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RESONANCE FOR QUASILINEAR ELLIPTIC HIGHER ORDER PARTIAL DIFFERENTIAL EQUATIONS AT THE FIRST EIGENVALUE

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1. Introduction. In this paper the author presents a resonance result on the Sobolev space $W^{m,p}(\Omega)$ where Ω is a bounded open connected subset of R^N meeting the cone property. We let $1 < p < \infty$ and Qu be the $2m$th order quasilinear differential operator in generalized divergence form

$$Qu = \sum_{1\leq|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi_m(u)), \tag{1.1}$$

for $u \in W^{m,p}$, where $\xi_m = \{D^\alpha u : 0 \leq |\alpha| \leq m\}$, and we make standard assumptions on A_α such as Carathéodory, uniform ellipticity, monotonicity, and a growth restriction. We shall study an equation of the following nature,

$$Qu(x) = g(x, u(x)) + h(x), \quad \text{for } u \in W^{m,p}(\Omega), \tag{1.2}$$

where $h(x) \in L^{p'}(\Omega)$, $p' = p/(p-1)$ and $g(x,t) : \Omega \times R \to R$ is Carathéodory. Subject to $mp > N$, we show the existence of a solution to (1.2) with g having superlinear growth in u but subject to a one-sided growth condition. Since Q lacks an $\alpha = 0$ order term, problem (1.2) is considered at resonance since $Qu = \lambda_1 u$ is solved by $\lambda_1 = 0$ and $u =$ constant, where λ_1 is defined as the first eigenvalue of Q. Shapiro [**9**, p. 365] provides a detailed explanation of this. This result primarily differs from that of Shapiro [**9**] in that our one-sided growth assumption on g is different from his, and since we approached the first eigenvalue of Q from values bigger than $\lambda_1 = 0$, in order for our results to hold, our Landesman-Lazer conditions must have reversed inequalities from those of Shapiro's theorem [**9**, p. 365]. Thus the theorem we will establish in this paper holds for a distinct class of functions that those meeting the hypothesis of Shapiro's Theorem 1. Examples meeting our conditions

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on g, but not covered by Shapiro [**9**] will be provided in the next section. However, we do point out that Shapiro [**9**] takes $h \in (W^{m,p})^*$, the dual of $W^{m,p}$, and that while his superlinear growth condition on g holds for a general p, its growth is governed by q where if $p < Nm^{-1}$ then $q = pN/(N - mp)$ and $q' = q/(q-1)$ for $p \geq Nm^{-1}$ with $q > p$. Thus his results, in this sense, are more general.

2. Preliminaries. In this section we introduce the necessary notation and establish preliminary results in order to prove the theorems in the following sections. We begin by letting $\Omega \subset R^N$, $N \geq 1$, be a bounded open connected set meeting the cone property, i.e., there exists a finite cone C such that each point x in Ω is a vertex of a finite cone C_x contained in Ω and congruent to C, see [**2**, p. 11] or [**1**, p. 66]. Thus, in particular, Ω cannot contain any cusps. The points of the open set Ω will be designated by $x = (x_1, \dots, x_N)$ and the elementary differential operators by $D^\alpha = \prod_{j=1}^N (\partial/\partial x_j)^{\alpha_j}$ for an ordered N-tuple $\alpha = (\alpha_1, \dots, \alpha_N)$ of nonnegative integers with the order of the operator D^α being written as $|\alpha| = \sum_{j=1}^N \alpha_j$. To write nonlinear partial differential operators in a convenient form, we introduce the vector space R^{s_m} whose elements are $\xi_m = \{\xi_\alpha : |\alpha| \leq m\}$ and divide each ξ_m into two parts $\xi_m = (\eta_{m-1}, \zeta_m)$ where $\eta_{m-1} = \{\eta_\beta : |\beta| \leq m-1\} \in R^{s_{m-1}}$ is the lower order part of ξ_m and $\zeta_m = \{\zeta_\alpha : |\alpha| = m\}$ is the part corresponding to the mth derivatives, i.e., the highest order terms. For $u \in W^{m,p}(\Omega)$, $\xi_m(u)(x) = \{D^\alpha u(x) : |\alpha| \leq m\}$. (Note $D^{(0,0,\dots,0)}u = u$.) Furthermore, the semi-linear form of the operator given by (1.1) is

$$Q(u,v) = \sum_{1 \leq |\alpha| \leq m} \int_\Omega A_\alpha(x, \xi_m(u)) D^\alpha v, \qquad \forall\, u, v \in W^{m,p}(\Omega). \tag{2.1}$$

We make the following usual assumptions on the coefficients of Q.

(A1) Each $A_\alpha : (\Omega \times R^{s_m}) \to R$ satisfies the Caratheodory conditions, i.e., $A_\alpha(x, \xi_m)$ is measurable for x in Ω for every fixed $\xi_m \in R^{s_m}$ and continuous in ξ_m for almost every fixed $x \in \Omega$.

(A2) There exist constants p with $1 < p < \infty$, $c \geq 0$ and a nonnegative function $\tilde{h} \in L^{p'}(\Omega)$ where $p' = p/(p-1)$ such that:

$|A_\alpha(x,\xi_m)| \le \tilde{h}(x) + c|\xi_m|^{p-1}$, $1 \le |\alpha| \le m$ for almost every $x \in \Omega$, for all $\xi_m \in R^{s_m}$.

(A3) $\sum_{|\alpha|=m}(A_\alpha(x,\eta_{m-1},\zeta_m) - A_\alpha(x,\eta_{m-1},\zeta'_m))(\zeta_\alpha - \zeta'_\alpha) > 0$ for almost every $x \in \Omega$, for all $(\eta_{m-1},\zeta_m) \in R^{s_m}$, $\zeta + m \ne \zeta'_m$ where $A_\alpha(x,\xi_m) = A_\alpha(x,\eta_{m-1},\zeta_m)$ with $\xi_m = (\eta_{m-1},\zeta_m)$. This is known as the monotonicity condition which will be needed when establishing results for $|\alpha| = m$.

(A4) There exists a positive constant $c_0 > 0$ such that

$$\sum_{1\le|\alpha|\le m} A_\alpha(x,\xi_m)\xi_\alpha \ge c_0\left\{\sum_{1\le|\alpha|\le m}|\xi_\alpha|^2\right\}^{p/2},$$

for almost every $x \in \Omega$, for all $\xi_m \in R^{s_m}$ and p is as given in (A2).

This is known as the uniform ellipticity condition.

Moreover, we make the following assumptions on $g(x,t)$.

(g1) $g(x,t)$ meets the usual Caratheodory conditions.

(g2) $g(x,t)$ grows superlinearly, that is, for all $\varepsilon > 0$, there exist a $g_\varepsilon \in L^{p'}(\Omega)$ such that $|g(x,t)| \le \varepsilon|t|^{p-1} + g_\varepsilon(x)$, $g_\varepsilon(x) \ge 0$, almost every $x \in \Omega$, for all $t \in R$, and $mp > N$.

(g3) $g(x,t)$ meets the following one-sided growth condition, $tg(x,t) \ge -c(x)|t| - d(x)$, $c(x), d(x) \ge 0$ for almost every $x \in \Omega$, and in $L^{p'}(\Omega)$, for all $t \in R$.

Before providing examples of functions meeting (g1)–(g3), we state the main theorem we will establish in this paper.

Theorem 2.1. *Let $mp > N$, and let $\Omega \subset R^N$ be a bounded domain with the cone property. Suppose g meets* (g1)–(g3)*, $h \in L^{p'}$, and Qu is given by* (1.1) *where $A_\alpha(x,\xi_m)$ satisfies* (A1)–(A4) *for $1 \le |\alpha| \le m$, and we set*

$$g_-(x) = \limsup_{t\to-\infty} g(x,t) \quad \textit{and} \quad g_+(x) = \liminf_{t\to+\infty} g(x,t).$$

Furthermore, suppose the following type of Landesman-Lazer condition prevails,

$$\int_\Omega g_-(x) < -\int_\Omega h(x) < \int_\Omega g_+(x), \tag{2.2}$$

then (1.2) *has a weak solution.*

By a *weak solution* we mean that there exists a $u \in W^{m,p}(\Omega)$ such that,

$$(2.3) \qquad Q(u,v) = \int_\Omega g(x,u)v + \int_\Omega hv, \quad \forall\, v \in W^{m,p}(\Omega),$$

where $Q(u,v)$ is given by (2.1). Examples of functions satisfying the hypothesis of Theorem 2.1 but not meeting those of Theorem 1 in Shapiro [**9**, p. 365], are the following.

