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DERIVATIONS ON BANACH PAIRS

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ABSTRACT. In this paper the well-known theorem by Johnson and Sinclair establishing the continuity of derivations on semi-simple Banach algebras is extended to the Banach pair framework. We establish the continuity of the component operators, D_+ and D_-, of every derivation (D_+, D_-) on a semi-simple Banach pair. We reduce the problem to primitive Banach pairs by showing the invariance of "almost all" primitive ideals under derivations.

0. Introduction. Given a complex Banach space, X, the Banach space $L(X)$ of all continuous linear operators on X is actually a Banach algebra with respect to the operator composition. However, such a product cannot be defined in the Banach space $L(H,K)$ of all continuous linear operators between two complex Hilbert spaces H and K. It is still possible to introduce the algebraic structure on this space by using the triple product $[abc] = ab^*c$, which satisfies

$$\|[abc]\| \leq \|a\|\,\|b\|\,\|c\|, \quad \forall\, a, b, c \in L(H,K).$$

However, both product and triple product are unable to explain the algebraic nature of the Banach space $A^+ = L(X^-, X^+)$ of all continuous linear operators between two complex Banach spaces X^- and X^+. In order to describe the algebraic structure of this space, we must consider a twin space, namely the Banach space $A^- = L(X^+, X^-)$. We need two triple products

$$[\cdot,\cdot,\cdot]_+ : A^+ \times A^- \times A^+ \longrightarrow A^+$$

and

$$[\cdot,\cdot,\cdot]_- : A^- \times A^+ \times A^- \longrightarrow A^-,$$

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defined by

$$[a^+b^-c^+]_+ = a^+b^-c^+ \quad \text{and} \quad [a^-b^+c^-]_- = a^-b^+c^-,$$

respectively, with the properties

$$\begin{aligned}
&[[a^+b^-c^+]_+d^-e^+]_+ = [a^+[b^-c^+d^-]_-e^+]_+ = [a^+b^-[c^+d^-e^+]_+]_+,\\
&[[a^-b^+c^-]_-d^+e^-]_- = [a^-[b^+c^-d^+]_+e^-]_- = [a^-b^+[c^-d^+e^-]_-]_-,\\
&\quad \|[a^+b^-c^+]_+\|_+77 \leq \|a^+\|_+\|b^-\|_-\|c^+\|_+,
\end{aligned}$$

and

$$\|[a^-b^+c^-]_-\|_- \leq \|a^-\|_-\|b^+\|_+\|c^-\|_-,$$

for all $a^+, b^+, c^+, d^+, e^+ \in A^+$ and $a^-, b^-, c^-, d^-, e^- \in A^-$. The couple (A^+, A^-) is the most typical example of a Banach pair. A *Banach pair* is defined as a couple (A^+, A^-) of complex Banach spaces, endowed with two triple products $A^+ \times A^- \times A^+ \to A^+$ and $A^- \times A^+ \times A^- \to A^-$, satisfying the above four conditions. For the sake of convenience, the norm on both A^+ and A^- will be denoted by $\|\cdot\|$ and similarly both triple products will be denoted by $[\cdot,\cdot,\cdot]$. It should be noted that all the linear structures considered in this paper are assumed to be complex.

Jordan pairs arise in a natural way in the geometry of bounded symmetric domains. Loos showed in [**10**] a strong dependence between bounded homogeneous circled domains, in finite-dimensional complex vector spaces, and certain Jordan ternary structure so-called *Jordan pairs*. It is easy to check that Banach pairs defined above become Jordan Banach pairs for the symmetrized product.

In the sequel the statements $\mathcal{P}(\sigma)$, where $\mathcal{P}$ is some relation, have a meaning that $\mathcal{P}(+)$ and $\mathcal{P}(-)$ are both valid.

Because of the Johnson-Sinclair theorem [**7**] on the continuity of derivations on semi-simple Banach algebras, our knowledge about the continuity of derivations on Banach algebras is fairly satisfactory. Even in the framework of the most familiar kinds of nonassociative algebras [**18, 19, 20**] and [**22**], similar results are known. The topic of automatic continuity has been successfully developed also in the triple system context [**1, 21**] and [**23**]. Now we contribute to this classical problem by establishing the continuity of derivations on Banach pairs.

Given a Banach pair (A^+, A^-), we say that a couple (D_+, D_-) of linear operators $D_+ : A^+ \to A^+$ and $D_- : A^- \to A^-$ is a *derivation* if the following "Leibnitz rule" holds:

$$D_\sigma([a^\sigma b^{-\sigma} a^\sigma]) = [D_\sigma(a^\sigma) b^{-\sigma} c^\sigma] + [a^\sigma D_{-\sigma}(b^{-\sigma}) c^\sigma] + [a^\sigma b^{-\sigma} D_\sigma(c^\sigma)],$$

for all $a^\sigma, b^\sigma, c^\sigma \in A^\sigma$.

An example of derivation on $(L(X^-, X^+), L(X^+, X^-))$ is given by

$$D_+(a^+) = a^+u^- - u^+a^+ \quad \text{and} \quad D_-(a^-) = a^-u^+ - u^-a^-, \\ \forall\, a^\sigma \in A^\sigma,$$

where u^- in $L(X^-)$ and u^+ in $L(X^+)$. It should be pointed out that operators of this type have been intensively studied since a long time ago [**4, 5, 6, 11, 12**] and [**15**], under the name of *generalized derivations*. Here we prove that every derivation on a "standard operator pair" is of this form, and therefore a theorem by Chernoff [**2**] is extended to the Banach pair framework.

The classical notion of semi-simplicity leads to the concept of a semi-simple Banach pair, defined in a natural way below. In the last section we prove that derivations on semi-simple Banach pairs consist of continuous operators. This result provides a generalization of the Johnson-Sinclair theorem. Moreover, we obtain a Kleinecke-type theorem for continuous derivations on Banach pairs.

It should be noted that, in order to prove the continuity of derivations on Banach pairs, we will follow the strategy established in our earlier paper [**20**] so the problem will be reduced to primitive Banach pairs. Therefore, it will be essential to prove the invariance of a sufficiently large number of primitive ideals under derivations. This will be shown in the third section. In this step we also extend results of Sinclair [**13**] and Thomas [**16**].

1. Banach Pairs.

1.1 *The radical.* Given a Banach pair (A^+, A^-) and linear subspaces $I^\sigma \subset A^\sigma$, the couple (I^+, I^-) will be called a *left-left ideal*, respectively *right-right ideal*, if $[A^\sigma A^{-\sigma} I^\sigma] \subset I^\sigma$, respectively $[I^\sigma A^{-\sigma} A^\sigma] \subset I^\sigma$. A *two-sided ideal* is a left-left and right-right ideal. Finally, (I^+, I^-) is said to be an *ideal* if it is a two-sided ideal such that $[A^\sigma I^{-\sigma} A^\sigma] \subset I^\sigma$.

Let (A^+, A^-) be a Banach pair. If (I^+, I^-) is an ideal whose components I^+ and I^- are closed in A^+ and A^-, respectively, then the quotient pair $(A^+/I^+, A^-/I^-)$ is a Banach pair in an obvious way.

A left-left ideal (I^+, I^-) in a Banach pair (A^+, A^-) will be called *modular* if there exist $u^+ \in A^+$ and $u^- \in A^-$ such that

$$[A^\sigma I^{-\sigma} u^\sigma] \subset I^\sigma,$$

and

$$a^\sigma - [a^\sigma u^{-\sigma} u^\sigma] \in I^\sigma,$$

for all $a^\sigma \in A^\sigma$. The couple (u^+, u^-) is called a *right modular unit* for (I^+, I^-).

We observe that, for a modular left-left ideal (I^+, I^-) in a Banach pair (A^+, A^-), we have that $I^+ = A^+$ if and only if $I^- = A^-$. Hence, the maximality of a modular left-left ideal (I^+, I^-) means the following: $I^\sigma \neq A^\sigma$ and, if (J^+, J^-) is another left-left ideal with $I^\sigma \subset J^\sigma$ and $(I^+, I^-) \neq (J^+, J^-)$, then $J^\sigma = A^\sigma$.

For a Banach pair (A^+, A^-) and $a^\sigma, b^\sigma, c^\sigma \in A^\sigma$ we introduce the multiplication operators by

$$L_{A^\sigma}(a^\sigma, b^{-\sigma})c^\sigma = M_{A^{-\sigma}}(a^\sigma, c^\sigma)b^{-\sigma} = R_{A^\sigma}(b^{-\sigma}, c^\sigma)a^\sigma = [a^\sigma b^{-\sigma} c^\sigma].$$

Hence $L_{A^\sigma}(a^\sigma, b^{-\sigma})$ and $R_{A^\sigma}(b^{-\sigma}, c^\sigma)$ are continuous linear operators from A^σ to A^σ. These operators are called *left* and *right multiplication operators*, respectively. Also $M_{A^{-\sigma}}(a^\sigma, c^\sigma)$ is a continuous linear operator from $A^{-\sigma}$ into A^σ which is called the *middle multiplication operator*.

Proposition 1.1.1. *The components I^+ and I^- of a maximal modular left-left ideal (I^+, I^-) in a Banach pair (A^+, A^-) are closed in A^+ and A^-, respectively.*

Proof. Let (u^+, u^-) be a right modular unit for (I^+, I^-). We have that $\|u^\sigma\| \neq 0$ and (v^+, v^-), where $v^+ = \|u^-\| \|u^+\|$ and $v^- = \|u^-\|^{-1} u^-$ is a right modular unit for (I^+, I^-).

Now we claim that $d(v^+, I^+) \geq 1$ where $d(v^+, I^+)$ denote the distance from v^+ to I^+. Otherwise there exists $a^+ \in I^+$ with $\|v^+ + a^+\| < 1$.

Let b^+ in A^+ be given by the following absolutely convergent series:

$$b^+ = \sum_{n=0}^{\infty} R_{A^+}(v^+ + a^+, v^-)^n (v^+ + a^+).$$

Since $[b^+ v^- (v^+ + a^+)] = b^+ - (v^+ + a^+)$, we obtain that

$$v^+ = (b^+ - [b^+ v^- v^+]) - [b^+ v^- a^+] - a^+$$

lies in I^+. Therefore, for every $a^+ \in A^+$,

$$a^+ = (a^+ - [a^+ v^- v^+]) + [a^+ v^- v^+] \in I^+$$

so $I^+ = A^+$, which gives a contradiction.

Since $d(v^+, I^+) \geq 1$, the couple $(\overline{I^+}, I^-)$ is a proper modular left-left ideal containing (I^+, I^-). Therefore, I^+ is closed in A^+.

