ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 1, Spring 1998

FOURIER ANALYSIS ON COSET SPACES

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ABSTRACT. Let G be a locally compact group with a closed subgroup H. We will define and study natural analogs of the Fourier and Fourier-Stieltjes algebras for the homogeneous space G/H of left cosets of H in G. In particular, we show that when H is compact, the Fourier algebra $A(G/H)$ of G/H can be used to study the nature of G/H in a manner similar to that of the group case.

1. Introduction. Let G be a locally compact group. Let $B(G)$ be the Fourier-Stieltjes algebra of G as defined by P. Eymard in [**6**]. In a recent article, Bekka, Lau and Schlichting investigated the self-adjoint translation invariant subalgebras of $B(G)$ [**3**]. In particular, they have characterized the self-adjoint two-sided translation invariant subalgebras of the Fourier algebra $A(G)$ [**3**, Theorem 2.1]. They showed that these spaces could be identified as the functions in $A(G)$ which are constant on cosets of some compact normal subgroup K of G. It follows that such algebras are isometrically isomorphic with the Fourier algebras of the quotient group G/K. Moreover, each compact normal subgroup determines a different subalgebra. It is an immediate consequence of this result that the structure of the quotient group G/K is reflected in algebra $A(G)$.

The result of Lau, Bekka and Schlichting can be viewed as a refinement of some earlier work of Takesaki and Tatsuuma [**24**]. In fact, Takesaki and Tatsuuma considered the left invariant self-adjoint subalgebras of $A(G)$ and succeeded in establishing a one-to-one correspondence between such space and all compact subgroups K of G. In this case we are dealing with those functions which are constant on left cosets of K. However, when K is not normal, no link has been made between the nature of these subalgebras of $A(G)$, the structure of the homogeneous space G/K of left cosets of K and the structure of G itself. This is precisely the goal of this paper. We will give what we believe

Received by the editors on August 1, 1995.
1991 AMS *Mathematics Subject Classification.* 43A85, 43A07, 43A15.
Key words and phrases. Locally compact group, cosets, Fourier algebra, amenable group.

are natural definitions for the Fourier and Fourier-Stieltjes algebra of a coset space, $A(G/H)$ and $B(G/H)$, respectively. We will then show that, for a compact subgroup K, it is possible to extend a number of classical results to this new setting. In particular, we are able to show that $A(G/K)$ is a regular Banach algebra with maximal ideal space G/K and to characterize the amenable cosets space G/K in terms of various structural properties of $A(G/K)$. We will be able to apply this analysis to study the Fourier algebra of a totally disconnected group.

2. Preliminaries and notation. Let G be a locally compact group. Let $C^*(G)$ denote the group C^*-algebra, the enveloping C^*-algebra of $L_1(G)$. We let Σ_G denote the equivalence classes of weakly continuous unitary representations of G. If $\pi \in \Sigma_G$ and ξ, $\eta \in \mathcal{H}_\pi$, where $\mathcal{H}_\pi$ is the Hilbert space associated with π, then the continuous function $u(x) = \langle \pi(x)\xi, \eta \rangle$ is called a coefficient function of π. The dual of $C^*(G)$ can be identified with $B(G)$, the space of all coefficient functions of G. $B(G)$ is a commutative Banach algebra with respect to the dual norm and pointwise multiplication called the Fourier-Stieltjes algebra of G.

For $\pi \in \Sigma_G$, we let A_π denote the closed linear span of the coefficient functions of π, and let B_π be the weak-$*$ closure of A_π. In the case of the left regular representation λ_G on $L_2(G)$, A_{λ_G} is actually a closed ideal in $B(G)$. A_{λ_G} is usually denoted by $A(G)$ and is called the Fourier algebra of G. The maximal ideal space $\Delta(A(G))$ of $A(G)$ can be identified with G. The dual of $A(G)$ is denoted by $VN(G)$. $VN(G)$ is the von Neumann subalgebra of $\mathcal{B}(L_2(G))$ generated by $\{\lambda(x) \mid x \in G\}$. (See [**1**] and [**6**].)

Let H be a closed subgroup of G. By G/H, we will denote the homogeneous space of left cosets of H. We will write $\tilde{x}$ to denote the left coset xH as an element of G/H. Let $\varphi : G \to G/H$ be the canonical map. Given a continuous function $\tilde{u}$ on G/H, we can identify $\tilde{u}$ with the continuous function u on G defined by $u = \tilde{u} \circ \varphi$. This provides us with an isomorphism between $C(G/H)$ and $C(G : H)$, the subalgebra of $C(G)$ consisting of functions which are constant on left cosets of H in G.

Let $\mathcal{A}$ be a commutative Banach algebra. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. A derivation of $\mathcal{A}$ on $\mathcal{X}$ is a linear map $D : \mathcal{A} \to \mathcal{X}$ such that $D(uv) = uD(v) + D(u)v$ for every $u, v \in \mathcal{A}$. $\mathcal{A}$ is said to be weakly amenable if every continuous derivation from $\mathcal{A}$ into a commutative Banach $\mathcal{A}$-bimodule is identically zero.