Example 2.2. Let $N = 1$, $\Omega = (0, 2\pi)$, and

$$g(x,t) = \begin{cases} |\sin x|(t^{p-1}/\log t) & \text{for } t \geq 2, \\ |\sin x|(2^{p-1}/\log 2)(t-1) & \text{for } 1 \leq t \leq 2, \\ 0 & \text{for } 0 \leq t \leq 1, \\ -g(x,-t) & \text{for } t < 0. \end{cases}$$

Also consider

Example 2.3. Let $N = 1$, $\Omega = (0, 2\pi)$, and

$$g(x,t) = \begin{cases} |\cos x| t^{p-1-\varepsilon} & \text{for } t \geq 0 \text{ and for} \\ & \quad p > 1+\varepsilon, \text{ where } 0 < \varepsilon < 1, \\ -g(x,-t) & \text{for } t < 0. \end{cases}$$

It is straightforward to verify that $g(x,t)$ in both illustrations is an odd (in t) continuous function that meets conditions (g1)–(g3). In particular, for both of these cases, we have that $g_-(x) = \lim_{t\to-\infty} g(x,t) = -\infty$ and $g_+(x) = \lim_{t\to\infty} g(x,t) = +\infty$. Hence, the Landesman-Lazer conditions (2.2) are certainly met, but not those conditions of Theorem 1 appearing in Shapiro [**9**]. He imposes conditions which would necessitate the existence of $h \in L^{p'}$ so that $+\infty < -\int_\Omega h(x) < -\infty$, which is absurd. The reversal in the inequalities in the Landesman-Lazer conditions occurred because, in order to establish his results, Shapiro [**9**] required that $\int_\Omega g_+(x) < -\int_\Omega h(x) < \int_\Omega g_-(x)$. On a final note, it is an easy matter to verify that our illustrations also do not meet his one-sided growth condition which is that $g(x,t)t \leq q(x)|t|$ for almost every $x \in \Omega$ and for all $t \in R$, for some $q(x) \geq 0$ for almost every $x \in \Omega$ and in $L^{p'}$.

For the proof of the theorem, we need the following fact established in Shapiro [**8**, pp. 1852–1854]. If $1 < p < \infty$ and Ω is a bounded open connected set with the cone property, then there exist a sequence $\{\phi_n\}_{n=1}^\infty$ in $W^{m,p}(\Omega)$ such that the following properties hold:

$$\begin{gathered}\{\phi_n\}_{n=1}^\infty \text{ is a complete orthonormal system} \\ \text{(CONS) in } L^2(\Omega); \\ \phi_1(x) = |\Omega|^{-1/2}; \\ \phi_n \in W^{m,2} \cap W^{m,p} \quad \text{for } n = 1, 2, \dots .\end{gathered} \tag{2.4}$$

Furthermore, from Shapiro [**8**, pp. 1852–1854] we see that if we let

$$S_J = \text{subspace of } W^{m,p}(\Omega) \text{ spanned by } \{\phi_1, \phi_2, \dots, \phi_J\}, \tag{2.5}$$

then, given $v \in W^{m,p}(\Omega)$, there exists $\{v_J\} \in S_J$ such that

$$\lim_{J\to\infty} \|v - v_J\|_{W^{m,p}} = 0. \tag{2.6}$$

We next define

$$g^n(x,t) = \begin{cases} n & \text{if } g(x,t) \geq n, \\ g(x,t) & \text{if } |g(x,t)| \leq n, \\ -n & \text{if } g(x,t) \leq -n. \end{cases} \tag{2.7}$$

Following the Galerkin method, see Kesavan [**4**], the theorem is proved by first showing that a solution, say u_J, exists for the following perturbed problem which is a nonresonance result in the finite dimensional space S_J. This proposition will be invoked when establishing results on $W^{m,p}(\Omega)$.

Proposition 2.4. *Let n be a fixed positive integer. Under the hypothesis of Theorem* 2.1, *we will show that there exists a weak solution, $u_J \in S_J$, of*

$$Qu - \frac{1}{n}\operatorname{sgn}(u)|u|^{p-1} = g^n(x,u) + h(x), \quad u \in S_J. \tag{2.8}$$

Observe that, for n a fixed large positive integer, $g^n(x,t)$ is bounded by n. Consequently, we are not assuming superlinear growth in establishing Proposition 2.4.

Thus, by a *weak solution* we mean a $u_J \in S_J$ such that

$$Q(u_J, v) - \frac{1}{n}\int_\Omega \operatorname{sgn}(u_J)|u_J|^{p-1}v = \int_\Omega g^n(x, u_J)v + \int_\Omega hv, \qquad \forall\, v \in S_J, \tag{2.9}$$

where

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}$$

Proof of Proposition 2.4. To establish the proposition, define for $\beta = (\beta_1^J, \beta_2^J, \dots, \beta_J^J) \in R^J$ the following,

$$\begin{aligned} F_1(\beta) = {} & -Q\bigg(\sum_{j=1}^J \beta_j^J \phi_j \phi_1\bigg) \\ & + \frac{1}{n}\int_\Omega \operatorname{sgn}\bigg(\sum_{j=1}^J \beta_j^J \phi_j\bigg)\bigg|\sum_{j=1}^J \beta_j^J \phi_j\bigg|^{p-1}\phi_1 \\ & + \int_\Omega g^n\bigg(x, \sum_{j=1}^J \beta_j^J \phi_j\bigg)\phi_1 + \int_\Omega h\phi_1 \\ (F_k(\beta))_{k=2}^J = {} & Q\bigg(\sum_{j=1}^J \beta_j^J \phi_j, \phi_k\bigg) \\ & - \frac{1}{n}\int_\Omega \operatorname{sgn}\bigg(\sum_{j=1}^J \beta_j^J \phi_j\bigg)\bigg|\sum_{j=1}^J \beta_j^J \phi_j\bigg|^{p-1}\phi_k \\ & - \int_\Omega g^n\bigg(x, \sum_{j=1}^J \beta_j^J \phi_j\bigg)\phi_k - \int_\Omega h\phi_k. \end{aligned} \tag{2.10}$$

Setting $F(\beta) = (F_1(\beta), \dots, F_J(\beta))$, multiplying both sides of (2.10) by β_k^J, summing on k, using the fact that ϕ_1 is a constant, see (2.4), and applying (2.1), we have

$$
\begin{aligned}
(F(\beta)\cdot\beta) &= \frac{1}{n}\int_\Omega \operatorname{sgn}\left(\sum_{j=1}^J \beta_j^J\phi_j\right)\left|\sum_{j=1}^J \beta_j^J\phi_j\right|^{p-1}\beta_1^J\phi_1 \\
&\quad + \int_\Omega g^n\biggl(x,\sum_{j=1}^J \beta_j^J\phi_j\biggr)\beta_1^J\phi_1 + \int_\Omega h\beta_1^J\phi_1 \\
&\quad + Q\biggl(\sum_{j=1}^J \beta_j^J\phi_j, \sum_{k=2}^J \beta_k^j\phi_k\biggr) \\
&\quad - \frac{1}{n}\int_\Omega \operatorname{sgn}\left(\sum_{j=1}^J \beta_j^J\phi_j\right)\left|\sum_{j=1}^J \beta_j^J\phi_j\right|^{p-1}\sum_{k=2}^J \beta_k^J\phi_k \\
&\quad - \int_\Omega g^n\biggl(x,\sum_{j=1}^J \beta_j^J\phi_j\biggr)\biggl(\sum_{k=2}^J \beta_k^J\phi_k\biggr) \\
&\quad - \int_\Omega h\biggl(\sum_{k=2}^J \beta_k^J\phi_k\biggr).
\end{aligned}
\tag{2.11}
$$

Note. For the remainder of this paper we will be using the L^p-norm unless otherwise indicated.