Finally, since (I^-, I^+) is a maximal modular left-left ideal in the Banach pair (A^-, A^+), we obtain that I^- is closed in A^-. $\square$

In [**3**], subsection 2.4 and 2.9, a *primitive* Banach pair is defined as a Banach pair (A^+, A^-) for which there exists a maximal modular left-left ideal (I^+, I^-) not containing properly a nonzero ideal of the pair. An ideal (P^+, P^-) is said to be *primitive* if the quotient pair $(A^+/P^+, A^-/P^-)$ is primitive. It is clear that a primitive ideal is the largest ideal contained in a maximal modular left-left ideal. Hence, from Proposition 1.1.1, it follows that primitive ideals in a Banach pair are necessarily closed.

The *radical*, $\operatorname{Rad}(A^+, A^-)$, of a Banach pair (A^+, A^-) is defined as the ideal (R^+, R^-) given by the intersection of all primitive ideals in (A^+, A^-). If (A^+, A^-) has no modular left-left ideal, we define $\operatorname{Rad}(A^+, A^-) = (A^+, A^-)$ and the Banach pair is called *radical*. When $\operatorname{Rad}(A^+, A^-) = (0, 0)$ the Banach pair is said to be *semi-simple*.

Example 1.1.2. For Banach spaces X^- and X^+, let (A^+, A^-) be a Banach subpair of the Banach pair $(L(X^-, X^+), L(X^+, X^-))$. Then, for every $(x_0^+, x_0^-) \in (X^+, X^-)$, the couple (I^+, I^-) given by $I^\sigma = \{a^\sigma \in A^\sigma : a^\sigma(x_0^{-\sigma}) = 0\}$ is a left-left ideal in (A^+, A^-). Assume now that every component A^σ contains the set $FL(X^{-\sigma}, X^\sigma)$ of those operators in $L(X^{-\sigma}, X^\sigma)$ with finite rank. Then there exist $u^+ \in A^+$ and $u^- \in A^-$ such that $u^\sigma(x_0^{-\sigma}) = x_0^\sigma$ and the ideal (I^+, I^-) is a maximal modular left-left ideal with (u^+, u^-) as a right modular unit. Therefore, $(0,0)$ is the largest ideal in (A^+, A^-) contained in (I^+, I^-) so the Banach pair (A^+, A^-) is primitive.

1.2 *Representations.* Given Banach spaces X^- and X^+ and letting $A^\sigma = L(X^{-\sigma}, X^\sigma)$, we observe that there exist continuous actions $(a^\sigma, x^{-\sigma}) \mapsto a^\sigma x^{-\sigma} = a^\sigma(x^{-\sigma})$ from $A^\sigma \times X^{-\sigma}$ to X^σ such that

$$[a^\sigma b^{-\sigma} c^\sigma] x^{-\sigma} = a^\sigma(b^{-\sigma}(c^\sigma x^{-\sigma})),$$

for all $a^\sigma, b^\sigma, c^\sigma \in A_\sigma$ and $x^\sigma \in X^\sigma$.

Let (A^+, A^-) be an arbitrary Banach pair. A couple of Banach spaces (X^+, X^-) is said to be an (A^+, A^-)-*module* if there exist two continuous bilinear mappings, $(a^+, x^-) \mapsto a^+ x^-$ from $A^+ \times X^-$ into X^+ and $(a^-, x^+) \mapsto a^- x^+$ from $A^- \times X^+$ into X^-, for which the above equality holds. The (A^+, A^-)-module (X^+, X^-) is called *irreducible* if $X^{-\sigma} A^\sigma \neq 0$ and the only (A^+, A^-)-submodules of (X^+, X^-) are $(0,0)$ and (X^+, X^-). We observe that (X^+, X^-) is irreducible if and only if every nonzero $x^{-\sigma}$ in $X^{-\sigma}$ satisfies

$$A^\sigma x^{-\sigma} = X^\sigma.$$

If (A^+, A^-) is a Banach pair acting on a couple (X^+, X^-) of Banach spaces then, for every $a^\sigma \in A^\sigma$, the mapping $\Phi^\sigma(a^\sigma)$ given by $\Phi^\sigma(a^\sigma) x^{-\sigma} = a^\sigma x^{-\sigma}$ defines a continuous linear operator from $X^{-\sigma}$ into X^σ. We note that Φ^σ is a continuous linear operator from A^σ into $L(X^{-\sigma}, X^\sigma)$. Moreover, the couple (Φ^+, Φ^-) is a *homomorphism* from (A^+, A^-) into $(L(X^-, X^+), L(X^+, X^-))$ in the following sense:

$$\Phi^\sigma([a^\sigma b^{-\sigma} c^\sigma]) = [\Phi^\sigma(a^\sigma)\Phi^{-\sigma}(b^{-\sigma})\Phi^\sigma(c^\sigma)],$$

for all $a^\sigma, b^\sigma, c^\sigma \in A^\sigma$.

A *representation* from a Banach pair (A^+, A^-) into a couple of Banach spaces (X^+, X^-) is defined as a couple (Φ^+, Φ^-) of continuous linear operators, $\Phi^\sigma : A^+ \to L(X^{-\sigma}, X^\sigma)$, satisfying the above equality. Such a representation is called *faithful* if $\ker \Phi^+ = 0$ and $\ker \Phi^- = 0$.

We observe that every representation (Φ^+, Φ^-) from a Banach pair (A^+, A^-) on a couple of Banach spaces (X^+, X^-) defines an (A^+, A^-)-module structure on (X^+, X^-) via

$$a^\sigma x^{-\sigma} = \Phi^\sigma(a^\sigma)(x^{-\sigma}), \quad a^\sigma \in A^\sigma, x^\sigma \in X^\sigma.$$

If these actions are irreducible, we say that the representation is *irreducible*.

Theorem 1.2.1. *An ideal (P^+, P^-) of a Banach pair (A^+, A^-) is primitive if and only if there exists an irreducible representation (Φ^+, Φ^-) from (A^+, A^-) on a suitable couple of Banach spaces (X^+, X^-) with $\ker \Phi^+ = P^+$ and $\ker \Phi^- = P^-$.*

Proof. Let (I^+, I^-) be a maximal modular left-left ideal of (A^+, A^-) with a right modular unit (u^+, u^-) such that (P^+, P^-) is the largest ideal of (A^+, A^-) contained in (I^+, I^-). Then I^+ and I^- are closed subspaces of A^+ and A^-, respectively. The couple of Banach spaces (X^+, X^-) given by $X^+ = A^+/I^+$ and $X^- = A^-/I^-$ is endowed with a structure of (A^+, A^-)-module defined by $a^\sigma(b^{-\sigma} + I^{-\sigma}) = [a^\sigma b^{-\sigma} u^\sigma] + I^\sigma$, $a^\sigma, b^\sigma \in A^\sigma$. The maximality of (I^+, I^-) implies that the (A^+, A^-)-module (X^+, X^-) is irreducible. Let (Φ^+, Φ^-) be the associated representation. Then $(\ker \Phi^+, \ker \Phi^-)$ is an ideal in (A^+, A^-) and we observe that $[\ker \Phi^\sigma u^{-\sigma} u^\sigma] \subset I^\sigma$ so that $\ker \Phi^\sigma \subset I^\sigma$. Actually it is easy to check that $(\ker \Phi^+, \ker \Phi^-)$ is the largest ideal in (A^+, A^-) contained in (I^+, I^-). Therefore, $\ker \Phi^+ = P^+$ and $\ker \Phi^- = P^-$.

On the other hand, by [**3**], if (Φ^+, Φ^-) is such a representation, then $(\ker \Phi^+, \ker \Phi^-)$ is a primitive ideal in (A^+, A^-). $\square$

By applying the above representation theorem to a primitive Banach pair, (A^+, A^-), we obtain a faithful and irreducible representation from (A^+, A^-) on a suitable couple (X^+, X^-) of Banach spaces.

Example 1.2.2. Every Banach algebra A gives rise to a Banach pair (A, A) if we define the two triple products by $[abc] = abc$. It is well known that every primitive ideal P in a Banach algebra A is the kernel of a continuous and irreducible representation Φ on a Banach space X.

Hence, the couple (Φ, Φ) is an irreducible representation of the Banach pair (A, A) on (X, X) and so the couple (P, P) is a primitive ideal in (A, A). In particular, the Banach pair (A, A) is semi-simple whenever the Banach algebra A is semi-simple.

Lemma 1.2.3. *Let (A^+, A^-) be a Banach pair and (X^+, X^-) be a couple of Banach spaces on which (A^+, A^-) is irreducibly represented with kernel (P^+, P^-). If (I^+, I^-) is an ideal for which either $I^+ \not\subset P^+$ or $I^- \not\subset P^-$, then (I^+, I^-) acts irreducibly on (X^+, X^-).*

Proof. Define linear subspaces Y^+ and Y^- of X^+ and X^-, respectively, by

$$Y^+ = \{x^+ \in X^+ : I^- x^+ = 0\} \quad \text{and} \quad Y^- = \{x^- \in X^- : I^+ x^- = 0\}.$$

If $x^+ \in Y^+$, then $(I^+(A^- x^+), 0)$ is an (A^+, A^-) submodule of (X^+, X^-) different from (X^+, X^-) so that $I^+(A^- x^+) = 0$. Therefore, $A^- Y^+ \subset Y^-$, and analogously we prove that $A^+ Y^- \subset Y^+$. Thus (Y^+, Y^-) is an (A^+, A^-)-submodule of (X^+, X^-).

If $(Y^+, Y^-) = (X^+, X^-)$, then $I^+ X^- = 0$ and $I^- X^+ = 0$ so that $I^+ \subset P^+$ and $I^- \subset P^-$, which contradicts our assumption. Hence $(Y^+, Y^-) = (0, 0)$. Therefore, for every nonzero element $x^+ \in X^+$, we have that $(A^+(I^- x^+), I^- x^+)$ is a nonzero (A^+, A^-)-submodule of (X^+, X^-) which shows that $I^- x^+ = X^-$. In a similar way we can prove that $I^+ x^- = X^+$ for every nonzero element $x^- \in X^-$, so (I^+, I^-) acts irreducibly on (X^+, X^-). □

1.3 *Density.* From now on we will consider a Banach pair (A^+, A^-) acting irreducibly on a couple (X^+, X^-) of Banach spaces. We observe that (A^-, A^+) is also a Banach pair acting irreducibly on (X^-, X^+). Therefore, there exists a duality between the components X^+ and X^- so, roughly speaking, every assertion on a component has a twin assertion on the twin component.

Lemma 1.3.1. *Let (A^+, A^-) be a Banach pair acting irreducibly on a couple (X^+, X^-) of Banach spaces. If $F : X^\sigma \to X^\sigma$ is a linear operator such that*

$$F(a^\sigma (b^{-\sigma} x^\sigma)) = a^\sigma (b^{-\sigma} F(x^\sigma)), \quad a^\sigma \in A^\sigma, b^{-\sigma} \in A^{-\sigma}, x^\sigma \in X^\sigma,$$

then F is a scalar multiple of the identity operator on X^σ.

