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SOME PROPERTIES OF NONLINEAR ADJOINT OPERATORS

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0. Introduction. In many branches of modern science and engineering nonlinear models more often are used. Nonlinear boundary value problems are studied to describe more precisely phenomena, for example, in the theory of plasticity, hydrodynamics, diffusion processes, biology, etc. In functional analysis this trend evokes the development of the nonlinear operator theory.

The present article is a slight contribution to this theory. Unlike the authors in [**1, 12–15, 17, 20, 21**], we consider a rather special class of operators from a Banach space into its dual involving nonlinearities of the power type. These operators, called polynomial and homogeneous operators, have some properties similar to linear operators. For example, for polynomial operators the continuity and the boundedness are equivalent. We generalize in a natural way some important notions known from linear analysis as the spectrum, numerical range, symmetry, self-adjointness and the normality. We show a number of their properties which can be useful for studying nonlinear operator equations, eigenvalue problems and other questions from nonlinear functional analysis and its applications.

1. Notations and definitions. Throughout this paper, let X, Y denote abstract (real or complex) Banach spaces and X^*, Y^* their dual spaces. By the symbol $\langle x^*, x\rangle$ we denote the value of a continuous linear functional $x^* \in X^*$ at a point $x \in X$. In case of Hilbert space X we use the same symbol for the inner product.

For the norm or weak convergence of the sequence $\{x_n\} \subset X$ to a point $x_0 \in X$ we use the symbols $x_n \to x_0$ or $x_n \stackrel{w}{\to} x_0$, respectively. Let $\mathbf{R}$ and $\mathbf{C}$ be the spaces of real and complex numbers, respectively. Further, we denote $S_1(0) = \{x \in X : \|x\| = 1\}$ the unit sphere in X.

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Definition 1.1. We shall say that an operator $F : X \to Y$ is a

(a) *positively homogeneous operator of the degree* k if there is a number $k \in \mathbf{R}$ such that

$$F(tx) = t^k.F(x)$$

for any $x \in X$ and any $t \in \mathbf{R}$, $t > 0$.

(b) *homogeneous operator of the degree* k on a real (respectively complex) space X if k is an integer and the equality

$$F(tx) = t^k.F(x)$$

holds for any $t \in \mathbf{R}$, respectively $t \in \mathbf{C}$, $t \neq 0$ and any $x \in X$.

(c) For any continuous positively homogeneous operator $F : X \to Y$ of the degree $k > 0$ we define the norm by

$$\|F\| = \sup_{x \in S_1(0)} \|F(x)\|.$$

Remark 1.2. It is easy to show that, for any continuous positively homogeneous operator $F : X \to Y$ and for arbitrary $x \in X$, the following estimation $\|F(x)\| \leq \|F\|.\|x\|^k$ holds.

Definition 1.3. We shall say that a positive homogeneous operator $F : X \to X^*$ is

a) *positive* if, for any $x \in X$, $x \neq 0$, it holds $\langle Fx, x\rangle > 0$,

b) *positively defined* if there is a number $c \in \mathbf{R}$, $c > 0$ such that $\inf \mathrm{Re}_{x \in S_1(0)} \langle Fx, x\rangle = c > 0$.

Definition 1.4. An operator $F : X \to Y$ is called *hemi-continuous* at $x_0 \in X$ if, for any sequence $\{t_n\} \subset \mathbf{R}$, $t_n \to 0$, and for any $h \in X$, it holds $(F(x_0 + t_n h) \overset{w}{\to} F(x_0)$.

We shall say that Gateaux derivative F' of a differentiable operator $F : X \to X^*$ is hemi-continuous at a point $x \in X$ if, for any sequence $\{t_n\} \subset [0, +\infty)$ such that $t_n \to 0$ and for arbitrary points $h \in X$, $y \in X$, the sequence $\{F'(x+t_nh)y\} \subset X^*$ converges in the weak*-topology of the space X^* to the point $F'(x)y \in X^*$.

Definition 1.5. We shall say that a homogeneous operator $P : X \to Y$ of the degree $k \geq 1$ is a *homogeneous polynomial operator* if there is a k-linear symmetric operator $\mathcal{P} : X \times X \times \cdots \times X \to X$, i.e., $\mathcal{P}(x_1, x_2, \ldots, x_k)$ is linear in any variable x_j, $j = 1, 2, \ldots, k$, and it does not change its values under arbitrary permutation of all variables) such that $\mathcal{P}(x, x, \ldots, x) = P(x)$ for any $x \in X$. Operator $\mathcal{P}$ is called the *polar operator* to operator P.