We assume that $\mathcal{A}$ is a semisimple commutative Banach algebra and as such we will identify $\mathcal{A}$ with its Gelfand transform. Given a closed subset A of $\Delta(\mathcal{A})$, we define the ideals $I(A)$, $j(A)$ and $J(A)$ as follows:

$$\begin{aligned} I(A) &= \{u \in \mathcal{A} \mid u(x) = 0 \text{ for every } x \in A\} \\ j(A) &= \{u \in I(A) \mid \operatorname{supp} y \text{ is compact}\} \\ J(A) &= \text{the norm closure of } j(A) \text{ in } I(A). \end{aligned}$$

A is said to be a set of spectral synthesis if $I(A) = J(A)$. A is said to be a set of weak spectral synthesis if, for each $u \in I(A)$, there exists a positive integer n such that $u^n \in J(A)$. We say that (weak) spectral synthesis fails for $\mathcal{A}$ if there exists a closed subset A of $\Delta(\mathcal{A})$ which is not a set of (weak) spectral synthesis.

A multiplier of $\mathcal{A}$ is a linear operator T on $\mathcal{A}$ for which $T(uv) = uT(v)$. We denote the space of all such maps by $\mathcal{M}(\mathcal{A})$. $\mathcal{M}(\mathcal{A})$ is a Banach algebra with respect to the operator norm.

3. The spaces $A(G : H)$ and $B(G : H)$.

Definition. Let H be a closed subgroup of the locally compact group G. Let $B(G : H) = \{u \in B(G) \mid u(xh) = u(x)$ for every $x \in G$, $h \in H\}$. Let $A(G : H) = \{u \in B(G : H) \mid \varphi(\operatorname{supp} u)$ be compact in $G/H\}^{-\|\cdot\|_{B(G)}}$.

We will begin with two straightforward results:

Proposition 3.1. (i) *$B(G : H)$ and $A(G : H)$ are closed subalgebras of $B(G)$. Moreover, $A(G : H)$ is a closed ideal in $B(G : H)$.*

(ii) *$B(G : H)$ is unital.*

(iii) *$A(G : H) \cap A(G) \neq \{0\}$ if and only if H is compact.*

(iv) *$A(G : H) = B(G : H)$ if and only if G/H is compact.*

When H is compact and normal, it is well known that $A(G : H)$ is isometrically isomorphic to $A(G/H)$. The next result shows that the assumption of compactness is not necessary. It is essentially due to Eymard [**6**, 2.9 Corollaire].

Proposition 3.2. *Let H be a closed normal subgroup of G. Then $B(G:H)$ and $A(G:H)$ are isometrically isomorphic to $B(G/H)$ and $A(G/H)$, respectively.*

Recall that a function $f \in C(G)$ is said to be weakly almost periodic if $O_L(f) = \{{}_xf \mid x \in G\}$ is relatively weakly compact. It is well known that $B(G) \subset WAP(G)$, the C^*-algebra of all continuous weakly almost periodic functions on G. Moreover, it is also well known that $WAP(G)$ has a unique translation invariant mean Ψ which is the weak-$*$ limit of a net Ψ_α of finite means. Each Ψ_α is a convex combination $\sum_{i=1}^{n_\alpha} a_{\alpha_i}\delta_{x_{\alpha_i}}$ of point masses. It follows that, for each $f \in WAP(G)$, the net $\sum_{i=1}^{n_\alpha} a_{\alpha_i}\,{}_{x_{\alpha_i}}f$ of convex combinations of translates of f converges pointwise to $\Psi(f)1_G$, the unique constant function in the closed convex hull of $O_L(f)$.

Theorem 3.3. *Let H be a closed subgroup of G. Then there exists a projection $P : B(G) \to B(G:H)$ with $\|P\| \leq 1$.*

Proof. Let Ψ be the unique invariant mean on $WAP(H)$. Let $u \in B(G)$. Let $\{x_\beta\}_{\beta\in\Gamma}$ be a complete set of coset representatives of H in G. Define

$$u_{x_\beta}(h) = u(x_\beta h) \quad \text{for } h \in H.$$

Then $u_{x_\beta} \in B(H) \subset WAP(H)$. Let

$$P(u)(x) = \Psi(u_{x_\beta}) \quad \text{for all } x \in x_\beta H.$$

By the previous remark, $P(u)(x)$, viewed as a function on G, is in the pointwise closure of the convex hull of the right translates of u by elements of H. Since translation is an isometry on $B(G)$ and since the unit ball in $B(G_d)$ is closed in the topology of pointwise convergence, $P(u) \in B(G_d)$ and $\|P(u)\|_{B(G_d)} \leq \|u\|_{B(G)}$. Moreover, by definition, $P(u)(xh) = P(u)(x)$ for every $h \in H$. Furthermore, since $x \mapsto u_x$ is continuous in $B(G)$ and hence in $\| \quad \|_\infty$, $P(u)$ is a continuous function.