Proof. We may assume that $\sigma = +$. Let $\mathcal{G}$ be the algebra of those linear operators $G : X^+ \to X^+$ such that

$$G(a^+(b^-x^+)) = a^+(b^-G(x^+)), \quad a^+ \in A^+, b^- \in A^-, x^+ \in X^+.$$

For a nonzero element G in $\mathcal{G}$, the couples $(G(X^+), A^-G(X^+))$ and $(\ker G, A^- \ker G)$ are (A^+, A^-)-submodules of (X^+, X^-). Hence it follows that $G(X^+) = X^+$ and $\ker G = 0$, so G is invertible and it is easy to check that G^{-1} lies in $\mathcal{G}$. Therefore $\mathcal{G}$ is a division algebra.

Now we fix a nonzero element x_0^+ in X^+. We obtain a norm on the vector space $\mathcal{G}$ by defining $|G| = \|Gx_0^+\|$, $G \in \mathcal{G}$. Let G and G' be in $\mathcal{G}$, and choose $a^+ \in A^+$ and $b^- \in A^-$ with $Gx_0^- = a^+(b^-x_0^+)$. We have that

$$\begin{aligned} |G'G| &= \|G'Gx_0^+\| = \|G'(a^+(b^-x_0^+))\| \\ &= \|a^+(b^-(G'x_0^+))\| \leq K\|a^+\|\|b^-\||G'|, \end{aligned}$$

for a suitable positive number K. Therefore, the right multiplication operator R_G, by the element G on $\mathcal{G}$, is continuous. If, given G in $\mathcal{G}$, we define $\|G\|$ as the usual operator norm of R_G, then $\|\cdot\|$ becomes an algebra norm on $\mathcal{G}$, and from the Gelfand-Mazur theorem the result follows. $\square$

Lemma 1.3.2. *Let (A^+, A^-) be a Banach pair acting irreducibly on a couple (X^+, X^-) of Banach spaces. If $x_1^\sigma, \dots, x_n^\sigma$ are linearly independent elements in the component X^σ, then there exists $a^{-\sigma}$ in $A^{-\sigma}$ such that $a^{-\sigma}x_1^\sigma \neq 0$ and $a^{-\sigma}x_k^\sigma = 0$ for $1 < k \leq n$.*

Proof. We may suppose that $\sigma = +$. For $n = 1$ the result is trivially true and we assume inductively that the result holds for n. Let's assume also that $a^-x^+ = 0$ whenever a^- lies in A^- and $a^-x_k^+ = 0$ for $1 \leq k \leq n+1$. We define

$$I^- = \{a^- \in A^- : a^-x_k^+ = 0 \text{ for } k = 2, \dots, n\}$$

and observe that (A^+, I^-) is a left-left ideal in (A^+, A^-). Hence, $(A^+I^-x_{n+1}^+, I^-x_{n+1}^+)$ is an (A^+, A^-)-submodule of (X^+, X^-), which is nonzero by the inductive hypothesis. Since $I^- x^+_{n+1} = X^-$, we define a linear operator $F : X^- \to X^-$ by $Fx^- = a^- x^+_1$ for $x^- = a^- x^+_{n+1}$ with $a^- \in I^-$. We observe that

$$F(a^-(b^+x^-)) = a^-(b^+G(x^-)), \quad a^- \in A^-, b^+ \in A^+, x^- \in X^-,$$

so the above lemma implies that there exists a number λ such that $a^-(x^+_1 - \lambda x^+_{n+1}) = 0$, for every $a^- \in I^-$. The inductive hypothesis applied to $x^+_1 - \lambda x^+_{n+1}, x^+_2, \dots, x^+_n$ shows that there exists $a^- \in A^-$ with $a^- x^+_k = 0$ for $k = 2, \dots, n$. Therefore, $a^- \in I^-$ but $a^-(x^+_1 - \lambda x^+_{n+1}) \neq 0$. This contradiction proves the result. □

From the above lemma it is easy to obtain the following density theorem.

Theorem 1.3.3. *Let (A^+, A^-) be a Banach pair acting irreducibly on a couple of Banach spaces (X^+, X^-). If $x^\sigma_1, \dots, x^\sigma_n$ are linearly independent elements in the component X^σ and $y^{-\sigma}_1, \dots, y^{-\sigma}_n$ are elements in $X^{-\sigma}$, then there exists $a^{-\sigma}$ in $A^{-\sigma}$ such that*

$$a^{-\sigma} x^\sigma_k = y^{-\sigma}_k \quad \textit{for } k = 1, \dots, n.$$

Now we argue as in [**7**, Lemma 2.1] in order to obtain the following improvement of it.

Lemma 1.3.4. *Let (A^+, A^-) be a Banach pair acting irreducibly on a couple (X^+, X^-) of infinite-dimensional Banach spaces. If $x^+_1, \dots, x^+_n, \dots$ are linearly independent elements in X^+, then, for every nonzero element $x^- \in X^-$, there exists $a^- \in A^-$ such that*

$$a^- x^+_1 = 0,$$
$$a^- x^+_2 = x^-,$$

and

$$a^- x^+_2, \dots, a^- x^+_n, \dots \quad \textit{are linearly independent.}$$

Proof. We apply inductively the above theorem to choose a sequence b_n^- in A^- such that

(i) $b_1^{-\sigma} x_1^\sigma = 0$,

(ii) $b_2^- x_2^+ = x^- - b_1^- x_2^+$,

(iii) $b_n^- x_m^+ = 0$ if $m < n$,

(iv) if $n \geq 3$ then $b_n^- x_n^+$ is not a linear combination of $c_n^- x_1^+, \dots, c_n^- x_{n-1}^+$ where $c_n^- = b_1^- + \cdots + b_{n-1}^-$,

(v) $\|b_n^-\| < 2^{-n}$ if $n \geq 3$.

Therefore the element $a^{-\sigma} = \sum_{n=1}^\infty b_n^{-\sigma}$ satisfies our requirements. □

We follow the pattern established in [**7**], by applying inductively the above result, to get suitable sequences whose intrinsic properties will solve a great deal of our continuity problem.

Previously we introduced a useful notation. If a Banach pair (A^+, A^-) acts on a couple of Banach spaces (X^+, X^-), then, given $(a^\sigma, a^{-\sigma})$ in (A^+, A^-), we define a continuous linear operator L_{X^σ} $(a^\sigma, a^{-\sigma})$ from X^σ into itself by

$$L_{X^\sigma}(a^\sigma, a^{-\sigma})(x^\sigma) = a^\sigma(a^{-\sigma} x^\sigma).$$

Corollary 1.3.5. *Let (A^+, A^-) be a Banach pair acting irreducibly on a couple (X^+, X^-) of infinite-dimensional Banach spaces. If $x_1^+, \dots, x_n^+, \dots$ are linearly independent elements in X^+, then, for every nonzero element x^+ in X^+, there exist sequences $\{a_n^+\}$ in A^+ and $\{a_n^-\}$ in A^- such that, for every natural number n,*

$$L_{X^+}(a_n^+, a_n^-) \cdots L_{X^+}(a_1^+, a_1^-) x_k^+ = 0 \quad \textit{for} \quad k \leq 2n,$$

and

$$L_{X^+}(a_n^+, a_n^-) \cdots L_{X^+}(a_1^+, a_1^-) x_{2n+1}^+ = x^+.$$

2. Derivations of primitive Banach pairs.

2.1 *Methods of automatic continuity.* Recall that the continuity of an operator F acting between Banach spaces X and Y may be determined by means of its so-called *separating subspace*

$$\mathcal{S}(F) = \{y \in Y : \text{ there exists } x_n \to 0 \text{ in } X \text{ and } F(x_n) \to y\}.$$

By the closed graph theorem it follows that F is continuous if and only if $\mathcal{S}(F) = 0$. Also, if G is a continuous linear operator from Y to another Banach space Z, it is well known [**14**, Lemma 1.3] that,

$$\mathcal{S}(GF) = \overline{G(\mathcal{S}(F))}.$$

Therefore the composition operator GF is continuous if and only if $G(\mathcal{S}(F)) = 0$.

It is easy to check that, for a derivation (D_+, D_-) on a Banach pair (A^+, A^-) the separating couple $(\mathcal{S}(D_+), \mathcal{S}(D_-))$ is an ideal consisting of closed subspaces.

Finally, in order to activate the power of suitable sequences that we will construct, we require the continuity principle stated by Thomas in [**16**].

Theorem 2.1.1. *Let X be a Banach space, $\{S_n\}$ a sequence of continuous linear operators from X into itself and $\{R_n\}$ a sequence of continuous linear operators whose domain is X but which may map into other Banach spaces Y_n. If F is a possibly discontinuous linear operator from X into itself such that*

$$R_n F S_1 \cdots S_m \quad \textit{is continuous for} \quad m > n,$$

then

$$R_n F S_1 \cdots S_n \quad \textit{is continuous for sufficiently large } n.$$

2.2 *Continuity of derivations on primitive Banach pairs.* The aim of this section is to establish the continuity of the component operators D_-, D_+ for every derivation (D_-, D_+) on a primitive Banach pair (A^+, A^-). First we observe that (D_-, D_+) is also a derivation on the primitive Banach pair (A^-, A^+). Therefore, to reach our goal, it suffices to prove the continuity of one component operator.

From Theorem 1.2.1, (A^+, A^-) is faithfully and irreducibly represented on a suitable couple (X^+, X^-) of Banach spaces.

The methods of the proof depend on the dimension of the Banach spaces X^+ and X^-.

