$$

we have immediately the reduction formula

$$\begin{aligned}Q\left[\begin{matrix}\alpha,1,1;\\ \mu,\nu,\delta;\end{matrix} x,x\right] &= \frac{1-\delta}{1+\alpha-\delta}\,{}_1F_2[1;\mu,\nu;x]\\ &\quad+\frac{\alpha}{1+\alpha-\delta}\,{}_2F_3[1,1+\alpha;\mu,\nu,\delta;x].\end{aligned}\tag{5.2}$$

Therefore, if we set $\nu=\pm\mu$ or $\nu=\pm(1-\mu)$ in equations (3.2), (4.3) and (4.4) we observe that the righthand members of these results may be expressed in terms of $Q[-z^2/4,-a^2z^2/4]$. Thus, when $a=\pm1$ we see that the integrals $\mathbf{C}_{e_{\mu,\pm\mu}}(\pm1,z)$, $\mathbf{C}_{s_{\mu,\pm\mu}}(\pm1,z)$, $\mathbf{C}_{c_{\mu,\pm\mu}}(\pm1,z)$ and $\mathbf{C}_{e_{\mu,\pm(1-\mu)}}(\pm1,z)$, $\mathbf{C}_{s_{\mu,\pm(1-\mu)}}(\pm1,z)$, $\mathbf{C}_{c_{\mu,\pm(1-\mu)}}(\pm1,z)$ reduce to generalized hypergeometric functions.

6. Reduction of special cases of $F^{1:1;0}_{1:2;1}$. We have seen in the previous section that $\mathbf{C}_{e_{\mu,\pm\mu}}(\pm1,z)$, $\mathbf{C}_{e_{\mu,\pm(1-\mu)}}(\pm1,z)$ and related integrals are reducible via the function Q to generalized hypergeometric functions. Since these integrals are given also in terms of $F^{1:1;0}_{1:2;1}$ by equations (3.1), (4.1) and (4.2), then the corresponding special cases of this Kampé de Fériet function should also be capable of reduction.

Indeed, if we equate the right members of equations (4.1), (4.3) and equations (4.2) and (4.4) and note that $\alpha_{-\nu} = -\alpha_\nu$, $\beta_{-\nu} = \beta_\nu$ if $\mathbf{C}_\nu(t) = \mathbf{J}_\nu(t)$ (or $\alpha_{-\nu} = \alpha_\nu$, $\beta_{-\nu} = -\beta_\nu$ if $\mathbf{C}_\nu(t) = \mathbf{E}_\nu(t)$), we obtain after some computation the following hypergeometric identities:

$$(6.1)\quad F^{1:1;0}_{1:2;1}\left[\begin{matrix}\frac{1+\mu}{2}: & 1; & -; \\ \frac{3+\mu}{2}: & \frac{2-\nu}{2}, \frac{2+\nu}{2}; & \frac{1}{2};\end{matrix}\ \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right]$$
$$= \cos(az) F^{0:3;1}_{2:2;0}\left[\begin{matrix}\text{———}: & \frac{1+\mu}{2}, \frac{2+\mu}{2}, 1; & 1; \\ \frac{2+\mu}{2}, \frac{3+\mu}{2}: & \frac{2-\nu}{2}, \frac{2+\nu}{2}; & -;\end{matrix}\ \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right]$$
$$+ \frac{az\sin(az)}{2+\mu}$$
$$\cdot F^{0:3;1}_{2:2;0}\left[\begin{matrix}\text{———}: & \frac{1+\mu}{2}, \frac{2+\mu}{2}, 1; & 1; \\ \frac{3+\mu}{2}, \frac{4+\mu}{2}: & \frac{2-\nu}{2}, \frac{2+\nu}{2}; & -;\end{matrix}\ \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right]$$