For any continuous k-linear operator $\mathcal{P}$ we define the norm by

$$\|\mathcal{P}\| = \sup_{x_1 \in S_1(0), i=1,2,\ldots,k.} \|\mathcal{P}(x_1, x_2, \ldots, x_k)\|$$

Definition 1.6 [**8**, Definition 3]. Let $D \subset X$ be an open set which is star-shaped with respect to the origin, i.e., for any $x \in D$ and all $t \in \langle 0, 1\rangle$, $tx \in D$ holds.

Let the operator $F : D \subset X \to X^*$ have *Gateaux-derivative* $F'(x)$ at any point $x \in D$, and let F satisfy the following conditions:

(1) $F(0) = 0$,

(2) The function $\langle F'(tx)h, x\rangle$ of the variable $t \in [0, 1]$ is integrable for arbitrary (but fixed) points $x \in D$, $h \in X$.

Let us suppose, further, that for any $x \in D$ there exists a unique point $x^*(x) \in X^*$ such that, for all $h \in X$, the following holds

$$\langle x^*(x), h\rangle = \int_0^1 \langle F'(tx)h, x\rangle\, dt.$$

Then the operator $F^* : D \subset X \to X^*$ defined for $x \in D$ by $F^*(x) = x^*(x)$ is called the *adjoint operator* to the operator F.

Remark 1.7. In the case of a real Banach space X, the adjoint operator F^* from Definition 1.6 can be written in the form

$$F^*(x) = \int_0^1 [F'(tx)]^*(x)\, dt,$$

where $[F'(tx)]^*$ denotes the adjoint operator to the continuous linear operator $F'(tx)$.

According to [**9**, Theorem 2.6], there exists the adjoint operator F^* to a nonlinear operator F if it satisfies the conditions (1) and (2) from Definition 1.6 and, moreover, F has at any point $x \in X$ a hemi-continuous Gateaux-derivative. Then F and F^* are both hemi-continuous and the following estimation holds

$$\|F^*(x)\| \le \|x\| \int_0^1 \|F'(tx)\| \, dt, \quad x \in X.$$

Theorem 1.8. *Let X be a Banach space, and let $F : X \to X^*$ satisfy the conditions* (1) *and* (2) *from Definition* 1.6. *Suppose, further, that F has on X a hemi-continuous Gateaux-derivative F'. Then there are operators $R, H : X \to X^*$ with the following properties*:

(i) $F(x) = H(x) + R(x)$ *for any $x \in X$.*

(ii) *Operator H is a potential operator, $H = \operatorname{grad} \varphi$, where the functional $\varphi : X \to \mathbf{R}$ is defined by*

$$\varphi(x) = \int_0^1 \langle F(tx), x \rangle \, dt \quad \text{for any } x \in X.$$

(iii) *The operator $R : X \to X^*$ is defined by*

$$R(x) = \int_0^1 \{F'(tx)(tx) - [F'(tx)]^*(tx)\} \, dt$$

for any $x \in X$.

Proof. It follows from [**8**, Theorem 2] and [**9**, Theorem 2.13]. □

Corollary 1.9. *Let X be a real Banach space. Let the operator $F : X \to X^*$ satisfy all assumptions of Theorem* 1.8. *Then, for any $x \in X$, the following hold.*

(1) $\langle F(x), x \rangle = \langle F^*(x), x \rangle = \langle H(x), x \rangle$. (*Here H is the operator defined in Theorem* 1.8.)

(2) *The operator F is positive (positively defined, coercive) if and only if the adjoint operator F^* is positive (positively defined, coercive).*

(3) *F is a potential operator if and only if $F = F^*$.*

Proof. Let $x \in X$ be an arbitrary point. According to Remark 1.7, we have

$$\begin{aligned}\langle F^*(x), x\rangle &= \Big\langle \int_0^1 [F'(tx)]^*(x)\,dt, x\Big\rangle \\ &= \int_0^1 \langle [F'(tx)]^*(x), x\rangle\,dt \\ &= \int_0^1 \langle x, F'(tx)(x)\rangle\,dt = \Big\langle x, \int_0^1 [F(tx)]'\,dt\Big\rangle \\ &= \langle x, F(x)\rangle = \langle F(x), x\rangle.\end{aligned}$$