It follows that $P(u) \in B(G)$. Finally, that P is a linear projection is obvious. $\square$

We note that, in general, we cannot hope that P maps $A(G)$ onto $A(G:H)$. In fact, if H is noncompact, then $P(u) = 0$ whenever $u \in A(G)$. This is fortunately not the case for a compact subgroup K. In this case it is routine to show that

$$P(u)(x) = P_K(u)(x) = \int_K u(xk)\, dk.$$

We have

Corollary 3.4. *Let K be a compact subgroup of G. Then P_K is a continuous projection of $B(G)$ onto $B(G:K)$. The restriction of P_K to $A(G)$ is a projection of $A(G)$ onto $A(G:K)$.*

If we let π be a continuous unitary representation of G, then the projection P in Theorem 3.3 maps A_π onto $A_\pi \cap B(G:H)$ and B_π onto $B_\pi \cap B(G:H)$. To see this, simply observe that A_π and B_π are both translation invariant.

In [**24**], Takesaki and Tatsuuma showed that the spaces $A(G:K)$ where K is a compact subgroup are precisely the norm-closed left translation invariant $*$-subalgebras of $A(G)$. In particular, they show that $A(G:K_1) = A(G:K_2)$ if and only if $K_1 = K_2$. We give a short proof of this last statement. If G is a [SIN]-group, then the assumption that the subgroups be compact is not necessary.

Proposition 3.5. *Let K_1 and K_2 be compact subgroups of G. Then $A(G:K_1) = A(G:K_2)$ if and only if $K_1 = K_2$. If G is a* [SIN]*-group, H_1 and H_2 are closed subgroups of G and $H_1 \neq H_2$, then $A(G:H_1) \neq A(G:H_2)$.*

Proof. Assume that $x_0 \in K_1$ and $x_0 \notin K_2$. Then there exists an open set $\tilde{U} \subset G/K_2$ with $\tilde{e} \in \tilde{U}$ and $\tilde{x}_0 \notin \tilde{U}$. Let $U = \varphi_{K_2}^{-1}(\tilde{U})$. Then U is an open neighborhood of K_2 not containing x_0. We can find a $u \in A(G)$ such that $u(x) = 1$ if $x \in K_2$ and $u(x) = 0$ if $x \notin U$. Let

$u_1 = P_{K_2}u$. Then $u_1 \in A(G:K_2)$. But $u_1(e) = 1$ while $u_1(x_0) = 0$ so $u_1 \notin A(G:K_1)$.

Let G be a [SIN]-group. Assume that $x_0 \in H_1$ and $x_0 \notin H_2$. Let V be an open symmetric neighborhood of e in G with $\bar{V}$ compact and $\overline{V^2H_2} \cap x_0H_2 = \varnothing$. By modifying an argument of Cowling and Rodway [**4**] in a manner similar to the proof of [**10**, Proposition 3.10], we get an extension u of 1_{H_2} in $B(G)$ with support in $\overline{V^2H_2}$. Let $u_1 = P_{H_2}u$. Observe that $\operatorname{supp} u \subseteq \overline{V^2H_2}$. Hence, $u_1 \in A(G:H_2)$ and, as before, $u_1(e) = 1$ while $u_1(x_0) = 0$. □

Corollary 3.6. *Let K_1 and K_2 be compact subgroups of G. Then $B(G:K_1) = B(G:K_2)$ if and only if $K_1 = K_2$. If G is a* [SIN]*-group, then $B(G:H_1) = B(G:H_2)$ if and only if $H_1 = H_2$ for any closed subgroups H_1 and H_2.*

Proof. In either case, the function u_1 constructed above is in $B(G:H_2)$ but not in $B(G:H_1)$. □

It is reasonable to believe that the previous two results hold for non-[SIN]-groups as well. However, we do not know how to construct the function u separating H_2 and the coset x_0H_2 as above. We note that any such procedure would also be useful in studying the ideal structure of $A(G)$ for arbitrary groups. In particular, one should then be able to give a complete characterization of the ideals of $A(G)$ with bounded approximate identities when G is amenable, see [**10**] and [**11**].

The analog of the Fourier algebra for the coset space G/H is often considered to be the space A_{π_H}, where π_H is the quasi-regular representation of G determined by H, see [**1**] and [**20**], for instance. However, this space may have two serious deficiencies. First A_{π_H} is not in general an algebra. Indeed, Arsac [**1**] has shown that, when K is a compact subgroup, then A_{π_k} is an algebra if and only if $A_{\pi_K} = A_{\pi_{K_1}}$, where $K_1 = \cap_{x\in G} xKx^{-1}$. Since K_1 is normal, $A_{\pi_{K_1}} = A(G/K_1)$, [**1**, 4.12 Théorème]. It follows immediately from these results that, for K compact, $A(G:K) \neq A_{\pi_K}$ unless K is normal. More can be said.

Proposition 3.7. *Let K be a compact subgroup of G. Then $A(G:K) = A_\pi$ for some $\pi \in \Sigma_G$ if and only if K is normal.*

Proof. If K is normal, this is obvious. Conversely, if $A(G:K) = A_\pi$, then $A(G:K)$ is a norm closed translation invariant $*$-subalgebra of $A(G)$. By [**3**, Theorem 2.1], $A(G:K) = A(G:K_1)$ for some compact normal subgroup K_1. It follows from Proposition 3.5 that $K = K_1$. $\square$

The second major deficiency in using A_{π_H} for the Fourier algebra of G/H is that it is possible for two distinct closed subgroups H_1 and H_2 to be such that $A_{\pi_{H_1}} = A_{\pi_{H_2}}$ even when these subgroups are compact [**1**, 4.14 Exemple 1]. As a result, we feel that the algebras $A(G:H)$ and $B(G:H)$ are more useful analogs for G/H of the Fourier and Fourier-Stieltjes algebras. Therefore, we define $A(G/H)$, the Fourier algebra of the coset space G/H, to be the subalgebra of $C(G/H)$ identified with $A(G:H)$ and $B(G/H)$, the Fourier-Stieltjes algebra of G/H, to be the subalgebra of $C(G/H)$ identified with $B(G:H)$. In addition, we define the norm on $B(G/H)$ in the obvious way. We can show, particularly when H is compact, that $A(G/H)$ and $B(G/H)$ have many of the desirable properties of $A(G)$ and $B(G)$, respectively. We begin with some simple observations.