and

$$(6.2)\quad F^{1:1;0}_{1:2;1}\left[\begin{matrix}\frac{2+\mu}{2}: & 1; & -; \\ \frac{4+\mu}{2}: & \frac{2-\nu}{2}, \frac{2+\nu}{2}; & \frac{3}{2};\end{matrix}\ \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right]$$
$$= \frac{2+\mu}{1+\mu}\frac{\sin(az)}{az}$$
$$\cdot F^{0:3;1}_{2:2;0}\left[\begin{matrix}\text{———}: & \frac{1+\mu}{2}, \frac{2+\mu}{2}, 1; & 1; \\ \frac{2+\mu}{2}, \frac{3+\mu}{2}: & \frac{2-\nu}{2}, \frac{2+\nu}{2}; & -;\end{matrix}\ \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right]$$
$$- \frac{\cos(az)}{1+\mu} F^{0:3;1}_{2:2;0}\left[\begin{matrix}\text{———}: & \frac{1+\mu}{2}, \frac{2+\mu}{2}, 1; & 1; \\ \frac{3+\mu}{2}, \frac{4+\mu}{2}: & \frac{2-\nu}{2}, \frac{2+\nu}{2}; & -;\end{matrix}\ \frac{-z^2}{4}, \frac{-a^2z^2}{4}\right].$$

Next, if we set $a = \pm 1$, $z = 2ix$, $\nu = \pm\mu$, $\nu = \pm(1-\mu)$ in equations (6.1) and (6.2), respectively, and then note equations (5.1) and (5.2), we deduce the following reduction formulas:

$$(6.3)\quad F\,\begin{matrix}1:1;0\\1:2;1\end{matrix}\left[\begin{matrix}\frac{1+\mu}{2}: & 1; & -;\\ \frac{3+\mu}{2}: & \frac{2-\mu}{2},\frac{2+\mu}{2}; & \frac{1}{2};\end{matrix}\,x^2,x^2\right]$$

$$= \cosh(2x)\left\{\frac{\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{2+\mu}{2},\frac{3+\mu}{2};x^2\right] + \frac{1+\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{2+\mu}{2},\frac{2-\mu}{2};x^2\right]\right\}$$

$$- \frac{2x\sinh(2x)}{2+\mu}\left\{\frac{\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{3+\mu}{2},\frac{4+\mu}{2};x^2\right] + \frac{1+\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{4+\mu}{2},\frac{2-\mu}{2};x^2\right]\right\}$$

$$(6.4)\quad F\,\begin{matrix}1:1;0\\1:2;1\end{matrix}\left[\begin{matrix}\frac{1+\mu}{2}: & 1; & -;\\ \frac{3+\mu}{2}: & \frac{3-\mu}{2},\frac{1+\mu}{2}; & \frac{1}{2};\end{matrix}\,x^2,x^2\right]$$

$$= \cosh(2x)\left\{\frac{-1+\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{2+\mu}{2},\frac{3+\mu}{2};x^2\right] + \frac{2+\mu}{1+2\mu}\,{}_2F_3\left[1,\frac{4+\mu}{2};\frac{2+\mu}{2},\frac{3+\mu}{2},\frac{3-\mu}{2};x^2\right]\right\}$$

$$- \frac{2x\sinh(2x)}{2+\mu}\left\{\frac{-1+\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{3+\mu}{2},\frac{4+\mu}{2};x^2\right] + \frac{2+\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{3+\mu}{2},\frac{3-\mu}{2};x^2\right]\right\}$$

and

$$(6.5)\quad F\,\begin{matrix}1:1;0\\1:2;1\end{matrix}\left[\begin{matrix}\frac{2+\mu}{2}: & 1; & -;\\ \frac{4+\mu}{2}: & \frac{2-\mu}{2},\frac{2+\mu}{2}; & \frac{3}{2};\end{matrix}\,x^2,x^2\right]$$

$$= \frac{2+\mu}{1+\mu}\frac{\sinh(2x)}{2x}\left\{\frac{\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{2+\mu}{2},\frac{3+\mu}{2};x^2\right] + \frac{1+\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{2+\mu}{2},\frac{2-\mu}{2};x^2\right]\right\}$$