Applying the assertion (i) of Theorem 1.8, we obtain

$$\langle F(x), x\rangle = \langle H(x), x\rangle + \langle R(x), x\rangle.$$

Now, due to the assertion (iii) of Theorem 1.8, the following holds

$$\begin{aligned}\langle R(x), x\rangle &= \Big\langle \int_0^1 F'(tx)(tx)\,dt, x\Big\rangle - \Big\langle \int_0^1 [F'(tx)]^*(tx)\,dt, x\Big\rangle \\ &= \int_0^1 \{F'(tx)(x), tx\rangle - \langle tx, F'(tx)(x)\rangle\}\,dt = 0,\end{aligned}$$

so that $\langle F(x), x\rangle = \langle H(x), x\rangle = \langle F^*(x), x\rangle$ and the assertion (1) is proven. □

The assertion (2) is a direct consequence of the assertion (1) and Definition 1.3. The assertion (3) follows from [**8**, Theorem 5].

Applying the above results to the case of a homogeneous operator, we obtain the following proposition, see [**8, 9**].

Proposition 1.10. *Let $F : X \to X^*$ be a homogeneous operator of the degree $k \geq 1$ having a hemi-continuous Gateaux-derivative F'. Then, for any $x \in X$, the following assertions hold*

(1) $$F^*(x) = \frac{1}{k}[F'(x)]^* x,$$

where F^* is the adjoint operator to the operator F and $[F'(x)]^*$ is the adjoint operator to the continuous linear operator $F'(x)$.

$$F(x) = H(x) + R(x). \tag{2}$$

Here $H, R : X \to X^*$ are hemi-continuous operators which can be written as

$$H(x) = \frac{1}{k+1}[F(x) + kF^*(x)],$$
$$R(x) = \frac{k}{k+1}[F(x) - F^*(x)].$$

(3) *H is a potential operator, $H = \operatorname{grad}\varphi$, where*

$$\varphi(x) = \frac{1}{k+1}\langle F(x), x\rangle$$

and the operator R fulfills the equality

$$\langle R(x), x\rangle = \frac{2ki}{k+1}\operatorname{Im}\{\langle F(x), x\rangle\}.$$

$$\|F^*(x)\| \leq \frac{1}{k}\|F'(x)\|.\|x\|. \tag{4}$$

(5) *If $S, T : X \to X^*$ are homogeneous operators with their adjoint operators S^*, T^*, then for any $\lambda \in \mathbf{C}$ the following holds*

$$(S - \lambda T)^* = S^* - \bar{\lambda}T^*.$$

Proof. It is obvious that the Gateaux derivative F' of a homogeneous operator F of the degree $k \geq 1$ is also a homogeneous operator of the degree $k-1$ because, for any $x, h \in X$ and $t \in \mathbf{C}$, $t \neq 0$, the following holds

$$\begin{aligned} F'(tx)h &= \lim_{\tau\to 0}\frac{F(tx+\tau x) - F(tx)}{\tau} \\ &= \lim_{\tau/t\to 0} t^{k-1}\frac{F(x+(\tau/t)h) - F(x)}{\tau/t} \\ &= t^{k-1}.F'(x)h. \end{aligned}$$

Now, using Remark 1.7 and Definition 1.1, we obtain

$$\begin{aligned} F^*(x) &= \int_0^1 [F'(tx)]^*(x)\,dt \\ &= \int_0^1 t^{k-1}[F'(x)]^*(x)\,dt \\ &= \frac{1}{k}[F'(x)]^*(x) \end{aligned}$$

and the assertion (1) is proven.

The assertion (2) follows easily from Theorem 1.8.

Indeed, according to (ii) from Theorem 1.8, we have

$$\begin{aligned} \varphi(x) &= \int_0^1 \langle F(tx), x\rangle\,dt \\ &= \int_0^1 \langle t^k F(x), x\rangle\,dt \\ &= \frac{1}{k+1}\langle F(x), x\rangle, \end{aligned}$$

and (iii) implies that

$$\begin{aligned} R(x) &= \int_0^1 F'(tx)(tx)\,dt - \int_0^1 [F'(tx)]^*(tx)\,dt \\ &= \int_0^1 t^k F'(x)(x)\,dt - \int_0^1 t^k [F'(x)]^*(x)\,dt \\ &= \frac{1}{k+1}F'(x)(x) - \frac{1}{k+1}[F'(x)]^*(x). \end{aligned}$$

Due to the definition of Gateaux derivatives, the following equation holds

$$\begin{aligned} F'(x)(x) &= \lim_{t\to 0}\frac{F(x+tx)-F(x)}{t} \\ &= \lim_{t\to 0}\frac{(1+t)^k-1}{t}\cdot F(x) \\ &= k.F(x). \end{aligned}$$