Moreover, since Q is linear on the second variable, see (2.1), applying Cauchy-Schwarz's inequality, (A4), the definition of g^n, and the following equality,

$$
-\sum_{j=1}^J \beta_j^J\phi_j + 2\sum_{k=2}^J \beta_k^J\phi_k = -\beta_1^J\phi_1 + \sum_{k=2}^J \beta_k^J\phi_k,
$$

we obtain

$$
\begin{aligned}
(F(\beta)\cdot\beta) &= Q\biggl(\sum_{j=1}^J \beta_j^J\phi_j, \sum_{k=2}^J \beta_k^J\phi_k\biggr) \\
&\quad - \frac{1}{n}\int_\Omega \operatorname{sgn}\left(\sum_{j=1}^J \beta_j^J\phi_j\right)\left|\sum_{j=1}^J \beta_j^J\phi_j\right|^{p-1}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\cdot\Big(-\beta_1^J\phi_1+\sum_{k=2}^J\beta_k^J\phi_k\Big)\\
&-\int_\Omega g^n\Big(x,\sum_{j=1}^J\beta_j^J\phi_j\Big)\Big(-\beta_1^J\phi_1+\sum_{k=2}^J\beta_k^J\phi_k\Big)\\
&-\int_\Omega h\Big(-\beta_1^J\phi_1+\sum_{k=2}^J\beta_k^J\phi_k\Big)\\
\geq{}& c_0\int_\Omega\bigg\{\sum_{1\leq|\alpha|\leq m}\Big|D^\alpha\sum_{k=2}^J\beta_k^J\phi_k\Big|^2\bigg\}^{p/2}\\
&-\frac{1}{n}\int_\Omega \operatorname{sgn}\Big(\sum_{j=1}^J\beta_j^J\phi_j\Big)\Big|\sum_{j=1}^J\beta_j^J\phi_j\Big|^{p-1}\\
&\qquad\qquad\cdot\Big(-\sum_{j=1}^J\beta_j^J\phi_j+2\sum_{k=2}^J\beta_k^J\phi_k\Big)\\
&-\|n\|_{p'}\Big\|\sum_{k=2}^J\beta_k^J\phi_k\Big\|-\|h\|_{p'}\Big\|\sum_{k=2}^J\beta_k^J\phi_k\Big\|\\
&-\|n\|_{p'}\|\beta_1^J\phi_1\|-\|h\|_{p'}\|\beta_1^J\phi_1\|.
\end{aligned}\tag{2.12}
$$

Inequality (2.12) reduces to

$$
\begin{aligned}
(F(\beta)\cdot\beta)\geq{}& c_0\int_\Omega\bigg\{\sum_{1\leq|\alpha|\leq m}\Big|D^\alpha\sum_{k=2}^J\beta_k^J\phi_k\Big|^2\bigg\}^{p/2}\\
&+\frac{1}{n}\int_\Omega \operatorname{sgn}\Big(\sum_{j=1}^J\beta_j^J\phi_j\Big)\Big|\sum_{j=1}^J\beta_j^J\phi_j\Big|^{p-1}\Big(\sum_{j=1}^J\beta_j^J\phi_j\Big)\\
&-\frac{2}{n}\int_\Omega \operatorname{sgn}\Big(\sum_{j=1}^J\beta_j^J\phi_j\Big)\Big|\sum_{j=1}^J\beta_j^J\phi_j\Big|^{p-1}\Big(\sum_{k=2}^J\beta_k^J\phi_k\Big)\\
&-(\|n\|_{p'}+\|h\|_{p'})\Big\|\sum_{k=2}^J\beta_k^J\phi_k\Big\|\\
&-(\|n\|_{p'}+\|h\|_{p'})\|\beta_1^J\phi_1\|.
\end{aligned}\tag{2.13}
$$

Using the fact that $\int_\Omega(\sum_{k=2}^J\beta_k^J\phi_k)\cdot\phi_1=0$, from the generalized Poincare's inequality, see [**5**, p. 32], we have that there exists a positive constant $k_1 = K_1(\Omega, p) > 0$ such that

$$c_0 \int_\Omega \bigg\{ \sum_{1\le|\alpha|\le m} \bigg| D^\alpha \sum_{k=2}^J \beta_k^J \phi_k \bigg|^2 \bigg\}^{p/2} \ge c_0 k_1 \int_\Omega \bigg| \sum_{k=2}^J \beta_k^J \phi_k \bigg|^p. \tag{2.14}$$

Letting $\delta = c_0 k_1$, applying (2.14) to (2.13), and using Hölder's inequality, we have

$$\begin{aligned}(F(\beta)\cdot\beta) \ge \delta \int_\Omega \bigg| \sum_{k=2}^J \beta_k^J \phi_k \bigg|^p &+ \frac{1}{n} \int_\Omega \bigg| \sum_{j=1}^J \beta_j^J \phi_j \bigg|^p \\ &- \frac{2}{n} \bigg\| \sum_{j=1}^J \beta_j^J \phi_j \bigg\|^{p-1} \bigg\| \sum_{k=2}^J \beta_k^J \phi_k \bigg\| \\ &- (\|n\|_{p'} + \|h\|_{p'}) \bigg\| \sum_{k=2}^J \beta_k^J \phi_k \bigg\| \\ &- (\|n\|_{p'} + \|h\|_{p'}) \|\beta_1^J \phi_1\|.\end{aligned} \tag{2.15}$$

By Young's inequality, see [**6**], one can show that, for n chosen big enough since $\delta > 0$ and $p > 1$, we have

$$\delta \bigg\| \sum_{k=2}^J \beta_k^J \phi_k \bigg\|^p + \frac{1}{n} \bigg\| \sum_{j=1}^J \beta_j^J \phi_j \bigg\|^p \ge \frac{3}{n} \bigg\| \sum_{j=1}^J \beta_j^J \phi_j \bigg\|^{p-1} \bigg\| \sum_{k=2}^J \beta_k^J \phi_k \bigg\|. \tag{2.16}$$

Inequality (2.16) will follow as a consequence of the following claim.

Claim 1. *Let $\delta > 0$, $p > 1$ and $p' = p/(p-1)$. Then there exists an n_0 such that for $n \ge n_0$,*

$$\delta \bigg\| \sum_{k=2}^J \beta_k^J \phi_k \bigg\|^p + \frac{1}{2n} \bigg\| \sum_{j=1}^J \beta_j^J \phi_j \bigg\|^p \ge \frac{3}{n} \bigg\| \sum_{j=1}^J \beta_j^J \phi_j \bigg\|^{p-1} \bigg\| \sum_{k=2}^J \beta_k^J \phi_k \bigg\|. \tag{2.17}$$

Proof of Claim 1. For simplicity of notation, let $A = \|\sum_{j=2}^J \beta_j^J \phi_j\|$ and $B = \|\sum_{j=1}^J \beta_j^J \phi_j\|^{p-1}$; then $B^{p'} = \|\sum_{j=1}^J \beta_j^J \phi_j\|^p$. Substituting these values in (2.17) and multiplying both sides by $(n/3)$, we see that (2.17) prevails if and only if the following holds

$$\frac{n}{3}\delta A^p + \frac{p'}{6}\frac{B^{p'}}{p'} \geq AB. \tag{2.18}$$

However, (2.18) holds if and only if the following does

$$\frac{6n\delta}{3p'}A^p + \frac{B^{p'}}{p'} \geq \frac{6}{p'}AB.$$

Setting $C = (6/p')A$ gives $A^p = (p'/6)^p C^p$. Thus the claim holds if and only if

$$I_n = \frac{6n\delta}{3p'}\left(\frac{p'}{6}\right)C^p + \frac{B^{p'}}{p'} \geq CB \tag{2.19}$$

is true. However, for n chosen large enough, it is the case that

$$\frac{6n\delta}{3p'}\left(\frac{p'}{6}\right)^p \geq \frac{1}{p}.$$

Thus, using the above inequality, we see that (2.19) holds if and only if

$$I_n \geq \frac{C^p}{p} + \frac{B^{p'}}{p'} \geq CB.$$

But this is Young's inequality, see [**6**]. Therefore, Claim 1 is established. Next, with $|\beta|^2 = (\beta_1^J)^2 + \cdots + (\beta_J^J)^2$, from (2.4) and since $mp > N$, by the Rellich-Kondrachov theorem [**1**, p. 144], we have that $\phi_i \in L^{p'}(\Omega)$ for all i, thus it follows that

$$\lim_{|\beta|\to\infty} \left\| \sum_{j=1}^{J} \beta_j^J \phi_j \|_p = \infty.$$