Lemma 2.2.1. *If either X^+ or X^- has a finite dimension, then D_+ is continuous.*

Proof. In this situation we observe that the middle multiplication operators on A^+ have finite rank. Thus, for all $a^-, c^- \in A^-$, the composition $D_- M_{A^+}(a^-, c^-)$ is continuous. Since $M_{A^+}(a^-, c^-)D_+ = D_- M_{A^+}(a^-, c^-) - M_{A^+}(D_- a^-, c^-) - M_{A^+}(a^-, D_- c^-)$, it follows that $M_{A^+}(a^-, c^-)D_+$ is continuous. Hence $\mathcal{S}(D_+) \subset \ker M_{A^+}(a^-, c^-)$ for all $a^-, c^- \in A^-$. Therefore $[A^- \mathcal{S}(D_+) A^-] = 0$ which shows that $\mathcal{S}(D_+) = 0$. Otherwise, by Lemma 1.2.3, $(\mathcal{S}(D_+), \mathcal{S}(D_-))$ acts irreducibly on (X^+, X^-) so

$$\begin{aligned} 0 &= [A^- \mathcal{S}(D_+) A^-] X^+ = A^-(\mathcal{S}(D_+)(A^- X^+)) \\ &= A^-(\mathcal{S}(D_+) X^-) = A^- X^+ = X^-, \end{aligned}$$

which is a contradiction. □

Lemma 2.2.2. *If both X^+ and X^- are infinite-dimensional, then D_+ is continuous.*

Proof. We choose a sequence $\{x_n^-\}$ of linearly independent vectors in X^-. For every nonzero $x^- \in X^-$, we apply Corollary 1.3.5 in order to obtain sequences $\{a_n^+\}$ in A^+ and $\{a_n^-\}$ in A^- satisfying

$$\begin{aligned} L_{X^-}(a_n^-, a_n^+) \cdots L_{X^-}(a_1^-, a_1^+) x_{2n+1}^- &= x^-, \\ L_{X^-}(a_n^-, a_n^+) \cdots L_{X^-}(a_1^-, a_1^+) x_k^- &= 0 \quad \text{for} \quad k \le 2n. \end{aligned}$$

We consider the sequence $\{S_n\}$ of continuous linear operators on A^+ given by $S_n = R_{A^+}(a_n^-, a_n^+)$, $n \in \mathbf{N}$, and the sequence $\{R_n\}$ of continuous linear operators from A^+ into X^+ given by $R_n(a^+) = a^+ x_{2n+1}^-$, $a^+ \in A^+$, $n \in \mathbf{N}$. We have, for all natural numbers m

and n,

$$\begin{aligned} R_n D_+ S_1 \cdots S_m &= R_n S_1 \cdots S_m D_+ \\ &+ R_n[R_{A^+}(D_- a_1^-, a_1^+) + R_{A^+}(a_1^-, D_+ a_1^+)] \cdots R_{A^+}(a_n^-, a_n^+) + \cdots \\ &\qquad + R_{A^+}(a_1^-, a_1^+) \cdots R_n[R_{A^+}(D_- a_n^-, a_n^+) + R_{A^+}(a_n^-, D_+ a_n^+)], \end{aligned}$$

which is continuous if $m > n$, since $R_n S_1 \cdots S_m = 0$. Thus, we can apply Theorem 2.1.1 in order to obtain that $R_n D_+ S_1 \cdots S_n$ is continuous for a sufficiently large n, so $R_n S_1 \cdots S_n D_+$ is continuous as well. This shows that $\mathcal{S}(D_+) \subset \ker(R_n S_1 \cdots S_n)$ so $\mathcal{S}(D_+)x^- = 0$. Hence $\mathcal{S}(D_+)X^- = 0$ and therefore, Lemma 1.2.3, $\mathcal{S}(D_+) = 0$. □

Putting together the above lemmas, we get the following theorem.

Theorem 2.2.3. *Every derivation (D^+, D^-) on a primitive Banach pair (A^+, A^-) consists of continuous linear operators D^+ and D^-.*

2.3. *Derivations on some operator pairs.* In this section we determine all the derivations on a relevant class of Banach pairs, namely, the standard operator pairs. To do this, we follow the pattern established earlier by Kaplansky in [**8**] and adapted by Chernoff in [**2**].

Given a couple (X^+, X^-) of Banach spaces, a Banach subpair (A^+, A^-) of the Banach pair $(L(X^-, X^+), L(X^+, X^-))$ is said to be a *standard operator pair* if every component A^σ contains the finite rank operators $FL(X^{-\sigma}, X^\sigma)$. From Example 1.1.2, it follows that such a pair is primitive and Theorem 2.2.3 shows that every derivation on it is continuous.

Theorem 2.3.1. *Let (A^+, A^-) be a standard operator pair on a couple (X^+, X^-) of Banach spaces, and let (D_+, D_-) be a derivation on (A^+, A^-). Then there exist continuous linear operators u^+ on X^+ and u^- on X^- such that, for all $a^+ \in A^+$ and $a^- \in A^-$,*

$$D_+ a^+ = a^+ u^- - u^+ a^+ \quad \textit{and} \quad D_- a^- = a^- u^+ - u^- a^-.$$

Proof. As is usual, given a continuous linear functional f on a Banach space X and an element y in a Banach space Y, we denote by $y \otimes f$ the continuous linear operator from X into Y given by $(y \otimes f)x = f(x)y$.

Fix elements $x^+ \in X^+$, $x^- \in X^-$ and continuous linear functionals f^+ on X^+ and f^- on X^- with $f^\sigma(x^\sigma) = 1$. Since $x^+ \otimes f^- = (x^+ \otimes f^-)(x^- \otimes f^+)(x^+ \otimes f^-)$, we have

$$\begin{aligned} D_+(x^+ \otimes f^-) = D_+&(x^+ \otimes f^-)(x^- \otimes f^-) \\ &+ (x^+ \otimes f^-)D_-(x^- \otimes f^+)(x^+ \otimes f^-) \\ &+ (x^+ \otimes f^+)D_+(x^+ \otimes f^-). \end{aligned}$$

By evaluating the above operators in the element x^-, we obtain that

$$f^+(D_+(x^+ \otimes f^-)x^-) + f^-(D_-(x^- \otimes f^+)x^+) = 0.$$

Now let

$$v^+ = (x^+ \otimes f^-)D_-(x^- \otimes f^+) - D_+(x^+ \otimes f^-)(x^- \otimes f^+) \in L(X^+)$$

and

$$v^- = (x^- \otimes f^+)D_+(x^+ \otimes f^-) - D_-(x^- \otimes f^+)(x^+ \otimes f^-) \in L(X^-).$$

We define a derivation (Δ_+, Δ_-) on (A^+, A^-) by

$$\Delta_+ a^+ = D_+ a^+ - (a^+ v^- - v^+ a^+)$$

and

$$\Delta_- a^- = D_- a^- - (a^- v^+ - v^- a^-).$$

Then $\Delta_-(x^- \otimes f^+)(x^+ \otimes f^-) + (x^- \otimes f^+)\Delta_+(x^+ \otimes f^-) = 0$. Therefore, for all $a^\sigma \in A^\sigma$,

$$\Delta_\sigma(a^\sigma(x^{-\sigma} \otimes f^{-\sigma})) = \Delta_\sigma(a^\sigma)(x^{-\sigma} \otimes f^{-\sigma}).$$

Now we define continuous linear operators w^+ on X^+ and w^- on X^- by

$$w^+(y^+) = \Delta_+(y^+ \otimes f^-)x^- \quad \text{and} \quad w^-(y^-) = \Delta_-(y^- \otimes f^+)x^+,$$

which satisfy

$$\Delta_\sigma(a^\sigma(x^{-\sigma}\otimes f^{-\sigma})) = w^\sigma a^\sigma(x^{-\sigma}\otimes f^{-\sigma}),$$

for all $a^\sigma \in A^\sigma$. Then, for all $a^\sigma \in A^\sigma$ and $y^\sigma \in X^\sigma$, we have

$$\begin{aligned} w^\sigma a^\sigma(y^{-\sigma}\otimes f^{-\sigma}) &= \Delta_\sigma[a^\sigma(y^{-\sigma}\otimes f^\sigma)(x^\sigma\otimes f^{-\sigma})] \\ &= (\Delta_\sigma a^\sigma)(y^{-\sigma}\otimes f^{-\sigma}) + a^\sigma\Delta_{-\sigma}(y^{-\sigma}\otimes f^\sigma)(x^{-\sigma}\otimes f^{-\sigma}) \\ &\quad + a^\sigma(y^{-\sigma}\otimes f^\sigma)\Delta_\sigma(x^{-\sigma}\otimes f^{-\sigma}), \end{aligned}$$

so that

$$\Delta_\sigma a^\sigma = w^\sigma - a^\sigma w^{-\sigma} - f^\sigma(w^\sigma x^\sigma)a^\sigma.$$

Now let $\lambda_+ = f^+(w^+x^+)$ and $\lambda_- = f^-(w^-x^-)$. Then $\lambda_+ + \lambda_- = 0$. If we define

$$u^+ = v^+ - w^+ + \frac{1}{2}\lambda_+ I_{X^+} \quad \text{and} \quad u^- = v^- - w^- + \frac{1}{2}\lambda_- I_{X^-},$$

then, for all $a^\sigma \in A^\sigma$, we conclude

$$D_\sigma a^\sigma = a^\sigma u^{-\sigma} - u^\sigma a^\sigma. \qquad \square$$

A representation theorem for derivations on standard operator algebras given by Chernoff in [**2**] follows from the above theorem.

Corollary 2.3.2. *Let A be a standard operator algebra on a Banach space X. Then every derivation D on A is such that*

$$Da = au - ua,$$

for a suitable continuous linear operator u on X. In particular, every derivation on $L(X)$ is inner.

Proof. Since the couple (D, D) is a derivation on the standard operator pair (A, A), there exist continuous linear operators u and v on X, such that

$$Da = av - ua = au - va,$$

for all $a \in A$. Therefore, $a(v-u) = -(v-u)a$ for all $a \in A$, which shows that $u = v$ and the result is proved. $\square$

3. Invariance of primitive ideals.

3.1 *Invariance principle.* Sinclair proved in [**13**] that every primitive ideal in a Banach algebra remains invariant by every continuous derivation on the algebra. Recently Thomas [**16**] showed that every, possibly discontinuous, derivation on a Banach algebra leaves invariant the primitive ideals in the algebra except perhaps a finite set of them which must have finite codimension. Now we translate both results to the pair framework. We start, following the strategy of Thomas, by showing that the continuity of a derivation pair is closely related with the invariance of primitive ideals under this derivation.

For a closed linear subspace M of a Banach space X we will denote by π_M the usual quotient map from X onto X/M.

Lemma 3.1.1. *Let T be a possibly discontinuous linear operator on a Banach space X, and let M be a closed linear subspace of X such that $\pi_M T^n$ is continuous for all $n \in \mathbf{N}$. Then there exists $C > 0$ such that*

$$\|\pi_M T^n\| \leq C^n, \quad \forall\, n \in \mathbf{N}.$$

Proof. We consider the closed linear subspace N of X given by

$$N = \{x \in M : T^n x \in M,\ \forall\, n \in \mathbf{N}\}.$$

From the continuity of $\pi_M T$ it follows that $\mathcal{S}(T) \subset \ker \pi_M = M$. For every natural number n, the linear operators $\pi_M T^n$ and $(\pi_M T^n)T$ are continuous so $(\pi_M T^n)(\mathcal{S}(T)) = 0$, that is, $T^n \mathcal{S}(T) \subset M$. Therefore, $\mathcal{S}(T) \subset N$ and the linear operator $T_N : X/N \to X/N$ given by $T_N(x+N) = T(x) + N$ is continuous. Now we consider the continuous linear operator ϕ from X/N onto X/M given by $\phi(x+N) = x+M$ and we observe that $\pi_M T^n = \phi T_N^n \pi_N$ for all $n \in \mathbf{N}$. Thus,

$$\|\pi_M T^n\| \leq \|\phi\| \|\pi_N\| \|T_N\|^n \leq \|T_N\|^n, \quad \forall\, n \in \mathbf{N}. \qquad \square$$

Now we state a useful duality between the invariance properties of the component operators of a derivation on a Banach pair.