Using this and the assertion (1), we obtain

$$\begin{aligned} R(x) &= \frac{k}{k+1}F(x) - \frac{k}{k+1}F^*(x) \\ &= \frac{k}{k+1}[F(x) - F^*(x)]. \end{aligned}$$

Further, for any $h \in X$, the following holds

$$\begin{aligned} \langle H(x), h\rangle &= \langle \operatorname{grad}\varphi(x), h\rangle \\ &= \frac{1}{k+1}[\langle F'(x)h, x\rangle + \langle F(x), h\rangle] \\ &= \frac{1}{k+1}\{\langle [F'(x)]^* x, h\rangle + \langle F(x), h\rangle\} \\ &= \frac{1}{k+1}\langle kF^*(x) + F(x), h\rangle. \end{aligned}$$

Hence,

$$H(x) = \frac{1}{k+1}[k.F^*(x) + F(x)]$$

and using (i) of Theorem 1.8 and the assertion (2) we obtain (3).

The assertion (4) follows immediately from Remark 1.7 and (5) is a consequence of the definition of the adjoint operator. □

Proposition 1.11. *Let $F : X \to X^*$ be a homogeneous operator of the degree $k \geq 1$ having a continuous Gateaux-derivative F'. Then the adjoint operator F^* is also homogeneous of the degree k and for the norms the following estimation holds:*

$$\max(\|F\|, \|F^*\|) \leq \frac{1}{k}\|F'\|.$$

Proof. According to Proposition 1.10 (1), the operator F^* is homogeneous of the degree k. Further, for a differentiable homogeneous operator F, $F(x) = (1/k)F'(x)(x)$ holds for any $x \in X$, see the proof of Proposition 1.10.

Using this and Definition 1.1 (c) and the relation

$$\frac{1}{k}F'(x)(x) = F(x),$$

see proof of Proposition 1.10, we obtain

$$\begin{aligned}\|F\| &= \sup_{x\in S_1(0)} \|F(x)\| \\ &= \frac{1}{k} \sup_{x\in S_1(0)} \|F'(x)(x)\| \\ &\le \frac{1}{k} \sup_{x\in S_1(0)} (\|F'(x)\|.\|x\|) \\ &\le \frac{1}{k}\|F'\|.\end{aligned}$$

Now, according to Theorem 1.10 (4), we have

$$\begin{aligned}\|F^*\| &= \sup_{x\in S_1(0)} \|F^*(x)\| \\ &\le \frac{1}{k} \sup_{x\in S_1(0)} \|F'(x)\|.\|x\| \\ &\le \frac{1}{k}\|F'\|.\end{aligned}$$

Hence, $\max(\|F\|, \|F^*\|) \le (1/k)\|F'\|$. □

Unlike the linear case, the equality $\|F\| = \|F^*\|$ does not hold generally as the following example shows.

Example 1.12. Let E_2 be a two-dimensional Euclidean space. Define for any $x = (\xi_1, \xi_2) \in E_2$ a homogeneous operator $F : E_2 \to E_2$ of the degree 3 by

$$F(x) = (f_1, f_2) = (\xi_2^3, \xi_1^3).$$

Then the linear operators $F'(x)$ and $[F'(x)]^*$ can be written as matrices

$$\begin{aligned}F'(x) &= \begin{pmatrix} 0 & 3\xi_2^2 \\ 3\xi_1^2 & 0 \end{pmatrix}, \\ [F'(x)]^* &= \begin{pmatrix} 0 & 3\xi_1^2 \\ 3\xi_2^2 & 0 \end{pmatrix}.\end{aligned}$$

According to Remark 1.7, we have

$$\begin{aligned} F^*(x) &= \int_0^1 [F'(tx)]^* x \, dt \\ &= \int_0^1 3t^2 (\xi_1^2\xi_2, \xi_1\xi_2^2) \, dt \\ &= (\xi_1^2\xi_2, \xi_1\xi_2^2). \end{aligned}$$

Using differential calculus, we obtain the norms of F and F^*:

$$\begin{aligned} \|F\| &= \sup_{x\in S_1(0)} \|F(x)\| \\ &= \max_{x\in S_1(0)} \sqrt{\xi_1^6 + \xi_2^6} = 1 \\ \|F^*\| &= \sup_{x\in S_1(0)} \|F^*(x)\| \\ &= \max_{x\in S_1(0)} \sqrt{\xi_1^4\xi_2^2 + \xi_2^4\xi_1^2} \\ &= \frac{1}{2}. \end{aligned}$$

It shows that $\|F\| \neq \|F^*\|$.