Let $AP(G/H)$, $WAP(G/H)$, denote the space of (weakly) almost periodic functions on G/H, see [**23**].

Proposition 3.8. *Let H be a closed subgroup of G. Then*

(i) *$B(G/H) \subseteq WAP(G/H)$ and $B(G/H) \cap AP(G/H)$ is the space identified with $B(G:H) \cap AP(G)$.*

(ii) *$B(G/H) \cap AP(G/H)$ is a complemented subalgebra of $B(G/H)$ with the Radon-Nikodym property.*

Proof. (i) This follows immediately from [**23**, Lemma 4.25].

(ii) Clearly $B(G/H) \cap AP(G/H)$ is an algebra. It is well known that $B(G) \cap AP(G)$ has the RNP [**17**] and that it is complemented in $B(G)$. In fact, $B(G) \cap AP(G)$ is of the form A_π where π is essentially the left regular representation of the almost periodic compactification of G. If P_π is the projection determined by π, then $P_\pi \circ P_H$ is the desired projection. $\square$

The significance of Proposition 3.8 can be seen in the following observation. Recall that, for any locally compact group G, $A(G)$ is sup-norm dense in $C_0(G)$. For a compact subgroup K, it is an immediate consequence of the regularity of $A(G)$ that $A(G/K)$ is again sup-norm dense in $C_0(G/K)$. However, M. Skantharajah has informed us that Ching Chou has shown that it is possible to have a noncompact subgroup H and a function $f \in C_0(G/H)$ for which $f \notin WAP(G/H)$. In particular, $A(G/H)$ need not be sup-norm dense in $C_0(G/H)$. Consequently, $A(G/H)$ may fail to separate points of G/H. For such groups, $A(G/H)$ will be of limited use in studying the nature of the coset space G/H. However, a careful examination of the proof of Theorem 3.5 yields the following:

Theorem 3.9. *Let G be a* [SIN]*-group with a closed subgroup H. Then $A(G/H)$ separates points in G/H.*

The next proposition extends an important result of Herz [**12**].

Proposition 3.10. *Let K be a compact subgroup of G. Let H be a closed subgroup of G such that $K \subseteq H$. Then every $\tilde{u} \in A(H/K)$ extends to a function $\tilde{u}_1 \in A(G/K)$ with $\|\tilde{u}\|_{A(H/K)} = \|\tilde{u}_1\|_{A(G/K)}$.*

Proof. Let $u \in A(H : K)$ be the function identified with $\tilde{u}$. By Herz's result, u extends to some $v \in A(G)$ of equal norm. Let $u_1 = P_K(v)$. Then since $\|P_K\| \leq 1$, $\tilde{u}_1$ is the desired extension. $\square$

It is not always possible to extend a $u \in B(H)$ to some $v \in B(G)$. However, this can be done whenever G is a [SIN]-group or if H is normal [**4**]. By modifying the above argument, we have

Proposition 3.11. *Let H Be a closed subgroup of G. Assume that either G is a* [SIN]*-group or that H is normal. Let H_1 be another closed subgroup containing H. Then every $u \in B(H_1/H)$ extends to a $v \in B(G/H)$ with the same norm.*

4. The structure of $A(G/K)$. Let K Be a compact subgroup of the locally compact group G. We assume that a measure $\mu_{G/K}$ has been chosen so that, for every $f \in L_1(G)$,

$$f(x) = \int_{G/H} \int_K f(xk)\, dk\, d\mu_{G/K}(\tilde{x}).$$

As in the previous section, we define a projection on $L_1(G)$ which we shall also denote by P_K by $P_K(f)(x) = \int_K f(xk)\, dk$. P_K maps $L_1(G)$ onto a closed subalgebra which is in general not self-adjoint. In fact, we have a "generic projection P_K" on $C^*(G)$, on $A(G)^* = VN(G)$ and on $B(G)^*$. We denote $P_k(C^*(G))$ by $C^*(G:K)$. $C^*(G:K)$ is simply the closure of $L_1(G:K)$ in $C^*(G)$ and is therefore a nonself-adjoint subalgebra of $C^*(G)$. Moreover, $C^*(G:K)^*$ is $B(G:K)$. We denote $P_K VN(G)$ by $VN(G:K)$. $VN(G:K)$ is the closure of $L_1(G:K)$ in $VN(G)$ with respect to the weak-$*$ topology. If $\Psi \in VN(G:K) = A(G:K)^*$, then the support (in the sense of Eymard) of Ψ consists of the union of cosets of K.