Applying (2.16) to (2.15), since $\delta > 0$, $n > 0$ and $p > 1$, then $(F(\beta)\cdot\beta) \to \infty$ as $|\beta| \to \infty$. Hence, there exists a $\rho > 0$ such that

$$(F(\beta)\cdot\beta) > 0 \quad \text{for } |\beta| \geq \rho. \tag{2.20}$$

However, in order to apply the corollary to Brouwer's fixed point theorem, see Kesavan [**4**, p. 219], we need to show that $F_i(\beta) \in C(R^J, R)$ for $i = 1, 2, \dots, J$. This follows from the definition of each $F_i(\beta)$, from (A1), (A2), (g1) and since g^n is bounded. Therefore,

$$F_i(\beta) \in C(R^J, R) \quad \text{for } i = 1, 2, \dots, J.$$

Thus we have that there exist $|\hat{\beta}| \le \rho$, $\hat{\beta} \in R^J$, such that

$$F_i(\hat{\beta}) = 0 \quad \text{for all } i = 1, 2, \dots, J.$$

Set $u_J = \sum_{i=1}^J \hat{\beta}_i^J \phi_i$ and observe from (2.10) that

$$Q(u_J, \phi_k) - \frac{1}{n}\int_\Omega \operatorname{sgn}(u_J)|u_J|^{p-1}\phi_k = \int_\Omega g^n(x, u_J)\phi_k + \int_\Omega h\phi_k, \quad \text{for } k = 1, 2, \dots, J.$$

This gives (2.9), and the proof of Proposition 2.4 is complete. □

3. Nonresonance $W^{m,p}(\Omega)$. We proceed along with the Galerkin approximation argument. By invoking Proposition 2.4 for each J, we will be able to obtain a sequence of solutions, u_J, which we will show to be uniformly bounded independent of J in $W^{m,p}$. Furthermore, this sequence will have a weak limit which will converge to a solution of the following proposition. This proposition, that we establish next, is a nonresonance result in the Sobolev space $W^{m,p}(\Omega)$.

Proposition 3.5. *Let n be a fixed positive integer. Under the hypothesis of Proposition* 2.4*, we will show that there exist $u_n \in W^{m,p}(\Omega)$ such that*

$$Q(u_n, v) - \frac{1}{n}\int_\Omega \operatorname{sgn}(u_n)|u_n|^{p-1}v = \int_\Omega g^n(x, u_n)v + \int_\Omega hv, \tag{3.1}$$

for all $v \in W^{m,p}(\Omega)$.

Proof of Proposition 3.5. Since n is a fixed positive integer, we invoke Proposition 2.4. This gives us a sequence $\{u_J\}_{J=1}^\infty$ such that $u_J \in S_J$

satisfies (2.9) for $J = 1, 2, \ldots$. Before we proceed with the proof, we show the following needed claim.

Claim 2. *The sequence*

$$\{\|u_J\|_{W^{m,p}}\}_{J=1}^{\infty} \quad \textit{is uniformly bounded.}$$

Proof of Claim 2. Suppose the claim is false. Then it suffices to assume that

$$\{\|u_J\|_{L^p}\}_{J=1}^{\infty} \longrightarrow \infty \quad \text{as } J \to \infty. \tag{3.2}$$

For, if $\|u_J\|_{L^p}$ is uniformly bounded, then we are done by the following argument. Take $v = u_J$ in (2.9) and apply (A4) to obtain

$$\begin{aligned} c_0 \int_\Omega \bigg\{ \sum_{1 \le |\alpha| \le m} |D^\alpha u_J|^2 \bigg\}^{p/2} &\le Q(u_J, u_J) \\ &= \frac{1}{n}\|u_J\|^p + \int_\Omega g^n(x, u_J)u_J + \int_\Omega h u_J \\ &\le \frac{1}{n}\|u_J\|^p + \|n\|_{p'}\|u_J\| + \|h\|_{p'}\|u_J\| \\ &\le k, \end{aligned}$$

for some $k > 0$ since n is fixed. Next, since $p > 1$, we see that there exists a constant $\delta > 0$ such that

$$\begin{aligned} c_0 \int_\Omega \bigg\{ \sum_{1 \le |\alpha| \le m} |D^\alpha u_J|^2 \bigg\}^{p/2} &\ge \delta c_0 \int_\Omega \sum_{1 \le |\alpha| \le m} |D^\alpha u_J|^p \\ &= \delta c_0 \sum_{1 \le |\alpha| \le m} \|D^\alpha u_J\|^p, \quad \forall\, n. \end{aligned}$$

The above two inequalities imply that

$$\sum_{1 \le |\alpha| \le m} \|D^\alpha u_J\|^p \le k, \quad \text{for some } k > 0.$$

But this, together with the assumption that the L^p-norm of this sequence is bounded, gives

$$\|u_J\|_{W^{m,p}} \le k, \quad \text{for some } k > 0,$$

thus, establishing Claim 2. □

We continue under the assumption that (3.2) holds.

For simplicity of notation, we let

$$\tilde{u}_J = -\beta_1^J \phi_1 + \sum_{j=2}^{J} \beta_j^J \phi_j,$$

where

$$u_J = \sum_{j=1}^{J} \beta_j^J \phi_j \tag{3.3}$$

and

$$u_{J2} = \sum_{j=2}^{J} \beta_j^J \phi_j.$$

Then, from (3.3), it follows that $\tilde{u}_J = -u_J + 2u_{J2}$. Thus, taking $v = \tilde{u}_J$ in (2.9), using (3.3) and (2.7), we obtain

$$\begin{aligned} Q(u_J, \tilde{u}_J) &= \frac{1}{n} \int_\Omega \operatorname{sgn}(u_J)|u_J|^{p-1}(-u_J + 2u_{J2}) \\ &\quad + \int_\Omega g^n(x, u_J)\tilde{u}_J + \int_\Omega h\tilde{u}_J \\ &= -\frac{1}{n} \int_\Omega |u_J|^p + \frac{2}{n} \int_\Omega \operatorname{sgn}(u_J)|u_J|^{p-1}(u_{J2}) \\ &\quad + \int_\Omega g^n(x, u_J)\tilde{u}_J + \int_\Omega h\tilde{u}_J \\ &\le -\frac{1}{n}\|u_J\|^p + \frac{2}{n}\|u_J\|^{p-1}\|u_{J2}\| \\ &\quad + \|n\|_{p'}\|\tilde{u}_J\| + \|h\|_{p'}\|\tilde{u}_J\|. \end{aligned} \tag{3.4}$$

Next, recall that, by Poincare's inequality, (A4) and (2.1), we have that there exists $\delta > 0$ such that

$$Q(u_J, \tilde{u}_J) \geq \delta \|u_{J2}\|^p. \tag{3.5}$$

Applying (3.5) to (3.4), we have

$$\delta \|u_{J2}\|^p \leq -\frac{1}{n}\|u_J\|^p + \frac{2}{n}\|u_J\|^{p-1}\|u_{J2}\| + (\|n\|_{p'} + \|h\|_{p'})\|\tilde{u}_J\|. \tag{3.6}$$

Now, since

$$|\beta_1^J \phi_1| = |\hat{u}_J(1)\phi_1| = \left| \frac{\phi_1}{|\Omega|^{1/2}} \int_\Omega u_J \right| \leq \frac{|\Omega|^{1/p'}}{|\Omega|}\|u_J\|$$

for $u_{J2} = u_J - \beta_1^J \phi_1$, then for some $k > 0$, $\|u_{J2}\| \leq k\|u_J\|$. Consequently, we have that $\|\tilde{u}_J\| \leq k'\|u_J\|$, for some $k' > 0$. Applying this to (3.6) and moving terms to the lefthand side, we obtain

$$\begin{aligned} \delta \|u_{J2}\|^p + \frac{1}{n}\|u_J\|^p - \frac{2}{n}\|u_J\|^{p-1}\|u_{J2}\| & \\ &\leq (\|n\|_{p'} + \|h\|_{p'})\|\tilde{u}_J\| \\ &\leq \tilde{k}\|u_J\|, \quad \text{for some } \tilde{k} > 0. \end{aligned} \tag{3.7}$$