2. Numerical range and spectrum of the adjoint operator. In this section some properties of the numerical range and spectrum of a couple of adjoint homogeneous operators are studied. For this reason we recall the necessary definitions which were introduced in [**7, 11**].

Definition 2.1. Let $S, T : X \to X^*$ be positively homogeneous operators of the degree k. A *numerical range* $W(S,T)$ *of the couple* (S,T) is defined as the set of complex numbers

$$W(S,T) = \left\{ \frac{\langle Sx, x\rangle}{\langle Tx, x :\rangle} : x \in S_1(0), \langle Tx, x\rangle \neq 0 \right\}.$$

It is evident that if X is a Hilbert space, S is linear continuous and T is the identity operator, then we obtain Hausdorff's and Toeplitz definition of the numerical range.

Proposition 2.2 [**11**, Proposition 3.14]. *Let $S, T : X \to X^*$ be positively homogeneous operators. Let S be continuous and T be positively defined. Then the following assertions hold*:

(1) *$W(S,T)$ is a bounded set, and for any $\lambda \in W(S,T)$, the following holds*

$$|\lambda| \leq \frac{\|S\|}{c}, \quad \text{where } c = \inf_{x \in S_1(0)} \mathrm{Re}\,\{\langle Tx, x\rangle\}.$$

(2) *If, in addition, S and T are polynomial operators, then $W(S,T)$ is a convex set, a generalization of Hausdorff and Toeplitz theorems on the convexity of the numerical range.*

Definition 2.3. Let $S, T : X \to Y$ be positively homogeneous operators. By the *approximative spectrum* (briefly spectrum) of the couple (S,T), we understand the set $\sigma(S,T)$ of complex numbers defined as follows

$$\sigma(S,T) = \{\lambda \in \mathbf{C} : \inf_{x \in S_1(0)} \|Sx - \lambda Tx\| = 0\}.$$

Definition 2.4. We shall say that $\lambda_0 \in \mathbf{C}$ is the *eigenvalue of a couple* (S,T) of positively homogeneous operators $S, T : X \to Y$ if there is a point $x_0 \in S_1(0) \subset X$ such that $Sx_0 - \lambda_0 Tx_0 = 0$. The point x_0 is called the *eigenvector of the couple* (S,T) related to λ_0.

The set of all eigenvalues of the couple (S,T) we denote by $\Lambda(S,T)$.

Relationship between numerical ranges and between sets of eigenvalues of couples of operators and their adjoint shows the following theorem.

Theorem 2.5. *Let $S, T : X \to X^*$ be homogeneous operators with their adjoint operators S^*, T^*, and let T be positive. Then the following assertions hold*:

(1) *If $\lambda \in W(S,T)$, then $\bar{\lambda} \in W(S^*,T^*)$.*

(2) *If $\lambda \in \Lambda(S,T)$, then $\bar{\lambda} \in \Lambda(S^*,T^*)$.*

Proof. According to Corollary 1.9 (2) the operator T^* is positive and thus the numerical range $W(S^*,T^*)$ is not empty. The positivity of T implies that $\operatorname{Im}\langle Tx,x\rangle = 0$ for any $x \in X$ so that, due to Corollary 1.9 (1), $\langle T^*x,x\rangle = \langle Tx,x\rangle$ holds. Further, from Proposition 1.10 (1) and its proof, we obtain

$$\begin{aligned}\langle S^*x,x\rangle &= \Big\langle \frac{1}{k}[S'(x)]^*x, x\Big\rangle = \frac{1}{k}\langle x, S'(x)x\rangle \\ &= \frac{1}{k}\langle x, kS(x)\rangle = \langle\overline{Sx,x}\rangle = \langle Sx,x\rangle - 2i.\operatorname{Im}\langle Sx,x\rangle.\end{aligned}$$

Hence
$$\frac{\langle S^*x,x\rangle}{\langle T^*x,x\rangle} = \frac{\langle Sx,x\rangle}{\langle Tx,x\rangle} - 2i.\operatorname{Im}\left\{\frac{\langle Sx,x\rangle}{\langle Tx,x\rangle}\right\}.$$
Let $\lambda \in W(S,T)$. Then we can choose a point $y \in S_1(0)$ such that $\lambda = \langle Sy,y\rangle/\langle Ty,y\rangle$. The following holds

$$\frac{\langle S^*y,y\rangle}{\langle T^*y,y\rangle} = \lambda - 2i.\operatorname{Im}\{\lambda\} = \bar{\lambda}.$$

Hence $\bar{\lambda} \in W(S^*,T^*)$ and the assertion (1) is proven.