For H noncompact, we have seen that $A(G/H)$ need not separate points of G/H. However, if G is a compact group, then it is known that, for any closed subgroup K, that $\Delta A(G/K) = G/K$. We will show that the assumption that G be compact is not necessary.

Theorem 4.1. *$A(G/K)$ is a regular commutative Banach algebra with $\Delta A(G/K) = G/K$.*

Proof. Let $\tilde{F} \subseteq G/K$ be closed. Let $\tilde{x}_0 \in G/K \backslash \tilde{F}$. There exists an open set $\tilde{U}$ containing $\tilde{x}_0$ such that $\tilde{U} \cap \tilde{F} = \varnothing$. There exits $u \in A(G)$ such that $u(x) = 1$ for each $x \in x_0K$ and $u(x) = 0$ if $\varphi^{-1}(\tilde{U})$. Let $v = P_K(u)$. Then $\tilde{v}(\tilde{x}_0) = 1$ and $\tilde{v}(\tilde{x}) = 0$ for every $\tilde{x} \in \tilde{F}$.

Let $\tilde{x}_0 \in G/K$. Then $\delta_{\tilde{x}_0}(\tilde{u}) = \tilde{u}(\tilde{x}_0)$ is clearly a continuous multiplicative linear functional on $A(G/K)$. Conversely, assume that $\tilde{\Phi} \in \Delta(A(G/K))$. We can identify $\tilde{\Phi}$ with $\Phi \in A(G:K)^*$. Since $A(G:K)$ is complemented in $A(G)$ there exists $\Gamma \in VN(G)$ with $P_K^*(\Phi) = \Gamma$ and $\Gamma|_{A(G:K)} = \Phi$. Since $\Phi \neq 0$, $\Gamma \neq 0$. Moreover, the support of Γ is the union of K-cosets. We can then proceed as in the proof of [**4**, 3.34 Théorème] to show that $\operatorname{supp}\Gamma = x_0K$ for some $x_0 \in G$. The set x_0K is a set of spectral synthesis for $A(G)$ [**10**]. Therefore, Γ is the weak-$*$ limit of operators of the form

$\Psi = \sum_{i=1}^{n} a_i L_{x_i}$ where $x_i \in x_0 K$. However, the restriction of L_{x_i} to $A(G:K)$ is δ_{x_0}. It follows that $\Gamma|_{A(G:K)} = \Phi = a\delta_{x_0}$ for some $a \in \mathbf{C}$. Since $\Phi \neq 0$ and Φ is multiplicative, $a = 1$. Hence, $\tilde{\Phi} = \delta_{\tilde{x}_0}$. Finally, the mapping $\tilde{x}_0 \mapsto \delta_{\tilde{x}_0}$ is a homeomorphism of G/K onto $\Delta(A(G/K))$. □

We can now extend some of the structural properties of the Fourier algebra $A(G)$ too that of the coset space G/K.

Theorem 4.2. *Let G be a locally compact group with compact subgroup K. Then the following are equivalent*:

(i) *G is amenable.*

(ii) *G/K is an amenable coset space.*

(iii) *$A(G/K)$ has a bounded approximate identity consisting of functions with compact support in G/K.*

(iv) *$A(G/K)$ weakly factorizes.*

Proof. The equivalence of (i) and (ii) is well known, see [**6**, p. 16].

(i) $\Rightarrow$ (iii). Let $\tilde{F} \subseteq G/K$ be compact. Then $F = \varphi^{-1}(\tilde{F})$ is compact in G. Let $\varepsilon > 0$. Since G is amenable, a standard application of Reiter's property P_2 establishes the existence of a function $u_{\tilde{F},\varepsilon}(x) \in A(G)$ with compact support such that $u_{\tilde{F},\varepsilon}(x) = 1/(1+\varepsilon)$ for every $x \in F$ and $\|u_{\tilde{F},\varepsilon}\|_{A(G)} \leq 1$. Let $v_{\tilde{F},\varepsilon} = P(u_{\tilde{F},\varepsilon})$. Clearly, $V_{\tilde{F},\varepsilon}(x) = 1/(1+\varepsilon)$ for every $x \in F$. Moreover, $\operatorname{supp} v_{\tilde{F},\varepsilon}$ is compact and $\|v_{\tilde{F},\varepsilon}\|_{A(G)} \leq 1$. The elements of $A(G/K)$ with compact support being dense in $A(G/K)$, it is now a routine matter to verify that $\{\tilde{v}_{\tilde{F},\varepsilon}\}$ determines a bounded approximate identity for $A(G/K)$.

(iii) $\Rightarrow$ (iv). This follows immediately from Cohen's factorization theorem, [**12**, p. 268].

(iv) $\Rightarrow$ (i). Let $F \subseteq G$ be compact. Let $\tilde{F} = \varphi(F)$ and $F' = \varphi^{-1}(\tilde{F})$. Clearly, F' is compact. Assume that $A(G/K)$ weakly factorizes. Since $A(G:K)$ is a self-adjoint algebra, there exist $u_{F'} \in A(G:K)$ such that $u(x) \geq 1$ on F' and $\|u_{F'}\|_{A(G)} \leq M$ where M can be chosen independently of F [**8**]. The remainder of the proof is standard, but we shall include it for completeness.