Lemma 3.1.2. *Let (D_+, D_-) be a derivation on a Banach pair (A^+, A^-). If (P^+, P^-) is a primitive ideal of (A^+, A^-), then*

$$D_+(P^+) \subset P^+ \quad \Longleftrightarrow \quad D_-(P^-) \subset P^-.$$

Proof. By Theorem 1.2.1 (P^+, P^-) is the kernel of an irreducible representation from (A^+, A^-) on a suitable couple (X^+, X^-) of Banach spaces.

Assume that $D_+(P^+) \subset P^+$. Then $[A^+(D_-(P^-) + P^-)A^+] \subset P^+$ so that

$$0 = [A^+(D_-(P^-) + P^-)A^+]X^- = A^+(D_-(P^-) + P^-)X^+.$$

Therefore, $D_-(P^-) \subset P^-$. Otherwise, by Lemma 1.2.3, $(P^+, D_-(P^-) + P^-)$ acts irreducibly on (X^+, X^-) contradicting that

$$0 = A^+(D_-(P^-) + P^-)X^+.$$

On the other hand, if $D_-(P^-) \subset P^-$, then, by considering the derivation (D_-, D_+) on the Banach pair (A^-, A^+) we obtain that $D_+(P^+) \subset P^+$. $\square$

A notion of quasinilpotence, which will be crucial in our reasoning, may be introduced in the Banach pair setting as follows.

Let (A^+, A^-) be a Banach pair. We observe that for $(a^+, a^-) \in A^+ \times A^-$ the sequences $\{\|R_{A^+}(a^-, a^+)^n a^+\|^{1/n}\}$ and $\{\|R_{A^-}(a^+, a^-)^n a^-\|^{1/n}\}$ converge. Moreover, for all $n \in \mathbf{N}$, we have that

$$\begin{aligned}\|R_{A^+}(a^-, a^+)^{n+2} a^+\| &\le \|a^+\|^2 \|R_{A^-}(a^+, a^-)^{n+1} a^-\| \\ &\le \|a^+\|^2 \|a^-\|^2 \|R_{A^+}(a^-, a^+)^n a^+\|.\end{aligned}$$

Hence,

$$\lim \|R_{A^+}(a^-, a^+)^n a^+\|^{1/n} = \lim \|R_{A^-}(a^+, a^-)^n a^-\|^{1/n}.$$

This number will be denoted by $r(a^+,a^-)$. The couple (a^+,a^-) is said to be *quasinilpotent* if $r(a^+,a^-)=0$. If (a^+,a^-) is quasinilpotent, then $\sum R_{A^\sigma}(a^{-\sigma},a^\sigma)^n a^\sigma$ is absolutely convergent and $b^\sigma = \sum_{n=1}^\infty R_{A^\sigma}(a^{-\sigma},a^\sigma)^n a^\sigma$ is such that $a^\sigma + b^\sigma - [a^\sigma a^{-\sigma} b^\sigma] = 0$. Therefore, (a^+,a^-) is right-quasiregular in the sense of [**3**].

Example 3.1.3. Given a continuous linear operator F on a Banach space X, let $r(F)$ be its usual spectral radius, i.e., $r(F) = \lim \|F^n\|^{1/n}$.

Let (X^+,X^-) b a couple of Banach spaces and (T,S) a couple of continuous linear operators, $T \in L(X^-,X^+)$ and $S \in L(X^+,X^-)$. Then

$$r(T,S) = \lim \|(TS)^n T\|^{1/n} \le \lim \|(TS)^n\|^{1/n} \|T\|^{1/n} = r(TS)$$

and

$$\begin{aligned} r(TS) &= \lim \|(TS)^{n+1}\|^{1/(n+1)} = \lim \|((TS)^n T)S\|^{1/(n+1)} \\ &\le \lim \|(TS)^n T\|^{1/(n+1)} \|S\|^{1/(n+1)} = r(T,S). \end{aligned}$$

Therefore, $r(TS) = r(T,S) = r(S,T) = r(ST)$.

Let (I^+,I^-) be a *quasinilpotent ideal* of the Banach pair (A^+,A^-) which means that (a^+,a^-) is quasinilpotent for all $a^+ \in I^+$ and $a^- \in I^-$. Obviously, (I^+,I^-) is a right quasiregular ideal, so is contained in the radical of (A^+,A^-), see [**3**].

Theorem 3.1.4. *Let (D_+,D_-) be a derivation on a Banach pair (A^+,A^-) and (P^+,P^-) a primitive ideal of (A^+,A^-). Then $D_+(P^+) \subset P^+$ and $D_-(P^-) \subset P^-$ if and only if either $\pi_{P^+} D_+^n$ or $\pi_{P^-} D_-^n$ are continuous operators for all $n \in \mathbf{N}$.*

Proof. For the necessity we observe that (D_+,D_-) gives rise to a derivation (D_{+P^+}, D_{-P^-}) on the primitive Banach pair $(A^+/P^+, A^-/P^-)$ which must be continuous by Theorem 2.2.3. Hence, $\pi_{P^\sigma} D_\sigma^n = D_{\sigma P^\sigma} \pi_{P^\sigma}$ is continuous for all $n \in \mathbf{N}$.

In order to prove the converse, suppose that the operators $\pi_{P^+} D_+^n$ are continuous for all $n \in \mathbf{N}$. Then, by the above lemma, there exists a positive number C such that $\|\pi_{P^+} D_+^n\| \le C^n$ for all $n \in \mathbf{N}$. Now we observe that, for all $a^+ \in P^+$, $a^- \in P^-$ and every $n \in \mathbf{N}$,

$$\pi_{P^+} D_+^{2n+1} R_{A^+}(a^-, a^+)^n a^+ = (2n+1)! \pi_{P^+} R_{A^+}(D_- a^-, D_+ a^+)^n D_+ a^+.$$

Hence,

$$\begin{aligned} \|\pi_{P^+} R_{A^+}(D_- a^-, D_+ a^+)^n D_+ a^+\|^{1/n} \\ &= (2n+1)!^{-1/n} \|_{\pi_{P^+}} D_+^{2n+1} R_{A^+}(a^-, a^+)^n a^+\|^{1/n} \\ &\leq (2n+1)!^{-1/n} \|\pi_{P^+} D_+^{2n+1}\|^{1/n} \|R_{A^+}(a^-, a^+)^n a^+\|^{1/n} \\ &\leq (2n+1)!^{-1/n} C \|R_{A^+}(a^-, a^+)^n a^+\|^{1/n} \end{aligned}$$

which converges to zero. This proves that $r(D_+(a^+) + P^+, D_-(a^-) + P^-) = 0$ and $((D_+(P^+) + P^+)/P^+, (D_-(P^-) + P^-)/P^-)$ is a quasinilpotent ideal of $(A^+/P^+, A^-/P^-)$. Hence it is contained in the radical of $(A^+/P^+, A^-/P^-)$ which equals (0,0). Therefore, $D_+(P^+) \subset P^+$ and $D_+(P^-) \subset P^-$.

Now if the operators $\pi_{P^-} D_-^n$ are continuous for all $n \in \mathbf{N}$ then, from the above, it follows that the derivation (D_-, D_+) on the Banach pair (A^-, A^+) is such that P^- and P^+ remain invariant under D_- and D_+, respectively. □

The above invariance principle leads to the following invariance property for continuous derivations on Banach pairs.

Corollary 3.1.5. *Let (D_+, D_-) be a derivation, with continuous component operators, on a Banach pair (A^+, A^-). Then, for every primitive ideal (P^+, P^-) in (A^+, A^-) we have*

$$D_+(P^+) \subset P^+ \quad \text{and} \quad D_-(P^-) \subset P^-.$$

The above result provides an extension of a well-known Sinclair theorem about invariance of primitive ideals.

Corollary 3.1.6. *Primitive ideals in a Banach algebra remain invariant under every continuous derivation on this algebra.*

Proof. We note that every derivation D on a Banach algebra A gives rise to a derivation on the Banach pair (A, A), namely the couple

(D, D). Now, by Example 1.2.2, every primitive ideal P in A leads to a primitive ideal in (A, A) by considering the couple (P, P). Therefore, from Corollary 3.1.5 we obtain that $D(P) \subset P$ for every derivation D on A and every primitive ideal P in A. $\square$

3.2. *Invariance theorem.* The aim of this section is to show that, given a derivation on a Banach pair (A^+, A^-), a sufficiently large number of primitive ideals in this pair remain invariant under it. Our methods will depend on the dimension of the component Banach spaces, X^+ and X^-, of the couples (X^+, X^-) on which (A^+, A^-) acts irreducibly.

Proposition 3.2.1. *Let (P^+, P^-) be a primitive ideal in a Banach pair (A^+, A^-) such that $(A^+/P^+, A^-/P^-)$ is faithfully and irreducibly represented on a couple (X^+, X^-) of infinite-dimensional Banach spaces. Then, for every derivation (D_+, D_-) on (A^+, A^-)*

$$D_+(P^+) \subset P^+ \quad \textit{and} \quad D_-(P^-) \subset P^-.$$

Proof. By Theorem 3.1.4 it suffices to show that the operators $\pi_{P^+} D_+^n$ are continuous for all $n \in \mathbf{N}$.