Hence, applying (2.17) to (3.7), we have that

$$\frac{1}{2n}\|u_J\|^p \leq \delta\|u_{J2}\|^p + \frac{1}{n}\|u_J\|^p - \frac{3}{n}\|u_J\|^{p-1}\|u_{J2}\| \leq \tilde{k}\|u_J\|. \tag{3.8}$$

Thus,

$$\|u_J\| \leq k, \quad \text{for some } k > 0. \tag{3.9}$$

However, (3.9) contradicts (3.2). Therefore, Claim 2 is established. □

Continuing along with the proof of Proposition 3.5, since it is well known that $W^{m,p}(\Omega)$ is a separable reflexive Banach space, [**1**, p. 47], and since $mp > N$, it consequently follows from the Rellich-Kondrachov compact embedding theorem for Sobolev spaces [**1**, p. 144] that there exists a subsequence of $\{u_J\}$ (which, for ease of notation, we take to be the full sequence), and a function u_n, such that

$$u_n \in W^{m,p}; \tag{3.10}$$

$$\lim_{J\to\infty} \|D^\alpha u_J - D^\alpha u_n\|_p = 0, \quad \text{for } |\alpha| \le m-1; \tag{3.11}$$

$$\lim_{J\to\infty} \int_\Omega D^\alpha u_J w = \int_\Omega D^\alpha u_n w \tag{3.12}$$
$$\text{for all } w \in L^{p'} \quad \text{and} \quad |\alpha| = m.$$

$$\lim_{J\to\infty} \int_\Omega h u_J = \int_\Omega h u_n; \tag{3.13}$$

$$\lim_{J\to\infty} \eta_{m-1}(u_J(x)) = \eta_{m-1}(u_n(x)) \quad \text{for a.e. } x \in \Omega, \tag{3.14}$$

where $\eta_{m-1}(u_n(x)) = \{D^\alpha u_n(x) : |\alpha| \le m-1\}$.

We next propose to show that there exists a subsequence of $\{u_{J_k}\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \zeta_m(u_{J_k}(x)) = \zeta_m(u_n(x)) \quad \text{for a.e. } x \in \Omega, \tag{3.15}$$

where $\zeta_m(u_{J_k}(x)) = \{D^\alpha u_n(x) : |\alpha| = m\}$. To show (3.15), it is sufficient to establish the following two facts: (1) there exists a subsequence $\{u_{J_k}\}_{k=1}^\infty$ such that

$$\begin{aligned}\lim_{k\to\infty} \sum_{|\alpha|=m} (A_\alpha(x, \eta_{m-1}(u_{J_k}), \zeta_m(u_{J_k})) \\ - A_\alpha(x, \eta_{m-1}(u_{J_k}), \zeta_m(u_n))) \\ \cdot (D^\alpha u_{J_k}(x) - D^\alpha u_n(x)) = 0 \quad \text{for a.e. } x \in \Omega,\end{aligned} \tag{3.16}$$

where $\xi_m(u_{J_k}) = (\eta_{m-1}(u_{J_k}), \zeta_m(u_{J_k}))$.

(2) With $\{u_{J_k}\}_{k=1}^\infty$ designating the same subsequence as in (3.16),

$$\{|\zeta_m(u_{J_k}(x))|\}_{k=1}^\infty \quad \text{is pointwise bounded for a.e. } x \in \Omega. \tag{3.17}$$

We shall soon see that both the proof of (3.16) and that of (3.17) are heavily dependent on the monotonicity assumption (A3). The proof that (3.16) and (3.17) imply (3.15) is due to Shapiro [**9**, p. 372]. However, we include it here for completeness. That is, there exists a finite constant $K(x)$ such that

$$|\zeta_m(u_{J_k}(x))| \leq K(x) \quad \text{for } k = 1, 2, \dots .$$

Thus, to see that (3.16) and (3.17) imply (3.15), let Ω_1 be the subset for which (3.14), (3.16) and (3.17) all hold simultaneously for $\{u_{J_k}\}_{k=1}^{\infty}$. Consequently,

$$\operatorname{meas}\Omega = \operatorname{meas}\Omega_1. \tag{3.18}$$

Suppose there exists $x_0 \in \Omega_1$ for which the equality in (3.15) does not hold. Hence, by (3.17), there exists a further subsequence $\{\zeta_m(u_{J_{k_l}}(x_0))\}_{l=1}^{\infty}$ and a $\zeta_m^* \in R^{s_m - s_{m-1}}$ with

$$\zeta_m^* \neq \zeta_m(u_n(x_0)) \tag{3.19}$$

such that $\lim_{l\to\infty} \zeta_m(u_{J_{k_l}}(x_0)) = \zeta_m^*$. Therefore, from (3.14)

$$\begin{aligned} \lim_{l\to\infty} \sum_{|\alpha|=m} (A_\alpha(x_0, \eta_{m-1}(u_{J_{k_l}}), \zeta_m(u_{J_{k_l}})) \\ - A_\alpha(x, \eta_{m-1}(u_{J_{k_l}}), \zeta_m(u_n)) \\ \cdot (D^\alpha u_{J_{k_l}}(x_0) - D^\alpha u_n(x_0)) \\ = \sum_{|\alpha|=m} [A_\alpha(x_0, \eta_{m-1}(u_n), \zeta_m^*) \\ - A_\alpha(x_0, \eta_{m-1}(u_n), \zeta_m(u_n))] \\ \cdot [\zeta_m^* - D^\alpha u_n(x_0)]. \end{aligned} \tag{3.20}$$

From (3.19) and (A3) we see that the righthand side of the equality in (3.20) is strictly positive. Hence, the limit on the lefthand side of the equality in (3.20) is strictly positive. However, x_0 is in Ω_1 and, from the choice of Ω_1 and (3.16), we see that the limit on the lefthand side of the equality in (3.20) is zero. We have arrived at a contradiction. Consequently, no such point like x_0 exists in Ω_1. From (3.18), we have that the Lebesgue measure of Ω_1 is the same as that of Ω. We conclude that (3.15) does indeed hold once (3.16) and (3.17) are established. To establish (3.16), we shall show separately that

$$\lim_{J\to\infty}\int_\Omega \sum_{|\alpha|=m} (A_\alpha(x,\eta_{m-1}(u_J),\zeta_m(u_n)) \cdot (D^\alpha u_J(x) - D^\alpha u_n(x)) = 0 \tag{3.21}$$

and

$$\lim_{J\to\infty}\int_\Omega \sum_{|\alpha|=m} (A_\alpha(x,\xi_m(u_J))(D^\alpha u_J(x) - D^\alpha u_n(x)) = 0. \tag{3.22}$$

The proof that (3.16) follows from (3.21) and (3.22) is again due to Shapiro [**9**, p. 373], but we put it here for ease of reading. We observe from the difference of the above two limits that

$$\lim_{J\to\infty}\int_\Omega \sum_{|\alpha|=m} [A_\alpha(x,\eta_{m-1}(u_J),\zeta_m(u_J)) - A_\alpha(x,\eta_{m-1}(u_J),\zeta_m(u_n))] \cdot [D^\alpha u_J(x) - D^\alpha u_n(x)] = 0. \tag{3.23}$$

But, by (A3), the integrand in this last limit is nonnegative for almost every $x\in\Omega$. Hence, the sequence

$$\left\{ \sum_{|\alpha|=m} [A_\alpha(x,\eta_{m-1}(u_J),\zeta_m(u_J)) - A_\alpha(x,\eta_{m-1}(u_J),\zeta_m(u_n))] \cdot [D^\alpha u_J(x) - D^\alpha u_n(x)] \right\}_{J=1}^{\infty}$$

converges in L^1-norm to zero, and (3.16) follows immediately from Rudin [**7**, p. 70]. We next show that, indeed, (3.21) and (3.22) hold.