Let $f \in C_{00}^{+}(G)$. Let $F = \operatorname{supp} f$ and $u_{F'}$ be chosen as above. Then

$$|\langle u_{F'}, f\rangle| \leq \|L_f\|_{Cv_2}\|u_{F'}\|_{A(G)} \leq M\|L_f\|_{Cv_2}.$$

But

$$\langle u_{F'}, f\rangle = \int_G u_{F'}(x)f(x)\,dx \geq \|f\|_1.$$

Therefore, $\|f\|_1 \leq M\|L_f\|_{Cv_2}$. It follows that $\|f\|_1^n \leq M\|L_f\|_{Cv_2}^n$ for every $f \in C_{00}^{+}(G)$ and, hence, that $\|f\|_1 = \|L_f\|_{Cv_2}$. This is well known to imply that G is amenable. □

Corollary 4.3. *Let G be an amenable locally compact group with a compact subgroup K. Then $\mathcal{M}(A(G/K)) = B(G/K)$ and the usual norms agree.*

Proof. By Theorem 4.2, $A(G/K)$ has a bounded approximate identity $\{\tilde{u}_\alpha\}$ where $\|u_\alpha\|_{A(G)} \leq 1$ for each α. If $\tilde{u} \in \mathcal{M}(A(G/K))$, then $\tilde{u}$ is a continuous function on G/K. Therefore, there exists $u \in C(G)$ such that $u = \tilde{u} \circ \varphi$ and $uv \in A(G:K)$ for every $v \in A(G:K)$. Now $uu_\alpha \in A(G:K)$ and $\|uu_\alpha\|_{B(G)} \leq \|\tilde{u}\|_{\mathcal{M}}\|u_\alpha\|_{A(G)} \leq \|\tilde{u}\|_{\mathcal{M}}$ for every α. Also $uu_\alpha(x)$ converges pointwise to $u(x)$. It follows that $u \in B(G:K)$ and that $\|u\|_{B(G)} \leq \|\tilde{u}\|_{\mathcal{M}}$. In particular, $\tilde{u} \in B(G/K)$ and $\|\tilde{u}\|_{B(G/K)} = \|\tilde{u}\|_{\mathcal{M}}$, since it is clear that $\|\tilde{u}\|_{B(G/K)} \geq \|\tilde{u}\|_{\mathcal{M}}$. □

Our next goal is to extend the automatic continuity result for derivations, the Fourier algebra of amenable groups given in [**9**]. We will need the following proposition.

Proposition 4.3. *Let K be a compact subgroup of G. Let $\tilde{E} \subset G/K$ be a set for which (weak) spectral synthesis fails in $A(G/K)$. Then (weak) spectral synthesis fails for $\varphi^{-1}(\tilde{E})$ in $A(G)$. In particular, if (weak) spectral synthesis fails for $A(G/K)$, then (weak) spectral synthesis fails for $A(G)$.*

Proof. Assume that $\tilde{E} \subset G/K$ is a set for which spectral synthesis fails in $A(G/K)$. Then there exists a $\tilde{v} \in I_{G/K}(\tilde{E})$ such that $\tilde{v} \notin J_{G/K}(\tilde{E})$. Let $v = \tilde{v} \circ \varphi$. Then $v \in I_G(A)$ where $A = \varphi^{-1}(\tilde{E})$.

Assume that $v \in J_G(A)$. Then there exists $\{v_n\}_{n=1}^{\infty} \subseteq j_G(A)$ such that $\lim_n \|v - v_n\|_{A(G)} = 0$. But then $\lim_n \|P(v - v_n)\|_{A(G)} = \lim_n \|v - P(v_n)\|_{A(G)} = 0$. However, $P(v_n) = \tilde{v}_n \circ \varphi$ for some $\tilde{v}_n \in A(G/K)$ and $\lim_n \|\tilde{v} - \tilde{v}_n\|_{A(G/K)} = \lim_n \|v - P(v_n)\|_{A(G)} = 0$. Observe that $\operatorname{supp} P(v_n) \subseteq (\operatorname{supp} v_n)K$. Hence $\operatorname{supp} \tilde{v}_n \subseteq \varphi(\operatorname{supp} v_n)$. It follows that $\tilde{v}_n \in j_{G/K}(\tilde{E})$. This is impossible since $\tilde{v} \notin J_{G/K}(\tilde{E})$.

A simple modification of the above argument shows that, if $\tilde{E} \subseteq G/K$ is not a weak spectral set for $A(G/K)$, then $\varphi^{-1}(\tilde{E})$ is not a weak spectral set for $A(G)$. □

We will give two corollaries of Proposition 4.3. The first corollary is of independent interest.

In [**19**], Malliavin showed that spectral synthesis fails for the Fourier algebra of any nondiscrete locally compact abelian group G. In a more recent article [**21**], Parthasarathy and Varma considered the failure of weak spectral synthesis in the Fourier algebra of G when G belongs to a class of infinite compact groups which contains all compact Lie groups. While there is every reason to believe that Malliavin's result extends to the noncommutative case, we do not know how to prove this. However, we have the following consequence of Proposition 4.3.