If the result fails, we may define

$$k' = \max\{k \in \mathbf{N} \cup \{0\} : \pi_{P^+} D_+^k \text{ is continuous}\}.$$

Now we choose linearly independent elements $x_1^-, \dots, x_n^-, \dots$ in X^-. By applying Corollary 1.3.5, we obtain, for every $x^- \in X^-$, sequences $\{a_n^+\}$ and $\{a_n^-\}$ in A^+ and A^-, respectively, such that

$$L_{X^-}(\pi_{P^-} a_n^-, \pi_{P^+} a_n^+) \cdots L_{X^-}(\pi_{P^-} a_1^-, \pi_{P^+} a_1^+) x_{2n+1}^- = x^-$$

and

$$L_{X^-}(\pi_{P^-} a_n^-, \pi_{P^+} a_n^+) \cdots L_{X^-}(\pi_{P^-} a_1^-, \pi_{P^+} a_1^+) x_k^- = 0$$

for $k \leq 2n$. Now we consider the sequence $\{S_n\}$ of continuous linear operators on A^+ given by $S_n = R_{A^+}(a_n^-, a_n^+)$, $n \in \mathbf{N}$, and the sequence $\{R_n\}$ of continuous linear operators from A^+ into X^+, given by

$$R_n(a^+) = (\pi_{P^+} D_+^{k'} a^+) x_{2n+1}^-, \quad a^+ \in A^+, \ n \in \mathbf{N}.$$

Finally we write

$$b_m^+ = L_{A^+}(a_m^+, a_{m-1}^-) \cdots L_{A^+}(a_2^+, a_1^-)(a_1^+).$$

Then, for all natural numbers m and n,

$$\begin{aligned}
&R_n D_+ S_1 \cdots S_m a^+ \\
&\quad = (\pi_{P^+} D_+^{k'+1})(R_{A^+}(a_1^-, a_1^+) \cdots R_{A^+}(a_m^-, a_m^+) a^+) x_{2n+1}^- \\
&\quad = \pi_{P^+} D_+^{k'+1}([a^+ a_m b_m^+]) x_{2n+1}^- \\
&\quad = \sum_{i+j+k=k'+1} \frac{(k'+1)!}{i!j!k!} \\
&\qquad\qquad \cdot [(\pi_{P^+} D_+^i a^+)(\pi_{P^-} D_-^j a_m^-)(\pi_{P^+} D_+^k b_m^+)] x_{2n+1}^- \\
&\quad = [(\pi_{P^+} D_+^{k'+1} a^+)(a_m^- + P^-)(b_m^+ + P^+)] x_{2n+1}^- \\
&\qquad + \sum_{\substack{i+j+k=k'+1 \\ i \le k'}} \frac{(k'+1)!}{i!j!k!} \\
&\qquad\qquad \cdot [(\pi_{P^+} D_+^i a^+)(\pi_{P^-} D_-^j a_m^-)(\pi_{P^+} D_+^k b_m^+)] x_{2n+1}^- \\
&\quad = (\pi_{P^+} D_+^{k'+1} a^+) L_{X^-}(a_m^- + P^-, a_m^+ + P^+) \\
&\quad \cdots L_{X^-}(a_1^- + P^-, a_1^+ + P^+) x_{2n+1}^- \\
&\qquad + \sum_{\substack{i+j+k=k'+1, \\ i \le k'}} \frac{(k'+1)!}{i!j!k!} \\
&\qquad\qquad \cdot [(\pi_{P^+} D_+^i a^+)(\pi_{P^-} D_-^j a_m^-)(\pi_{P^+} D_+^k b_m^+)] x_{2n+1}^- .
\end{aligned}$$

Since $L_{X^-}(a_m^- + P^-, a_m^+ + P^+) \cdots L_{X^-}(a_1^- + P^-, a_1^+ + P^+) x_{2n+1}^- = 0$, if $m > n$, the operator $R_n D_+ S_1 \cdots S_m$ is continuous. By Theorem 2.1.1, it follows that the operator $R_n D_+ S_1 \cdots S_n$ is continuous for sufficiently large n so that the operator from A^+ into X^+ given by

$$a \longmapsto (\pi_{P^+} D_+^{k'+1} a^+) x^-,$$

is continuous. Therefore, $\mathcal{S}(\pi_{P^+} D_+^{k'+1}) X^- = 0$ so that $\mathcal{S}(\pi_{P^+} D_+^{k'+1}) = 0$ which contradicts the definition of k'. □

For a derivation in a Banach pair (A^+, A^-) we study the invariance of the primitive ideals (P^+, P^-) for which the quotient pair

$(A^+/P^+, A^-/P^-)$ acts faithfully and irreducibly on a couple of Banach spaces (X^+, X^-), one of them having finite dimension. In essence, to do this we will follow the pattern of [**17**], earlier established in [**18**]. However, several additional difficulties will have to be considered.

It will be crucial, in the sequel, to consider ideals (I^+, I^-) for which there exists (u^+, u^-) in (A^+, A^-) such that

$$a^+ - [a^+u^-u^+] \in I^+ \quad \text{and} \quad a^- - [u^-u^+a^-] \in I^-,$$

for all $a^+ \in A^+$ and $a^- \in A^-$.

Assume that (u^+, u^-) is such a couple for the ideal (I^+, I^-) and let (J^+, J^-) be an ideal in (A^+, A^-) such that $I^+ + J^+ = A^+$. Since $[A^-A^+A^-] = [A^-(I^+ + J^+)A^-] \subset I^- + J^-$, we have that $[u^-u^+a^-] \in I^- + J^-$ whenever a^- lies in A^-. Therefore, for every $a^- \in A^-$, we obtain

$$a^- = (a^- - [u^-u^+a^-]) + [u^-u^+a^-] \in I^- + J^-,$$

which shows that $I^- + J^- = A^-$.

Lemma 3.2.2. *Let $(P_1^+, P_1^+), \dots, (P_n^+, P_n^+)$ be ideals of a Banach pair (A^+, A^-) satisfying the following properties*:

1. *There exist $u_1^+, \dots, u_n^+ \in A^+$ and $u_1^-, \dots, u_n^- \in A^-$ such that*

$$a^+ - [a^+u_k^-u_k^+] \in P_k^+ \quad \text{and} \quad a^- - [u_k^-u_k^+a^-] \in P_k^-,$$

for all $a^+ \in A^+$, $a^- \in A^-$ and $k = 1, \dots, n$.

2. *$P_i^+ + P_j^+ = A^+$ and so $P_i^- + P_j^- = A^-$ if $i \neq j$.*

Then the homomorphism from (A^+, A^-) onto $(\oplus_{k=1}^n(A^+/P_k^+), \oplus_{k=1}^n(A^-/P_k^-))$ given by

$$a^+ \longmapsto (a^+ + P_1^+, \dots, a^+ + P_n^+)$$

and

$$a^- \longmapsto (a^- + P_1^-, \dots, a^- + P_n^-)$$

is onto.

Proof. We observe that A^+ and A^- are Banach algebras with respect to the products

$$a^+ \cdot b^+ = [a^+ u^- b^+] \quad \text{and} \quad a^- \cdot b^- = [a^- u^+ b^-],$$

and the norms

$$|a^+| = \|u^-\|\|a^+\| \quad \text{and} \quad |a^-\| = \|u^+\|\|a^-\|,$$

respectively. Also P_k^+ is an ideal in A^+ with u_k^+ as a right modular unit, for $k = 1, \dots, n$. Similarly, P_k^- is an ideal in A^- with u_k^- as a left modular unit.

Now we observe that the reasoning used in [**18**, Lemma 2.2] and [**17**, Lemma 3.2.1] may be followed even if we have either right modular units or left modular units instead of the units required in those lemmas, respectively. Therefore, our result is shown. □

Lemma 3.2.3. *Let (A^+, A^-) be a Banach pair faithfully and irreducibly represented on a couple (X^+, X^-) of Banach spaces, with* $\dim X^- < \infty$ *and* $\dim X^- \le \dim X^+$*. Then there exist elements u^+ in A^+ and u^- in A^- such that*

$$L_{X^-}(u^-, u^+) = I_{X^-}.$$

Equivalently, for all $a^+ \in A^+$ and $a^- \in A^-$,

$$[a^+ u^- u^+] = a^+ \quad \textit{and} \quad [u^- u^+ a^-] = a^-.$$

Proof. Let $\{x_1^-, \dots, x_n^-\}$ be a basis in X^- and $x_1^+, \dots, x_n^+$ be linearly independent elements in X^+. From Theorem 1.3.3 there exist elements $u^+ \in A^+$ and $u^- \in A^-$ with $u^\sigma x_k^{-\sigma} = x_k^\sigma$ for $k = 1, \dots, n$. Therefore, it is clear that $L_{X^-}(u^-, u^+) = I_{X^-}$.

For all $a^+ \in A^+$ and $a^- \in A^-$, we have

$$(a^+ - [a^+ u^- u^+])X^- = 0 \quad \text{and} \quad (a^- - [u^- u^+ a^-])X^+ = 0.$$

Hence, $[a^+ u^- u^+] = a^+$ and $[u^- u^+ a^-] = a^-$. On the other hand, if $[a^+ u^- u^+] = a^+$, for every $a^+ \in A^+$, then $A^+(L_{X^-}(u^-, u^+) - I_{X^-})x^- = 0$ so that $L_{X^-}(u^-, u^+) = I_{X^-}$. □

A Banach pair (A^+, A^-), whose product is nonzero, having no nontrivial ideals is said to be *simple*.

Corollary 3.2.4. *Let (A^+, A^-) be a Banach pair faithfully and irreducibly represented on a couple (X^+, X^-) of Banach spaces, one of them having finite dimension. Then (A^+, A^-) is simple.*

Proof. Let (I^+, I^-) be an ideal of (A^+, A^-). From Lemma 1.2.3 it follows that (X^+, X^-) is a faithful and irreducible (I^+, I^-)-module.

If $\min\{\dim X^+, \dim X^-\} = \dim X^-$, then we apply the above lemma to obtain elements $u^+ \in I^+$ and $u^- \in I^-$ with $[a^+u^-u^+] = a^+$, for all $a^+ \in A^+$. Since $u^+ \in I^+$ for all a^+ in A^+, we have that $a^+ = [a^+u^-u^+]$ lies in I^+ so $A^+ = I^+$ and $A^- = I^-$.

If the above equality does not hold, by considering the faithful and irreducible (I^-, I^+)-module, (X^-, X^+), the result follows. $\square$

Lemma 3.2.5. *Let $(P_1^+, P_1^+), \dots, (P_n^+, P_n^+)$ be pairwise different primitive ideals of a Banach pair (A^+, A^-) such that, for every $k = 1, \dots, n$, the quotient pair $(A^+/P_k^+, A^-/P_k^-)$ is faithfully and irreducibly represented onto a couple (X_k^+, X_k^-) of Banach spaces with $\dim X_k^- < \infty$ and $\dim X_k^- \leq \dim X_k^+$. Then the homomorphism from (A^+, A^-) to $(\oplus_{k=1}^n (A^+/P_k^+), \oplus_{k=1}^n (A^-/P_k^-))$ given by*

$$a^+ \longmapsto (a^+ + P_1^+, \dots, a^+ + P_n^+)$$

and

$$a^- \longmapsto (a^- + P_1^-, \dots, a^- + P_n^-)$$

is onto.