To prove the assertion (2) we take $\lambda \in \Lambda(S,T)$ and find an eigenvector $x_1 \in S_1(0)$ such that $Sx_1 - \lambda Tx_1 = 0$. Now, according to Proposition 1.10 (5), it follows that

$$(Sx_1 - \lambda Tx_1)^* = S^*x_1 - \bar{\lambda}T^*x_1 = 0,$$
$$\text{hence } \bar{\lambda} \in \Lambda(S^*,T^*).$$

Similarly, as in the theory of linear operators, it is possible to define the spectral radius and the numerical radius of a couple of homogeneous operators. □

Definition 2.6. Let $S,T : X \to X^*$ be positively homogeneous operators. Supposing the sets $\sigma(S,T)$ and $W(S,T)$ are nonempty and bounded, we define the number $r_{\sigma(S,T)} = \sup_{\lambda\in\sigma(S,T)} |\lambda|$ as the *spectral radius* of the couple (S,T) and the number

$$r_{W(S,T)} = \sup_{\lambda\in W(S,T)} |\lambda|$$

as the *numerical radius* of the couple (S, T).

Proposition 2.7. *If $S, T) : X \to X^*$ are homogeneous operators, then the following inequality holds*

$$r_{\sigma(S,T)} \le r_{W(S,T)} \le r_{W(S^*,T^*)}.$$

If, moreover, S is a continuous and T a positively defined operator, then

$$r_{W(S,T)} \le \frac{\|S\|}{c},$$

where

$$c = \inf_{x \in S_1(0)} \mathrm{Re}\,\{\langle Tx, x\rangle\}.$$

Proof. The proof follows easily from Theorem 2.5 and Proposition 2.2. □

It is well known that, for normal linear operator S and the identity operator I on a Hilbert space, the following important equality holds

$$r_{W(S,I)} = \|S\|.$$

(See, for example, [**18**, Theorem 6.2-E].) It also holds for some nonlinear operators, see [**11**, Example 3.18].

Further, results in [**3**] imply that the above equality also holds for a special class of homogeneous operators called symmetric polynomial operators. It will be natural to generalize the notions of normality and symmetry on the class of nonlinear operators.

Definition 2.8. Let an operator $F : X \to X^*$ have Gateaux derivative $F'(x_0)$ at a given point $x_0 \in X$. We shall say that the operator F is *normal at the point* x_0 if, for any $h \in X$ the equality

$$\|F'(x_0)h\| = \|[F'(x_0)]^* h\|$$

holds.

Here $[F'(x_0)]^*$ is the adjoint to the linear continuous operator $F'(x_0)$.

We shall say that F is a *normal operator on an open set* $D \subset X$ if it is normal at any point $x \in D$.

Remark 2.9. If X is a Hilbert space and $F : X \to X$ is a Gateaux-differentiable operator. Then F is normal at a point x_0 if and only if the linear operators $F'(x_0)$ at $[F'(x_0)]^*$ are mutually commuting. (See [**18**, Theorem 6.2-D].)

Lemma 2.10. *Let $F : X \to X^*$ be a homogeneous operator of the degree $k \geq 1$ having hemi-continuous Gateaux-derivative, and let F be normal at a point $x_0 \in X$. Then F and its adjoint operator F^* satisfy the following equality*

$$\|F(x_0)\| = \|F^*(x_0)\|.$$

Proof. According to Proposition 1.10 the following holds

$$F^*(x_0) = \frac{1}{k}[F'(x_0)]^* x_0.$$

Further, $F(x_0) = (1/k)F'(x_0)x_0$, see the proof of Proposition 1.10, because F is homogeneous of the degree k and Gateaux-differentiable. Now the normality of F at x_0 implies that the equality

$$\|F(x_0)\| = \frac{1}{k}\|F'(x_0)x_0\| = \frac{1}{k}\|[F'(x_0)]^* x_0\| = \|F^*(x_0)\|$$

holds. □

Corollary 2.11. *Let $F : X \to X^*$ be a normal homogeneous operator on X with its adjoint operator F^*. Then $\|F\| = \|F^*\|$.*

On the contrary, a homogeneous operator satisfying the last equality need not be normal as the next example shows.