Corollary 4.4. *Let G be a locally compact group for which $A(G)$ admits* (*weak*) *spectral synthesis. Then G is totally disconnected.*

Proof. Let G_0 be the connected component of the identity e in G. If (weak) spectral synthesis fails for $A(G_0)$, then (weak) it is routine to verify that spectral synthesis fails for $A(G)$.

Assume that $A(G_0)$ admits (weak) spectral synthesis. If $G_0 \neq \{e\}$, then G_0 has a proper compact normal subgroup K such that G_0/K is a nontrivial connected Lie group. However, every nontrivial connected Lie group contains a closed nondiscrete abelian subgroup H. But (weak) spectral synthesis fails for $A(H)$, [**19**] and [**21**, Theorem 3.1], and thus for $A(G_0/K)$. We can now appeal to Proposition 4.3 to see that (weak) spectral synthesis fails for $A(G_0/K)$ and hence for $A(G_0)$. This contradiction shows that $G_0 = \{e\}$ and therefore that G is totally disconnected. □

If G is assumed to have additional structure, then this result may indeed extend Malliavin's theorem. For example, if G is either locally finite or locally solvable, then any infinite compact open subgroup would contain a nondiscrete abelian subgroup. This is clearly impossible if $A(G)$ satisfies (weak) synthesis.

Corollary 4.5. *Let G be a locally compact group with a compact subgroup K. Then each singleton $\{\hat{x}\} \subset G/K$ is a set of spectral synthesis for $A(G/K)$. Furthermore, if G is amenable, then every finite subset of G/K is a set of spectral synthesis.*

Proof. The first statement follows immediately from Lemma 4.3 and the fact that K (and hence every coset of K) is a set of spectral synthesis for $A(G)$. If G is amenable, then it can be shown that any set of the form $A = \cup_{k=1}^{n} x_k K$ is a set of spectral synthesis for $A(G)$, see [**11**, Theorem 3.11]. Hence every finite set in G/K is a set of spectral synthesis. □

Proposition 4.6. *Let G be an amenable locally compact group with compact subgroup K. Let $\{\hat{x}_1, \dots, \hat{x}_n\}$ be a finite subset of G/K. Then $I = I_{G/K}\{\hat{x}_1, \dots, \hat{x}_n\}$ has a bounded approximate identity $\{\hat{u}_\alpha\}$ in $A(G/K) \cap C_{00}(G/K)$.*

Proof. Since G is amenable, $A(G/K)$ has a bounded approximate identity in $C_{00}(G/K)$ by Theorem 4.2.

Let $\hat{F}$ be a compact subset of G/K with $x_1 \notin \hat{F}$. Then there exists $u \in B(G)$, the Fourier-Stieltjes algebra of G, such that $u(x) = 1$ if $x \in x_1 K$, $u(x) = 0$ on $\varphi^{-1}(\hat{F})$ and $\|u\|_{B(G)} = 1$. Let $v = P(u)$. Again, $v(x) = 1$ on $x_1 K$, $v(x) = 0$ on $\varphi^{-1}(\hat{F})$ and $\|v\|_{B(G)} = 1$. Since $\{\hat{x}_1\}$ is a set of spectral synthesis, by proceeding as in the proof of [**10**, Proposition 3.2], we can construct a bounded approximate identity for $I_{G/K}\{\hat{x}_1\}$ with the desired properties. Similarly, we get such a bounded approximate identity for each ideal $I_{G/K}\{\hat{x}_i\}$ for $i = 2, \dots, n$. That I also has a bounded approximate identity with the desired properties is standard. □

Corollary 4.7. *Let G be a locally compact group with compact subgroup K. Then the following are equivalent*:

(i) *G is amenable.*

(ii) *If I is a cofinite ideal of $A(G/K)$, then $I = I(\{x_1, \dots, x_n\})$ where $n = \operatorname{codim}(I)$.*

(iii) *Every cofinite ideal I in $A(G/K)$ has a bounded approximate identity.*

(iv) *Each homomorphism of $A(G/K)$ with finite dimensional range is continuous.*

Proof. Assume that G is amenable. It follows from Proposition 3 and Theorem 4 that every closed cofinite ideal of $A(G/K)$ has a bounded approximate identity. Then, from Cohen's factorization theorem, we have that each closed, cofinite ideal of $A(G/K)$ is idempotent. But, by [**5**, Theorem 2.3], (i) implies each of (ii), (iii) and (iv). Conversely, each of (ii), (iii) and (iv) imply that $A(G/K)$ weakly factorizes [**5**, Theorem 2.3]. Thus G is amenable by Theorem 4.2. $\square$

The following result is a generalization of [**9**, Theorem 1].

Theorem 4.8. *Let G be a locally compact group, and let K be a compact subgroup of G. Then the following are equivalent*:

(i) *G is amenable.*

(ii) *Every derivation from $A(G/K)$ into a Banach $A(G/K)$-bimodule is continuous.*

Proof. $A(G/K)$ is a Silov algebra. It follows from Corollary 4.5 that each closed primary ideal in $A(G/K)$ has codimension 1. Moreover, by Proposition 4.6, each maximal ideal has a bounded approximate identity. It follows from [**2**, Corollary 2.10] that every derivation from $A(G/K)$ into a Banach $A(G/K)$-bimodule is continuous.