Proof. By applying Lemma 3.2.3 there exist $u_1^+, \dots, u_n^+$ in A^+ and $u_1^-, \dots, u_n^-$ in A^- satisfying the first requirement in Lemma 3.2.2. We observe that $((P_i^+ + P_j^+/P_i^+), (P_i^- + P_j^-/P_i^-))$ and $((P_i^+ + P_j^+/P_j^+), (P_i^- + P_j^-/P_j^-))$ are ideals of the Banach pairs $((A^+/P_i^+), (A^-/P_i^-))$ and $((A^+/P_j^+), (A^-/P_j^-))$ respectively. Since Banach pairs are simple, Corollary 3.2.4, it follows that $P_i^+ = P_j^+$ and $P_i^- = P_j^-$ whenever $P_i^+ + P_j^+ \neq A^+$ for $i \neq j$, contradicting our assumption. Therefore, the second assertion in Lemma 3.2.2 is fulfilled and our result follows from that lemma. □

Lemma 3.2.6. *Let R, S and T be linear operators on a finite-dimensional Banach space X. Then there exist complex numbers λ and μ (which may be chosen with arbitrarily small modulus) such that $R+\lambda S+\mu T+\lambda\mu I_X$ is invertible.*

Proof. Consider the nonzero complex polynomial in two complex variables given by

$$p(\lambda,\mu)=\det(R+\lambda S+\mu T+\lambda\mu I_X).$$

Since the set $\{(\lambda,\mu)\in\mathbf{C}^2 : p(\lambda,\mu)=0\}$ has no interior points, our result follows. □

Lemma 3.2.7. *Let (A^+,A^-) be a Banach pair and $\{(P_n^+,P_n^-)\}$ be a sequence of pairwise different primitive ideals of (A^+,A^-) for which each quotient pair $(A^+/P_n^+,A^-/P_n^-)$ is faithfully and irreducibly represented into a couple of Banach spaces (X_n^+,X_n^-) with $\dim X_n^-<\infty$ and $\dim X_n^-\le\dim X_n^+$. Then there exist sequences $\{a_n^+\}$ in A^+ and $\{a_n^-\}$ in A^- such that $a_m^+\in P_n^+$ and $a_m^-\in P_n^-$, whenever $m>n$, and $L_{X_n^-}(a_m^-+P_n^-,a_m^++P_n^+)$ is invertible, if $m\le n$.*

Proof. By Lemma 3.2.3, there exist sequences $\{u_n^+\}$ in A^+ and $\{u_n^-\}$ in A^- such that $L_{X_n^+}(u_n^-+P_n^-,u_n^++P_n^+)=I_{X_n^-}$ for all $n\in\mathbf{N}$.

Let k be a natural number. By applying Lemma 3.2.5 we obtain elements $b_k^+,b_{k+1}^+,\dots,$ in A^+ and $b_k^-,b_{k+1}^-,\dots$ in A^- such that, for $j=k,k+1,\dots,$

$$b_j^\sigma+P_i^\sigma=P_i^\sigma \quad\text{for}\quad i=1,\dots,j-1$$

and

$$b_j^\sigma+P_j^\sigma=u_j^\sigma+P_j^\sigma.$$

For $j\ge k+1$, given complex numbers $\lambda_k,\lambda_{k+1},\dots,\lambda_j$ and $\mu_k,\mu_{k+1},$

$\dots, \mu_j$, we have

$$\begin{aligned}
L_{X_j^-}\Bigg(\sum_{i=k}^{j}(\lambda_i b_i^- + P_j^-), \sum_{i=k}^{j}(\mu_i b_i^+ + P_j^+)\Bigg)& \\
= L_{X_j^-}&\Bigg(\sum_{i=k}^{j-1}(\lambda_i b_i^- + P_j^-), \sum_{i=k}^{j-1}(\mu_i b_i^+ + P_j^+)\Bigg) \\
&+ \lambda_j L_{X_j^-}\Bigg(u_j^- + P_j^-, \sum_{i=k}^{j-1}(\mu_i b_i^+ + P_j^+)\Bigg) \\
&+ \mu_j L_{X_j^-}\Bigg(\sum_{i=k}^{j-1}(\lambda_i b_i^- + P_j^-), u_j^+\Bigg) \\
&+ \lambda_j \mu_j I_{X_j^-}.
\end{aligned}$$

From Lemma 3.2.6 we get inductively two sequences of complex numbers $\{\lambda_n\}_{n=k}^{\infty}$ and $\{\mu_n\}_{n=k}^{\infty}$ such that

$$\lambda_k = \mu_k = 1,$$
$$L_{X_j^-}\Bigg(\sum_{i=k}^{j}(\lambda_i b_i^- + P_j^-), \sum_{i=k}^{j}(\mu_i b_i^+ + P_j^+)\Bigg)$$

is invertible if $j \geq k+1$, and

$$\|\lambda_j b_j^-\|, \|\mu_j b_j^+\| \leq 2^{k-j} \quad \text{if } j \geq k.$$

Therefore, by defining

$$a_k^+ = \sum_{j=k}^{\infty} \mu_j b_j^+ \quad \text{and} \quad a_k^- = \sum_{j=k}^{\infty} \lambda_j b_j^-,$$

the sequences $\{a_n^+\}$ and $\{a_n^-\}$ have the required properties. $\square$

Proposition 3.2.8. *Let (D_+, D_-) be a derivation on a Banach pair (A^+, A^-). Let $\mathcal{P}$ be the set of the primitive ideals (P^+, P^-) of (A^+, A^-) such that*:

1. *the quotient pair* $(A^+/P^+, A^-/P^-)$ *is faithfully and irreducibly represented on a couple* (X^+, X^-) *of Banach spaces, one of them having finite dimension,*

2. $D_+(P^+) \not\subset P^+$, *or equivalently,* $D_-(P^-) \not\subset P^-$.

Then $\mathcal{P}$ *is a finite set.*

Proof. If the result does not hold, by considering the Banach pair (A^-, A^+) instead of (A^+, A^-) if necessary, we get a sequence $\{(P_n^+, P_n^-)\}$ of pairwise different primitive ideals of (A^+, A^-) for which $D_+(P_n^+) \not\subset P_n^+$. Also there exists a faithful and irreducible representation (Φ_n^+, Φ_n^-) from the quotient pair $(A^+/P_n^+, A^-/P_n^-)$ on a couple of Banach spaces (X_n^+, X_n^-) with $\dim X_n^- < \infty$ and $\dim X_n^- \le \dim X_n^+$.

By Theorem 3.1.4, for every natural number n there exists $k_n \in \mathbf{N}$ such that $\pi_{P_n^+} D_+^{k_n}$ is discontinuous. Indeed, k_n may be chosen such that $\pi_{P_n^+} D_+^k$ is discontinuous if $k = k_n$ but continuous if $k < k_n$.

Let $\{a_n^+\}$ and $\{a_n^-\}$ be the sequences obtained from the above lemma. We consider the sequences of continuous linear operators, $\{S_n\}$ and $\{R_n\}$ given by

$$S_n = R_{A^+}(a_n^-, a_n^+)$$

and

$$R_n = \Phi_n^+ \pi_{P_n^+} D_+^{k_n - 1}.$$

We observe that, for all $b^+ \in A^+$ and $m, n \in \mathbf{N}$,

$$\begin{aligned}
&\Phi_n^+ \pi_{P_n^+} R_{A^+}(a_1^-, a_1^+) \cdots R_{A^+}(a_m^-, a_m^+) b^+ \\
&\qquad = \Phi_n^+ R_{A^+/P_n^+}(a_1^- + P_n^-, a_1^+ + P_n^+) \\
&\qquad\quad \cdots R_{A^+/P_n^+}(a_m^- + P_n^-, a_m^+ + P_n^+)(b^+ + P_n^+) \\
&\qquad = \Phi_n^+(b^+ + P_n^+) \Phi_n^-(a_m^- + P_n^-) \Phi_n^+(a_m^+ + P_n^+) \\
&\qquad\quad \cdots \Phi_n^-(a_1^- + P_n^-) \Phi_n^+(a_1^+ + P_n^+) \\
&\qquad = \Phi_n^+(b^+ + P_n^+) L_{X_n^-}(a_m^- + P_n^-, a_m^+ + P_n^+) \\
&\qquad\quad \cdots L_{X_n^-}(a_1^- + P_n^-, a_1^+ + P_n^+),
\end{aligned}$$

which equals zero if $m > n$ since in such a case $a_m^+ \in P_n^+$. Moreover, by writing $b_m^+ = L_{A^+}(a_m^+, a_{m-1}^-) \cdots L_{A^+}(a_2^+, a_1^-)(a_1^+)$, we have

$$\begin{aligned}
&R_n D_+ S_1 \cdots S_m a^+ \\
&\quad = \Phi_n^+ \pi_{P_n^+} D_+^{k_n} R_{A^+}(a_1^-, a_1^+) \\
&\quad\quad \cdots R_{A^+}(a_m^-, a_m^+) a^+ \\
&\quad = \Phi_n^+ \pi_{P_n^+} [a^+ a_m^- b_m^+] \\
&\quad = \Phi_n^+ \pi_{P_n^+} \bigg(\sum_{i+j+k=k+n} \frac{k_n!}{i!j!k!} [(D_+^i a^+)(D_-^j a_m^-)(D_+^k b_m^+)] \bigg) \\
&\quad = \Phi_n^+ \pi_{p_n^+} [(D_+^{k_n} a^+) a_m^- b_m^+] \\
&\quad\quad + \Phi_n^+ \bigg(\sum_{\substack{i+j+k=k_n, \\ i<k_n}} \frac{k_n!}{i!j!k!} [(\pi_{P_n^+} D_+^i a^+)(\pi_{P_n^-} D_-^j a_m^-)(\pi_{P_n^+} D_+^k b_m^+)] \bigg) \\
&\quad = \Phi_n^+ \pi_{p_n} + R_A + (a_1^-, a_1^+) \cdots R_A + (a_m^-, a_m^+)(D_+^{k_n} a^+) \\
&\quad\quad + \Phi_n^+ \bigg(\sum_{\substack{i+j+k=k_n \\ i<k_n}} \frac{k_n!}{i!j!k!} [(\pi_{P_n^+} D_+^i a^+)(\pi_{P_n^-} D_-^j a_m^-)(\pi_{P_n^+} D_+^k b_m^+)] \bigg) \\
&\quad = \Phi_n^+ \pi_{P_n^+} (D_+^{k_n} a^+) L_{X_n^-}(a_m^- + P_n^-, a_m^+ + P_n^+) \\
&\quad\quad \cdots L_{X_n^-}(a_1^- + P_n^-, a_1^+ + P_n^+) \\
&\quad\quad + \Phi_n^+ \bigg(\sum_{\substack{i+j+k=k_n \\ i<k_n}} \frac{k_n!}{i!j!k!} [(\pi_{P_n^+} D_+^i a^+)(\pi_{P_n^-} D_-^j a_m^-)(\pi_{P_n^+} D_+^k b_m^+)] \bigg).
\end{aligned}$$

Thus $R_n D_+ S_1 \cdots S_m a^+$ is continuous if $m > n$. By Theorem 2.1.1, the operator $R_n D_+ S_1 \cdots S_n a^+$ and, therefore, also the operator

$$\begin{aligned}
a^+ \longmapsto (\Phi_n^+ \pi_{P_n^+} (D_+^{k_n} a^+)) L_{X_n^-}(a_n^- + P_n^-, a_n^+ + P_n^+) \\
\cdots L_{X_n^-}(a_1^- + P_n^-, a_1^+ + P_n^+),
\end{aligned}$$

from A^+ into $L(X_n^-, X_n^+)$ is continuous for a sufficiently large n. Since the operators $L_{X_n^-}(a_n^- + P_n^-, a_n^+ + P_n^+), \dots, L_{X_n^-}(a_1^- + P_n^-, a_1^+ + P_n^+)$ are invertible, it follows that $\Phi_n^+ \pi_{P_n^+} D_+^{k_n}$ is continuous, so that $\mathcal{S}(\pi_{P_n^+} D_+^{k_n}) \subset \ker \Phi_n^+ = 0$. Thus $\pi_{P_n^+} D_+^{k_n}$ is continuous, contradicting the choice of k_n, and our result is proved. $\square$

Since every primitive ideal has to be like one of those considered in either Proposition 3.2.1 or 3.2.8, the following result is immediate.