Equation (3.21) is also established in Shapiro [**9**, pp. 373–374], but it is here for completeness. Observe that

$$\begin{aligned}\int_\Omega A_\alpha(x,\eta_{m-1}(u_J),\zeta_m(u_n))[D^\alpha u_J(x) - D^\alpha u_n(x)] &= \int_\Omega [A_\alpha(x,\eta_{m-1}(u_J),\zeta_m(u_n)) - A_\alpha(x,\eta_{m-1}(u_n),\zeta_m(u_n))] \cdot [D^\alpha u_J - D^\alpha u_n] \\ &\quad + \int_\Omega A_\alpha(x,\eta_{m-1}(u_n),\zeta_m(u_n))[D^\alpha u_J(x) - D^\alpha u_n].\end{aligned} \tag{3.24}$$

From $u \in W^{m,p}$ and (A2), we see that $A_\alpha(x, \eta_{m-1}(u_n), \zeta_m(u_n)) \in L^{p'}$ for $|\alpha| = m$. Consequently, it follows from (3.12) that the second integral on the righthand side of the equality in (3.24) converges to zero as $k \to \infty$ for $|\alpha| = m$. Therefore, (3.21) will follow once we show that

$$\lim_{J\to\infty} \int_\Omega [A_\alpha(x, \eta_{m-1}(u_j), \zeta_m(u_n)) - A_\alpha(x, \eta_{m-1}(u_n), \zeta_m(u_n))] \cdot [D^\alpha u_J - D^\alpha u_n] = 0$$

for $|\alpha| = m$. From (3.10) and Hölder's inequality, we see that this last limit will follow once we show

$$\lim_{J\to\infty} \int_\Omega [A_\alpha(x, \eta_{m-1}(u_J), \zeta_m(u_n)) - A_\alpha(x, \eta_{m-1}(u_n), \zeta_m(u_n))]^{p/(p-1)} = 0 \tag{3.25}$$

for $|\alpha| = m$. To see that (3.25) holds, we observe from (3.14) and (A1) that the integrand in (3.25) converges to zero as $J \to \infty$ for almost every $x \in \Omega$. Also, we see from (3.11) and (A2) that the integrand in (3.25) is absolutely equi-integrable, i.e., given $\varepsilon > 0$, there exists δ such that meas $E < \delta$ implies

$$\int_E |A_\alpha(x, \eta_{m-1}(u_J), \zeta_m(u_n)) - A_\alpha(x, \eta_{m-1}(u_n), \zeta_m(u_n))|^{p/(p-1)} < \varepsilon$$

for $|\alpha| = m$ and $J = 1, 2, \dots$. Consequently, we conclude from Egoroff's theorem [**6**] that (3.25) holds. But this establishes (3.21).

To establish (3.22), we observe from (A2) and Claim 2 that there exists a constant $k_5 > 0$ such that

$$\int_\Omega |A_\alpha(x, \xi_m(u_J))|^{p/(p-1)} \leq k_5, \quad \text{for } 1 \leq |\alpha| \leq m \quad \text{and} \quad J = 1, 2, \dots . \tag{3.26}$$

Consequently, we obtain from (3.11) and Hölder's inequality that

$$\lim_{J\to\infty} \int_\Omega A_\alpha(x, \xi_m(u_J))(D^\alpha u_J(x) - D^\alpha u_n(x)) = 0 \quad \text{for } 1 \leq |\alpha| \leq m-1.$$

Hence, (3.22) will follow once we show

$$\lim_{J\to\infty}\int_\Omega \sum_{1\le|\alpha|\le m} A_\alpha(x,\xi_m(u_J))(D^\alpha u_J(x)-D^\alpha u_n(x))=0. \tag{3.27}$$

To establish (3.27), we first observe from (3.10) and (2.5) that there exists $\{P_J u_n\}_{J=1}^\infty$ with $P_j u_n \in S_J$ such that

$$\lim_{J\to\infty}\|P_J u_n-u_n\|_{W^{m,p}}=0. \tag{3.28}$$

We therefore obtain from (3.26), (3.28) and Hölder's inequality that

$$\lim_{J\to\infty}\int_\Omega A_\alpha(x,\xi_m(u_J))(D^\alpha P_J u_n(x)-D^\alpha u_n(x))=0$$
$$\text{for } 1\le|\alpha|\le m.$$

Consequently, (3.27) will follow once we show

$$\lim_{J\to\infty}\int_\Omega \sum_{1\le|\alpha|\le m} A_\alpha(x,\xi_m(u_J))(D^\alpha u_J(x)-D^\alpha P_J u_n(x))=0. \tag{3.29}$$

To establish (3.29), we invoke (2.9) and obtain that

$$\begin{aligned} Q(u_J,u_J-P_J u_n) &= \int_\Omega g^n(x,u_J)(u_J-P_J u_n)\\ &\quad+\int_\Omega h(u_J-P_J u_n)\\ &\quad+\frac{1}{n}\int_\Omega \operatorname{sgn}|u_J|^{p-1}(u_J-P_J u_n). \end{aligned} \tag{3.30}$$

Next we observe from $h\in L^{p'}$, (3.13) and (3.28) that

$$\lim_{J\to\infty}\int_\Omega h(u_J-P_J u_n)=0. \tag{3.31}$$

Likewise, from Hölder's inequality, Claim 2, (3.11) and (3.28), we obtain

$$\frac{1}{n}\lim_{J\to\infty}\int_\Omega \operatorname{sgn}(u_J)|u_J|^{p-1}(u_J-P_J u_n)=0. \tag{3.32}$$

Then, we see from (2.7) and (g2) with $\varepsilon = 1$ that

$$|g^n(x, u_J)| \le g_1(x) + k^{p-1} \quad \text{for a.e. } x \in \Omega \quad \text{and} \quad J = 1, 2, \dots, \tag{3.33}$$

where $g_1 \in L^{p'}$ and k is the bound from Claim 2. Therefore, $\|g^n(x, u_J)\|_{p'}$ is bounded independent of J. Hence, to show that the first integral on the righthand side of (3.30) converges to 0 as J goes to infinity, we rewrite it as

$$\int_\Omega g^n(x, u_J)(u_J - P_J u_n) = \int_\Omega g^n(x, u_J)(u_J - u_n) + \int_\Omega g^n(x, u_J)(u_n - P_J u_n). \tag{3.34}$$

From (3.28), we see that $\lim_{J\to\infty} \|u_n - P_J u_n\|_p = 0$. Hence, from (3.33) and by Hölder's inequality, we have that

$$\lim_{J\to\infty} \int_\Omega g^n(x, u_J)(u_n - P_J u_n) = 0. \tag{3.35}$$

Similarly, from (3.11), we have that $\|u_J - u_n\| \to 0$. This, together with (3.33) and Hölder's inequality, gives that the first integral on the righthand side of (3.34) also converges to 0. This last fact, coupled with (3.35), says that (3.34) converges to 0 as J goes to ∞. The above fact, in conjunction with (3.30)–(3.32), gives that

$$\lim_{J\to\infty} Q(u_J, u_J - P_J u_n) = 0. \tag{3.36}$$

Next, from (2.1), we see that $Q(u_J, u_J - P_J u_n)$ is the same as the integral on the lefthand side of the equality in (3.29). Hence, the limit in (3.36) equals the limit in (3.29), and (3.29) is established. Consequently, (3.22) prevails, and since (3.22) and (3.21) imply (3.16), equation (3.16) is also established. The proof that (3.17) holds is a standard argument done by Shapiro [**9**, pp. 374–376]. One simply replaces his uniform ellipticity condition by ours, namely (A4). Therefore, (3.15) is established.