If G is nonamenable, then $A(G/K)$ fails to factorize by Theorem 4.2. Therefore, since $A(G/K)^2$ is dense in $A(G/K)$, $A(G/K)^2$ is not closed in $A(G/K)$. Let Φ be a discontinuous linear functional on $A(G/K)$ with $\Phi(u) = 0$ for every $u \in A(G/K)^2$. Let X be a one-dimensional space, and let $u.x = x.u = 0$ for every $u \in A(G/K)$. Then $D : A(G/K) \to X$ defined by $D(u) = \Phi(u)x$ is a discontinuous derivation, see [**2**, Example 1]. $\square$

5. Weak amenability of $A(G)$. In [**9**], we showed that $A(G)$ is a weakly amenable Banach algebra whenever G is a discrete. Recently, B. Johnson has shown that, if G is the rotation group on $\mathbf{R}^3$, then $A(G)$ is not weakly amenable [**16**]. Johnson's surprising result leaves us to ask for which G is $A(G)$ weakly amenable? In fact, very little is known about this class of groups. As an application of the material of the previous sections, we will show that class for which $A(G)$ is weakly amenable contains all totally disconnected groups. For compact groups this follows from [**16**, Theorem 7.1].

Theorem 5.1. *Let H be an open subgroup of G. then $A(G/H)$ is weakly amenable.*

Proof. Let $D : A((G/H)) \to X$ be a continuous derivation into a commutative Banach $A(G/H)$-bimodule. Let $\tilde{u}$ be an idempotent in $A(G/H)$. Then $D(\tilde{u}) = D(\tilde{u}^n) = n(D(\tilde{u}))$ for $n \geq 2$. It follows that $D(\tilde{u}) = 0$. Since H is open, the linear span of the idempotents is dense in $A(G/H)$. Hence, D is identically zero and $A(G/H)$ is weakly amenable. $\square$

Lemma 5.2. *Let G be a totally disconnected locally compact group. Let $u \in A(G)$ and $\varepsilon > 0$. Then there exists an open compact subgroup K and a $v \in A(G : K)$ such that $\|u - v\|_{A(G)} < \varepsilon$.*

Proof. The map $x \mapsto {}_x u$ is continuous from G into $A(G)$. Therefore, there exists a neighborhood V of e such that, if $x \in V$, then $\|u - {}_x u\|_{A(G)} < \varepsilon$. Let K be an open compact subgroup contained in V. Let $v = P_K(u) = \int_K {}_k u \, dk$ be the vector-valued integral of translates of u. Then

$$\begin{aligned}\|u - v\|_{A(G)} &= \left\| \int_K (u - {}_k u)\, dk \right\|_{A(G)} \\ &\leq \int_K \|u - {}_k u\|_{A(G)}\, dk \leq \varepsilon. \qquad \square\end{aligned}$$

Theorem 5.3. *Let G be a totally disconnected locally compact group. Then $A(G)$ is weakly amenable.*

Proof. Let $D : A(G) \to X$ be a continuous derivation into a commutative Banach $A(G)$-bimodule. Then the restriction of D to $A(G : K)$, where K is a compact open subgroup of G, determines a derivation of $A(G/K)$ in the obvious way. By Theorem 5.1, D is zero on each $A(G : K)$. However, Lemma 5.2 shows that every $u \in A(G)$ can be approximated arbitrarily closely by some $v \in A(G : K)$ for some open compact subgroup K. It follows that $D = 0$. □

In essence, what we have shown is that, for a locally compact totally disconnected group G, the span of the idempotents in $A(G)$ is dense. In fact, this can easily be seen to characterize totally disconnected groups. The idempotents in $A(G)$ are characteristic functions of open compact subsets in the coset ring of G. Let $\mathcal{K} = \cap\{K \mid K$ is an open compact subgroup$\}$. If G is not totally disconnected, then $\mathcal{K} \neq \{e\}$. However, the idempotents in $A(G)$ are constant on $\mathcal{K}$. It follows that their span cannot be dense in $A(G)$.

Proposition 5.4. *Let G_1 and G_2 be such that $A(G_i)$ is weakly amenable for $i = 1, 2$. Then $A(G_1 \times G_2)$ is also weakly amenable.*

Proof. Since $A(G_1)$ and $A(G_2)$ are both weakly amenable, so is the projective tensor product $A(G_1)\hat{\oplus}A(G_2)$ [**12**]. The map $u \otimes v \to w$, where $w(g_1, g_2) = u(g_1)v(g_2)$ extends to a continuous homomorphism from $A(G_1)\hat{\oplus}A(G_2)$ onto a dense subalgebra of $A(G_1 \times G_2)$. It follows that $A(G_1 \times G_2)$ is also weakly amenable. □

Corollary 5.5. *Let $G = G_1 \times G_2$ where G_1 is abelian and G_2 is totally disconnected, then $A(G)$ is weakly amenable.*

Proof. It is well known that, if G is abelian, then $A(G)$ is in fact amenable. The result now follows from Theorem 5.3 and Proposition 5.4. □

Note added in proof. We wish to thank Professor A. Derighetti for making us aware of [**17**] where Noël Lohoué obtains our result, Corollary 4.4, in the case of spectral synthesis.

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