Example 2.12. Let us consider a homogeneous polynomial operator $F : E_2 \to E_2$ of the degree 2 on a two-dimensional Euclidean space E_2

defined for any $x = (x_1, x_2) \in E_2$ by the equality

$$F(x) = (x_1 x_2, 0).$$

Then

$$F'(x) = \begin{pmatrix} x_2 & x_1 \\ 0 & 0 \end{pmatrix}$$

and

$$[F'(x)]^* = \begin{pmatrix} x_2 & 0 \\ x_1 & 0 \end{pmatrix}.$$

Using Proposition 1.10 (1) we obtain the adjoint operator F^* to F in the form

$$F^*(x) = \frac{1}{2}[F'(tx)]^* x = \frac{1}{2}(x_1 x_2, x_1^2).$$

Now we easily find that

$$\begin{aligned} \|F\| &= \sup_{x \in S_1(0)} \|F(x)\| = \sup_{x \in S_1(0)} \sqrt{x_1^2 x_2^2} \\ &= \sup_{x \in S_1(0)} |x_1 x_2| = \frac{1}{2} \\ \|F^*\| &= \sup_{x \in S_1(0)} \|F^*(x)\| = \frac{1}{2} \sup_{x \in S_1(0)} \sqrt{x_1^2 x_2^2 + x_1^4} \\ &= \frac{1}{2} \sup_{x \in S_1(0)} |x_1| = \frac{1}{2}. \end{aligned}$$

Hence, $\|F\| = \|F^*\|$ but the operator F is not normal on E_2. Indeed, it is not normal at a point $x = (x_1, x_2) \in E_2$ such that $x_1 \neq 0$ because, for arbitrary points $x = (x_1, x_2) \in E_2$ and $h = (h_1, h_2) \in E_2$, we have

$$\begin{aligned} F'(x)h &= (x_2 h + x_1 h_2, 0), \\ [F'(x)]^* h &= (x_2 h_1, x_1 h_1), \end{aligned}$$

so that

$$\begin{aligned} \|F'(x)h\| &= |x_2 h_1 + x_1 h_2|, \\ \|[F'(x)]^* h\| &= |h_1|.\|x\|. \end{aligned}$$

For the normality of the operator F the equality $\|F'(x)h\| = \|[F'(x)]^* h\|$ should be fulfilled, see Definition 2.8. But, if $x_1 \neq 0$, then this equality cannot be satisfied for all $h \in E_2$ because, putting $h = (0, 1)$, we obtain a contradiction.

Proposition 2.13. *Let $S, T : X \to X^*$ be Gateaux-differentiable homogeneous operators with their adjoint operators S^*, T^*. Let S and T be normal on X. Then the following assertions hold*:

(1) *A number λ belongs to spectrum $\sigma(S, T)$ if and only if the conjugate number $\bar{\lambda}$ belongs to spectrum $\sigma(S^*, T^*)$.*

(2) *If λ_0 is an eigenvalue of the couple (S, T) with an eigenvector $x_0 \in S_1(0)$, then the number $\bar{\lambda}_0$ is an eigenvalue of the couple (S^*, T^*) with the same eigenvector x_0.*

Proof. Let $\lambda \in \sigma(S, T)$. Then, according to Definition 2.3 there exists a sequence $\{x_n\} \subset S_1(0)$ such that $\|Sx_n - \lambda T x_n\| \to 0$.

It follows from properties of the adjoint operator, Proposition 1.10 (5), that the adjoint to operator $(S - \lambda T)$ is the operator $(S^* - \bar{\lambda} T^*)$. Using Lemma 2.10 we obtain

$$\|Sx_n - \lambda T x_n\| = \|S^* x_n - \bar{\lambda} T^* x_n\|,$$

so that

$$\|S^* x_n - \bar{\lambda} T^* x_n\| \to 0 \quad \text{and thus} \quad \bar{\lambda} \in \sigma(S^*, T^*).$$

This proves the assertion (1). The assertion (2) follows from Proposition 1.10 (5) and Lemma 2.10 analogously. □

Proposition 2.14. *Let $S, T : X \to X^*$ be homogeneous operators of the degree $k \geq 1$. Let S have hemi-continuous Gateaux-derivative and T be a positive operator. Suppose λ_0 is an eigenvalue of the couple (S, T) with an eigenvector x_0. Then*

$$Re\, \lambda_0 = \frac{\langle H x_0, x_0 \rangle}{\langle T x_0, x_0 \rangle},$$

where H is the potential operator from Theorem 1.8.