$$- \frac{\cosh(2x)}{1+\mu}\left\{\frac{\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{3+\mu}{2},\frac{4+\mu}{2};x^2\right] + \frac{1+\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{4+\mu}{2},\frac{2-\mu}{2};x^2\right]\right\}$$

$$
\begin{aligned}
(6.6)\quad & F\begin{matrix}1:1;0\\1:2;1\end{matrix}\left[\begin{matrix}\frac{2+\mu}{2}: & 1; & -;\\ \frac{4+\mu}{2}: & \frac{3-\mu}{2},\frac{1+\mu}{2}; & \frac{3}{2};\end{matrix}\, x^2, x^2\right]\\
&= \frac{2+\mu}{1+\mu}\frac{\sinh(2x)}{2x}\left\{\frac{-1+\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{2+\mu}{2},\frac{3+\mu}{2};x^2\right]\right.\\
&\qquad\left.+\frac{2+\mu}{1+2\mu}\,{}_2F_3\left[1,\frac{4+\mu}{2};\frac{2+\mu}{2},\frac{3+\mu}{2},\frac{3-\mu}{2};x^2\right]\right\}\\
&\quad -\frac{\cosh(2x)}{1+\mu}\left\{\frac{-1+\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{3+\mu}{2},\frac{4+\mu}{2};x^2\right]\right.\\
&\qquad\left.+\frac{2+\mu}{1+2\mu}\,{}_1F_2\left[1;\frac{3+\mu}{2},\frac{3-\mu}{2};x^2\right]\right\}
\end{aligned}
$$

We remark that the restriction $\mathrm{Re}\,\mu > -1$ (or $\mathrm{Re}\,\mu > -2$) may be waived in equations (6.1)–(6.6) by appealing to the principle of analytic continuation. We note also that other reduction formulas for special cases of $F\,{}^{1:1;0}_{1:2;1}[x,x]$ are given in [**12**].

Acknowledgment. The author thanks the referee for several helpful suggestions.

REFERENCES

1. H. Exton, *Handbook of hypergeometric integrals*, Wiley/Halsted, New York, 1978.

2. Y.L. Luke, *The special functions and their approximations*, Vol. I, Academic Press, San Diego, 1969.

3. A.R. Miller, *An incomplete Lipschitz-Hankel integral of* K_0, J. Math. Anal. Appl. **133** (1988), 195–208.

4. ———, *A Lipschitz-Hankel integral of* K_0, J. Franklin Institute **325** (1988), 787–795.

5. ———, *Incomplete Lipschitz-Hankel integrals of Bessel functions*, J. Math. Anal. Appl. **140** (1989), 476–484.

6. ———, *A note on incomplete integrals of cylindrical functions*, J. Franklin Institute **326** (1989), 571–577.

7. ———, *Incomplete Lipschitz-Hankel integrals of Macdonald functions*, J. Math. Anal. Appl. **145** (1990), 504–515.

8. ———, *Representations for incomplete Lipschitz-Hankel integrals of cylindrical functions*, J. Franklin Institute **328** (1991), 163–170.

9. ———, *Incomplete Lipschitz-Hankel integrals of Struve functions*, J. Franklin Institute **329** (1992), 563–581.

10. ———, *Reduction formulae for Kampé de Fériet functions* $F^{p:2;1}_{q:1;0}$, J. Math. Anal. Appl. **151** (1990), 428–437.

11. A.R. Miller and H.M. Srivastava, *Reduction formulas for Kampé de Fériet functions associated with certain classes of incomplete Lipschitz-Hankel type integrals of cylindrical functions*, J. Franklin Institute **329** (1992), 155–170.

12. ———, *Further reducible cases of certain Kampé de Fériet functions associated with incomplete integrals of cylindrical functions*, Appl. Math. Comp. **68** (1995), 199–216.

13. H.M. Srivastava and H.L. Manocha, *A treatise on generating functions*, Halsted, New York, 1984.

1616 Eighteenth St. NW, Washington, DC 20009-2530