Theorem 3.2.9. *Let (D_+, D_-) be a derivation on a Banach pair (A^+, A^-). Then*

$$D_+(P^+) \subset P^+ \quad \textit{and} \quad D_-(P^-) \subset P^-$$

for every primitive ideal (P^+, P^-) of (A^+, A^-) except for a finite set of them, for which the quotient pair $(A^+/P^+, A^-/P^-)$ is faithfully and irreducibly represented on a couple of Banach spaces (X^+, X^-), one of them having finite dimension.

The above invariance properties for primitive ideals in Banach pairs imply the known invariance properties for primitive ideals in Banach algebras given in [**16**]. Indeed, for a derivation D on a Banach algebra A and a primitive ideal P of A, we apply the above invariance theorem to the primitive ideal (P, P), considering the derivation (D, D) on the Banach pair (A, A), and the theorem about invariance of primitive ideals proved by Thomas follows.

Corollary 3.2.10. *Let D be a derivation on a Banach algebra A. All the primitive ideals of A, except perhaps a finite set of them which must necessarily have finite codimensions, remain invariant under D.*

4. Continuous derivations.

4.1 *Automatic continuity theorem.* Now we prove that, for a derivation on a semi-simple Banach pair, there exists a family of invariant primitive ideals whose intersection is zero. From this, we deduce the continuity of the component operators of that derivation.

Lemma 4.1.1. *Let $(P_1^+, P_1^-), \dots, (P_n^+, P_n^-)$ be primitive ideals of a Banach pair (A^+, A^-) which do not remain invariant under a derivation (D_+, D_-). If (Q^+, Q^-) is an ideal of (A^+, A^-) with $P_1^+ \cap \dots \cap P_n^+ \cap Q^+ = 0$, then $Q^+ = 0$.*

Proof. Assume that $n = 1$, and let (X_1^+, X_1^-) be a couple of

Banach spaces on which (A^+, A^-) is irreducibly represented with kernel (P_1^+, P_1^-).

We claim that $Q^+X_1^- = 0$. Otherwise $Q^+ \not\subset P_1^+$ and, by Lemma 1.2.3, (Q^+, Q^-) acts irreducibly on (X_1^+, X_1^-). Since

$$\begin{aligned}[(D_+P_1^+)Q^-Q^+] \subset & D_+[P_1^+Q^-Q^+] - [P_1^+(D_-Q^-)Q^+] \\ & - [P_1^+Q^-(D_+Q^+)] \subset P_1^+ \cap Q^+ = 0,\end{aligned}$$

we have

$$0 = [(D_+P_1^+)Q^-Q^+]X^- = D_+(P_1^+)(Q^-(Q^+X^-)) = D_+(P_1^+)X^-.$$

Thus, $D_+(P_1^+) \subset P_1^+$ and, by Lemma 3.1.2, $D_-(P_1^-) \subset P_1^-$ contradicting our assumption. Therefore, $Q^+X = 0$ and so $Q^+ \subset P_1^+$ which implies that $Q^+ = 0$.

Assume that the result holds for a natural number n. Let (P_1^+, P_1^-), $\dots, (P_n^+, P_n^-), (P_{n+1}^+, P_{n+1}^-)$, and (Q^+, Q^-) be ideals of (A^+, A^-) satisfying the requirements of the lemma. Then we apply the above step to the ideals

$$(P_{n+1}^+, P_{n+1}^-) \quad \text{and} \quad (P_1^+ \cap \cdots \cap P_n^+ \cap Q^+, P_1^- \cap \cdots \cap P_n^- \cap Q^-)$$

to obtain that $P_1^+ \cap \cdots \cap P_n^+ \cap Q^+ = 0$. By the induction assumption it follows that $Q^+ = 0$. $\square$

Theorem 4.1.2. *Every derivation on a semi-simple Banach pair consists of continuous operators.*

Proof. Let (D_+, D_-) be a derivation on a semi-simple Banach pair (A^+, A^-). By Theorem 3.2.9, $D_+(P^+) \subset P^+$ and $D_-(P^-) \subset P^-$ for all primitive ideal (P^+, P^-), except perhaps for a finite set of them $(P_1^+, P_1^-), \dots, (P_n^+, P_n^-)$. If (Q^+, Q^-) is the ideal of (A^+, A^-) obtained as the intersection of all the invariant primitive ideals in (A^+, A^-), then

$$\begin{aligned}&(P_1^+ \cap \cdots \cap P_n^+ \cap Q^+, P_1^- \cap \cdots \cap P_n^- \cap Q^-) \\ &\qquad = \operatorname{Rad}(A^+, A^-) = (0, 0).\end{aligned}$$

Now we apply the above lemma to obtain that $Q^+ = 0$.

From Theorem 3.1.3 it follows that $\mathcal{S}(D_+)$ is contained in P^+ for every invariant primitive ideal (P^+, P^-) in (A^+, A^-) and so $\mathcal{S}(D_+) \subset Q^+ = 0$. Therefore, D_+ is continuous.

Finally, by considering the derivation (D_-, D_+) on the semi-simple Banach pair (A^-, A^+) the continuity of D_- follows. $\square$

Now the classical Johnson-Sinclair theorem can be obtained by considering, for every derivation D on a semi-simple Banach algebra A, the derivation (D, D) on the semi-simple Banach pair (A, A), Example 1.2.2.

Corollary 4.1.3. *Every derivation on a semi-simple Banach algebra is continuous.*

4.2. *A Kleinecke type theorem.* The continuity of the component operators of a derivation on a Banach pair is a useful tool for inquiring about the ranges of the components. We prove a result about derivations with continuous components whose second iterations annihilate a couple of elements in a Banach pair.

Theorem 4.2.1. *Let (A^+, A^-) be a Banach pair and (D_+, D_-) be a derivation on (A^+, A^-) with continuous components. If $D_+^2 a^+ = 0$ and $D_-^2 a^- = 0$ for some a^+ in A^+ and a^- in A^-, then the couple $(D_+ a^+, D_- a^-)$ is quasinilpotent.*

Proof. Consider the subalgebra $RM(A^+, A^-)$ of $L(A^+, A^+)$ generated by all right multiplication operators on A^+, and let A be the Banach algebra obtained as the operator-norm closure of $RM(A^+, A^-)$ in $L(A^+, A^+)$. For all $b^+ \in A^+$ and $b^- \in A^-$ we have

$$D_+ R_{A^+}(b^-, b^+) - R_{A^+}(b^-, b^+) D_+ = R_{A^+}(D_- b^-, b^+) + R_{A^+}(b^-, D_+ b^+).$$

Hence the subalgebra of those elements T in $RM(A^+, A^-)$ for which $D_+ T - T D_+$ lies in $RM(A^+, A^-)$ equals $RM(A^+, A^-)$ since it contains all right multiplication operators on A^+. Therefore, we can define a derivation Δ on $RM(A^+, A^-)$ by

$$\Delta(T) = D_+ T - T D_+$$

for all $T \in RM(A^+, A^-)$. Since this derivation is continuous, we extend it to a (unique) derivation on A, which we also denote by Δ.

Now we observe that

$$\Delta(R_{A^+}(D_-a^-, a^+) + R_{A^+}(a^-, D_+a^+)) = 2R_{A^+}(D_-a^-, D_+a^+)$$

and

$$\Delta^2(R_{A^+}(D_-a^-, a^+) + R_{A^+}(a^-, D_+a^+)) = 0.$$

By the classical Kleinecke theorem, it follows that $R_{A^+}(D_-a^-, D_+a^+)$ is a quasinilpotent operator in $L(A^+, A^+)$. Therefore,

$$\begin{aligned} r(D_+a^+, D_-a^-) &= \lim \|R_{A^+}(D_-a^-, D_+a^+)^n D_+a^+\|^{1/n} \\ &\leq \lim(\|R_{A^+}(D_-a^-, D_+a^+)^n\|^{1/n} \|D_+a^+\|^{1/n}) \\ &= 0. \qquad \square \end{aligned}$$

Let $S \in L(X^+)$ and $T \in L(X^-)$ be continuous linear operators on Banach spaces X^+ and X^-. By an operator *intertwining* with the couple (S, T) we mean a linear operator F from X^+ to X^- such that

$$TF = FS.$$

Corollary 4.2.2. *Let $S \in L(X^+)$ and $T \in L(X^-)$ where X^+ and X^- are Banach spaces. If $F^+ \in L(X^-, X^+)$ and $F^- \in L(X^+, X^-)$ are such that the operators $TF^- - F^-S$ and $SF^+ - F^+T$ are intertwining with (S, T), then the operators $(TF^- - F^-S)(SF^+ - F^+T)$ and $(SF^+ - F^+T)(TF^- - F^-S)$ are quasinilpotent.*

Proof. We consider the derivation (D_+, D_-) on the Banach pair $(L(X^-, X^+), L(X^+, X^-))$ given by

$$D_+(F^+) = SF^+ - F^+T \quad \text{and} \quad D_-(F^-) = TF^- - F^-S,$$

for all $F^+ \in L(X^-, X^+)$ and $F^+ \in L(X^+, X-)$. This derivation whose components are continuous is such that $D_+^2(F^+) = 0$ and $D_-^2(F^-) = 0$. From Theorem 4.2.1 it follows that the couple $(SF^+ - F^+T, TF^- -$

$F^{-}S)$ is quasinilpotent which proves the result having Example 3.1.3 in mind. □

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