It remains to show that Claim 2 and (3.10)–(3.15), along with the fact that $\{u_J\}_{J=1}^{\infty}$ satisfies (2.8), imply that (3.1) holds. To show this, we let $v \in \cup_{J=1}^{\infty} S_J$. Then it follows from (3.11) and (3.14) that

$$\lim_{k\to\infty} \int_\Omega \operatorname{sgn}(u_{J_k})|u_{J_k}|^{p-1} v = \int_\Omega \operatorname{sgn}(u_n)|u_n|^{p-1} v. \tag{3.37}$$

Also, from (g2) with $\varepsilon = 1$,

$$|g^n(x, u_J)v| \le |vg_1| + |v|k^{p-1} \quad \text{for a.e. } x \in \Omega, \tag{3.38}$$

where $g_1 \in L^{p'}$ and k is the bound from Claim 2. Thus we see that $|vg_1| \in L^1(\Omega)$. Hence we conclude from the Lebesgue dominated convergence theorem, (g1), (3.14) and (3.38), that

$$\lim_{k\to\infty} \int_\Omega g^n(x, u_{J_k})v = \int_\Omega g^n(x, u_n)v. \tag{3.39}$$

Next we see from (A2) in conjunction with Claim 2 and Hölder's inequality that

$$\{A_\alpha(x, \xi_m(u_{J_k}))D^\alpha v\}_{k=1}^{\infty} \tag{3.40}$$

is uniformly equi-integrable for $1 \le |\alpha| \le m$. Also (A1) along with (3.14) and (3.15) yields

$$\lim_{k\to\infty} A_\alpha(x, \xi_m(u_{J_k}))D^\alpha v(x) = A_\alpha(x, \xi_m(u_n))D^\alpha v(x)$$

for almost every $x \in \Omega$ and $1 \le |\alpha| \le m$. This fact, along with (3.40), (2.1) and Egoroff's theorem, gives $\lim_{k\to\infty} Q(u_{J_k}, v) = Q(u_n, v)$. From Proposition 2.4, (3.37), (3.39) and this last limit, we have that

$$Q(u_n, v) - \frac{1}{n}\int_\Omega \operatorname{sgn}(u_n)|u_n|^{p-1} v = \int_\Omega hv + \int_\Omega g^n(x, u_n)v, \quad \forall\, v \in \bigcup_{J=1}^{\infty} S_J. \tag{3.41}$$

It is a straightforward density argument to conclude that (3.41) also holds for all $v \in W^{m,p}(\Omega)$. Hence, (3.1) is established, and the proof of Proposition 3.5 is complete. $\square$

4. Resonance $W^{m,p}(\Omega)$. In this section we prove Theorem 2.1 which allows for g to grow superlinearly under the restriction that $mp > N$.

Proof of Theorem 2.1. Employing the familiar Galerkin approximation scheme, we first invoke Proposition 3.5 and obtain a sequence $\{u_n\}_{n=1}^{\infty}$ such that

$$u_n \in W^{m,p} \quad \text{satisfies (3.1) for } n = 1, 2, \dots . \tag{4.1}$$

We claim that

$$\{\|u_n\|_{W^{m,p}}\}_{n=1}^{\infty} \quad \text{is uniformly bounded.} \tag{4.2}$$

Suppose claim (4.2) is false. Then, without loss of generality, we can assume that

$$\lim_{n\to\infty} \|u_n\|_{W^{m,p}} = \infty. \tag{4.3}$$

Next we let $\tilde{u}_n$ be as defined by (3.3) except that we replace J by n everywhere. Thus, letting v be $\tilde{u}_n$ in (3.1), and applying (g2), we obtain

$$\begin{aligned} Q(u_n, \tilde{u}_n) &= \frac{1}{n}\int_\Omega \operatorname{sgn}(u_n)|u_n|^{p-1}(\tilde{u}_n) \\ &\quad + \int_\Omega g^n(x, u_n)\tilde{u}_n + \int_\Omega h\tilde{u}_n \\ &\le \frac{1}{n}\|u_n\|^{p-1}\|\tilde{u}_n\| + \varepsilon\|u_n\|^{p-1}\|\tilde{u}_n\| \\ &\quad + \|g_\varepsilon\|_{p'}\|\tilde{u}_n\| + \|h\|_{p'}\|\tilde{u}_n\|. \end{aligned} \tag{4.4}$$

By the definition of $\tilde{u}_n$ and applying arguments similar to those used between (3.6) and (3.7), we obtain $\|\tilde{u}_n\| \le k\|u_n\|$, for some $k > 0$. Therefore, it follows that $\|g_\varepsilon\|_{p'}\|\tilde{u}_n\| + \|h\|_{p'}\|\tilde{u}_n\| \le \tilde{k}\|u_n\|$, where $\tilde{k} = k(\|g_\varepsilon\|_{p'} + \|h\|_{p'})$. Next, using (A4), the fact that $Q(u_n, \tilde{u}_n) = Q(u_n, u_n)$, in conjunction with (4.4), we obtain

$$\begin{aligned} c_0\int_\Omega \Big\{\sum_{1\le|\alpha|\le m} |D^\alpha u_n|^2\Big\}^{p/2} \le \frac{k}{n}\|u_n\|^p + k\varepsilon\|u_n\|^p + \tilde{k}\|u_n\| \\ \text{for some } \tilde{k} > 0. \end{aligned} \tag{4.5}$$

Next, setting

$$v_n = \frac{u_n}{\|u_n\|_{W^{m,p}}}, \tag{4.6}$$

and dividing both sides of (4.5) by $\|u_n\|^p_{W^{m,p}}$, and since $\|u_n\|_{W^{m,p}} \to \infty$ and $\varepsilon > 0$ is arbitrary, we have

$$\lim_{n\to\infty} c_0 \int_\Omega \left\{ \sum_{1\le|\alpha|\le m} |D^\alpha v_n|^2 \right\}^{p/2} = 0. \tag{4.7}$$

Thus, since $c_0 > 0$, we have that

$$\lim_{n\to\infty} \sum_{1\le|\alpha|\le m} \|D^\alpha v_n\|^p = 0. \tag{4.8}$$

From (4.6), we see that $\|v_n\|_{W^{m,p}} = 1$ for $n = 1, 2, \ldots$. Hence,

$$\|v_n\|^p_{W^{m,p}} = 1. \tag{4.9}$$

Hence, since $1 = \|v_n\|^p_{W^{m,p}} = \|v_n\|^p_p + \sum_{1\le|\alpha|\le m} \|D^\alpha v_n\|^p_p$, we infer from (4.8) that

$$\lim_{n\to\infty} \|v_n\|_p = 1. \tag{4.10}$$

Clearly, $\{\|v_n\|^p_{W^{m,p}}\}_{n=1}^\infty$ is a uniformly bounded sequence, thus there exists a subsequence and a function v_0 with the following properties:

$$v_n \longrightarrow v_0 \in W^{m,p}(\Omega), \quad \text{weakly}; \tag{4.11}$$

$$\lim_{n\to\infty} \|D^\alpha v_n - D^\alpha v_0\|_p = 0, \quad \text{for } |\alpha| \le m-1; \tag{4.12}$$

$$\lim_{n\to\infty} \int_\Omega D^\alpha v_n w = \int_\Omega D^\alpha v_0 w \tag{4.13}$$

$$\text{for all } w \in L^{p'} \quad \text{and} \quad |\alpha| = m$$

$$\lim_{n\to\infty} \int_\Omega h v_n = \int_\Omega h v_0, \quad \text{since} \quad h \in L^{p'}. \tag{4.14}$$

$$\lim_{n\to\infty} D^\alpha v_n(x) = D^\alpha v_0(x) \tag{4.15}$$

$$\text{for a.e. } x \in \Omega \quad \text{and} \quad |\alpha| \le m-1.$$

Next, from (4.8) and (4.13), and Hölder's inequality, we have

$$\int_\Omega D^\alpha v_0 w = 0 \quad \text{for } w \in L^{p'}, \quad 1 \le |\alpha| \le m.$$

Consequently, $D^\alpha v_0 = 0$ almost everywhere in Ω for $1 \le |\alpha| \le m$. Since Ω is a bounded open connected set meeting the cone property, we conclude that $v_0 =$ constant almost everywhere in Ω. From (4.10) and (4.12) we obtain that

$$\|v_0\|_p = 1.$$

Hence, this constant is a nonzero either positive or negative quantity. We shall assume that it is positive. Since a similar argument prevails for the case when the constant is negative. Let

$$v_0 = c_4 \quad \text{a.e.} \quad x \in \Omega, \quad c_4 = (\text{meas}\,\Omega)^{-1/p}. \tag{4.16}$$