Proof. According to Theorem 1.8, $S(x) = H(x) + R(x)$ holds, where H is the potential operator and R is the operator satisfying, according to Proposition 1.10 (3), the following relation

$$\langle Rx, x\rangle = \frac{2ki}{k+1}\mathrm{Im}\,\{\langle Sx, x\rangle\}.$$

Then, for an eigenvalue λ_0, we obtain

$$\begin{aligned}\lambda_0 = \frac{\langle Sx_0, x_0\rangle}{\langle Tx_0, x_0\rangle} &= \frac{\langle Hx_0, x_0\rangle}{\langle Tx_0, x_0\rangle} + \frac{\langle Rx_0, x_0\rangle}{\langle Tx_0, x_0\rangle}\\ &= \frac{\langle Hx_0, x_0\rangle}{\langle Tx_0, x_0\rangle} + \frac{2ki}{k+1}\frac{\mathrm{Im}\,\langle Sx_0, x_0\rangle}{\langle Tx_0, x_0\rangle}.\end{aligned}$$

Both expressions $\langle Tx_0, x_0\rangle$ and $\langle Hx_0, x_0\rangle$ are real because T is positive and H is potential. Hence, $\mathrm{Re}\,\lambda_0 = \langle Hx_0, x_0\rangle / \langle Tx_0, x_0\rangle$. □

Definition 2.15. Let $\omega : \langle 0, +\infty) \to \langle 0, +\infty)$ be a continuous increasing function and such that $\omega(0) = 0$, $\lim_{t\to+\infty}\omega(t) = +\infty$. Then the mapping $J : X \to 2^{X^*}$ defined on a real Banach space X by $J(0) = 0$,

$$J(x) = \{x^* \in X^* : \langle x, x*\rangle = \omega(\|x\|).\|x\|, \|x^*\| = \omega(\|x\|), x \neq 0\}$$

is called the *duality mapping* on X with the gauge function ω.

Proposition 2.16. *Let X be a rotund Banach space, and let $S : X \to X^*$ be a homogeneous polynomial operator of the degree $k \geq 1$. Suppose $J : X \to X^*$ is a duality mapping with the gauge function $\omega(t) = c.t^k$ where $c > 0$. Consider eigenvalues $\lambda_1, \lambda_2 \in \Lambda(S, J)$ of the couple (S, J) with eigenvectors $x_1, x_2 \in S_1(0)$. Then the following inequality is satisfied*

$$|\lambda_1 - \lambda_2| \leq \frac{\|S\| + k.\|\mathcal{S}\|}{c}.\|x_1 - x_2\|.$$

(*Here $\mathcal{S}$ is the polar k-linear operator to operator P from Definition* 1.5).

Proof. According to the above assumptions, we have $Sx_j = \lambda_j Jx_j$, $j = 1, 2$. Using this and Definition 2.15 of the duality mapping, we obtain

$$|\lambda_1 - \lambda_2| = \left| \frac{\langle Sx_1, x_1 \rangle}{\langle Jx_1, x_1 \rangle} - \frac{\langle Sx_2, x_2 \rangle}{\langle Jx_2, x_2 \rangle} \right| = \frac{1}{c} |\langle Sx_1, x_1 - x_2 \rangle + \langle Sx_1 - Sx_2, x_2 \rangle|.$$

Using Definition 1.5 and the properties of k-linear polar operator $\mathcal{S}$, we obtain the following equality

$$S(x_1) - S(x_2) = \sum_{j=1}^{k} \mathcal{S}(x_1 - x_2, x_1^{j-1}, x_2^{k-j}),$$

where the symbol $\mathcal{S}(x_1 - x_2, x_1^{j-1}, x_2^{k-j})$ denotes that the point x_1 recurs $(j-1)$-times and the point x_2 recurs $(k-j)$-times. Finally, triangular inequality and properties of duality mapping, see Definition 2.15, imply the following estimation

$$\begin{aligned} |\lambda_1 - \lambda_2| &\leq \frac{1}{c} \{ \|S\|.\|x_1\|^k.\|x_1 - x_2\| + \|Sx_1 - Sx_2\|.\|x_2\| \} \\ &\leq \frac{1}{c} \left\{ \|S\|.\|x_1 - x_2\| + \left\| \sum_{j=1}^{k} \mathcal{S}(x_1 - x_2, x_1^{j-1}, x_2^{k-j}) \right\| \right\} \\ &\leq \frac{\|S\| + k.\|\mathcal{S}\|}{c}.\|x_1 - x_2\| \quad \text{for } x_1, x_2 \in S_1(0). \end{aligned}$$

□

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