Next we invoke (4.1) with $v = v_0 = c_4 > 0$ almost everywhere $x \in \Omega$ to obtain

$$Q(u_n, v_0) - \frac{1}{n}\int_\Omega \text{sgn}\,(u_n)|u_n|^{p-1} v_0 = \int_\Omega g^n(x, u_n) v_0 + \int_\Omega h v_0.$$

From the definition of Q, we have that $Q(u_n, v_0) = 0$. Therefore,

$$\int_\Omega g^n(x, u_n) v_0 + \int_\Omega h v_0 \le 0.$$

Since, by hypothesis, $mp > N$, we have from the compact embedding theorems that

$$\lim_{n\to\infty} v_n = v_0 \quad \text{uniformly}. \tag{4.18}$$

Applying (g3) and using (4.18), we have for $n \ge n_0$ that

$$g^n(x, u_n) \ge -c(x) - \frac{d(x)}{|u_n|}.$$

Therefore, for $n \ge n_0$ and since $v_0 > 0$ almost everywhere, the following holds

$$g^n(x, u_n) v_0 \ge -v_0 c(x) - d(x)\frac{v_0}{|u_n|}.$$

Thus, we can apply Fatou's lemma to the following quantity,

$$\liminf_{n\to\infty}\left(\int_\Omega\left(g^n(x,u_n)v_0+c(x)v_0+d(x)\frac{v_0}{|u_n|}\right)\right.$$
$$\left.-\int_\Omega\left(c(x)v_0+d(x)\frac{v_0}{|u_n|}\right)\right)\le-\int_\Omega hv_0.$$

Now, since $u_n=v_n\|u_n\|_{W^{m,p}}\to+\infty$, we obtain

$$\int_\Omega\liminf_{n\to\infty}g^n(x,u_n)+\int_\Omega\liminf_{n\to\infty}\left(c(x)v_0+d(x)\frac{v_0}{|u_n|}\right)$$
$$-\liminf_{n\to\infty}\int_\Omega\left(c(x)v_0+d(x)\frac{v_0}{|u_n|}\right)\le-\int_\Omega hv_0.$$

However,

$$\liminf_{n\to+\infty}g^n(x,u_n)\ge g_+(x).$$

Therefore,

$$\int_\Omega g_+(x)v_0\le-\int_\Omega hv_0.$$

But this yields a contradiction to the Landesman-Lazer conditions (2.2). Hence, we cannot have (4.3) holding. Thus there exists a constant $K_6>0$ such that

$$\|u_n\|_{W^{m,p}}\le K_6\quad\text{for } n=1,2,\dots. \tag{4.19}$$

As before, there exists a subsequence of $\{u_n\}$ (which for ease of notation we take to be the full sequence) and a function u such that

$$u_n\to u\in W^{m,p},\quad\text{weakly;} \tag{4.20}$$

$$\lim_{n\to\infty}\|D^\alpha u_n-D^\alpha u\|_p=0,\quad\text{for }|\alpha|\le m-1; \tag{4.21}$$

$$\lim_{n\to\infty}\int_\Omega D^\alpha u_n w=\int_\Omega D^\alpha uw \tag{4.22}$$

$$\text{for all } w\in L^{p'}\quad\text{and}\quad|\alpha|=m.$$

$$\lim_{n\to\infty}\int_\Omega hu_n=\int_\Omega hu; \tag{4.23}$$

$$\lim_{n\to\infty}\eta_{m-1}(u_n(x))=\eta_{m-1}(u(x)) \tag{4.24}$$

$$\text{for a.e. } x\in\Omega, \tag{4.24}$$

where $\eta_{m-1}(u_n(x)) = \{D^\alpha u_n(x) : |\alpha| \le m-1\}$.

We next propose to show that there exists a subsequence of $\{u_{n_k}\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \zeta_m(u_{n_k}(x)) = \zeta_m(u(x)) \quad \text{for a.e. } x \in \Omega, \tag{4.25}$$

where $\zeta_m(u_{n_k}(x)) = \{D^\alpha u(x) : |\alpha| = m\}$. As in the proof of Proposition 3.5, once (4.25) is established, it will be an easy matter to establish Theorem 2.1 from (4.19)–(4.25). However, the proof that (4.25) holds, is parallel to the proof of (3.15). One simply replaces u_J by u_n, u_{J_k} by u_{n_k} and u_n by u.

To complete the proof of the theorem, we have to show that (4.19)–(4.25) along with (4.1) gives (2.3). In order to accomplish this, let $v \in W^{m,p}(\omega)$ be given. Then it follows from (4.1), (3.1) and (2.1) that

$$\sum_{1\le|\alpha|\le m} \int_\Omega A_\alpha(x, \xi_m(u_{n_k}))D^\alpha v - \frac{1}{n}\int_\Omega \operatorname{sgn}(u_{n_k})|u_{n_k}|^{p-1}v = \int_\Omega g^{n_k}(x, u_{n_k})v + \int_\Omega hv. \tag{4.26}$$

From (4.19), we see that $\|u_{n_k}\|_p \le K_6$ for $k = 1, 2, \dots$. Hence, it follows from Hölder's inequality and $v \in W^{m,p}$ that

$$\lim_{k\to\infty} \frac{1}{n_k}\int_\Omega \operatorname{sgn}(u_{n_k})|u_{n_k}|^{p-1}v = 0. \tag{4.27}$$

Next, from (g2) with $\varepsilon = 1$, we see that

$$|g^n(x, u_n)| \le g_1(x) + |u_n|^{p-1}, \quad \text{for } k = 1, 2, \dots, \tag{4.28}$$

where $g_1 \in L^{p'}$. Also we see from Hölder's inequality that

$$\int_E |u_n|^{p-1}|v| \le \left\{\int_E |u_n|^p\right\}^{(p-1)/p}\left\{\int_E |v|^p\right\}^{1/p}, \tag{4.29}$$

where E is a measurable subset of Ω. From Claim 2 and (4.19), we see that the first integral on the righthand side of the inequality in (4.29) is uniformly bounded in n. Hence, it follows from (4.28) and (4.29) that

$$\{g^n(x, u_n)v\}_{n=1}^\infty \quad \text{is absolutely equi-integrable,} \tag{4.30}$$

and from (g1), (2.7) and (4.24), we have that

$$\lim_{n\to\infty} g^n(x,u_n)v(x) = g(x,u)v(x) \quad \text{a.e. in } \Omega. \tag{4.31}$$

Therefore, from (4.30), (4.31) and, by Vitali's theorem, we have that

$$\lim_{k\to\infty} \int_\Omega g^{n_k}(x,u_{n_k})v = \int_\Omega g(x,u)v. \tag{4.32}$$

Next, with $\{u_{n_k}\}_{k=1}^{\infty}$, the subsequence given in (4.25), we obtain from (A1), (4.24) and (4.25) that

$$\begin{aligned}\lim_{k\to\infty} & A_\alpha(x,\xi_m(u_{n_k}(x)))D^\alpha v(x) \\ &= A_\alpha(x,\xi_m(u(x)))D^\alpha v(x), \quad \text{a.e. in } \Omega,\end{aligned} \tag{4.33}$$

for $1 \le |\alpha| \le m$. Also, we see from (4.19), (A2) and Hölder's inequality that

$$\{A_\alpha(x,\xi_m(u_{n_k}(x))D^\alpha v\}_{n=1}^{\infty} \tag{4.34}$$

is absolutely equi-integrable for $1 \le |\alpha| \le m$. Hence, it follows from (4.33), (4.34) and Vitali's theorem that

$$\lim_{k\to\infty} \int_\Omega A_\alpha(x,\xi_m(u_{n_k}(x)))D^\alpha v = \int_\Omega A_\alpha(x,\xi_m(u))D^\alpha v, \tag{4.35}$$

for $1 \le |\alpha| \le m$. From (4.26), (4.27) and (4.35), we obtain that

$$\sum_{1\le|\alpha|\le m} \int_\Omega A_\alpha(x,\xi_m(u))D^\alpha v = \int_\Omega g(x,u)v + \int_\Omega hv, \qquad \forall\, v \in W^{m,p}(\Omega).$$

But, from (2.1), we see that this last equality is the same as (2.3), and the proof of Theorem 2.1 is complete. □

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