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CONVEXITY, SCHUR-CONVEXITY AND BOUNDS FOR THE GAMMA FUNCTION INVOLVING THE DIGAMMA FUNCTION

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ABSTRACT. We consider inequalities for the ratio $\Gamma(x+\beta)/\Gamma(x)$, with bounds expressed in terms of the digamma function or its derivatives. We show that this type of inequalities naturally arises from convexity or Schur-convexity of some functions. As a result, we re-derive, generalize or improve several inequalities due to Gautschi, Kershaw and Alzer. Also we present some new inequalities of the type introduced by Gurland.

1. Introduction. Many authors investigated inequalities for the ratio

$$Q(x,\beta)=\frac{\Gamma(x+\beta)}{\Gamma(x)},\quad x>0,\beta>0, \tag{1}$$

see the bibliography in [**2**]. In this paper we consider the bounds for (1) that involve the digamma function $\Psi=\Gamma'/\Gamma$ or its derivatives.

The first result in this area is due to Gautschi [**5**]:

$$Q(x,\beta)<\exp(\beta\Psi(x+\beta)),\quad 0<\beta<1, x>0. \tag{2}$$

Kershaw [**9**] improves and complements Gautschi's bound to

$$\exp(\beta\Psi(x+\beta-1+\sqrt{1-\beta}))<Q(x,\beta)<\exp(\beta\Psi(x+\beta/2)), \tag{3}$$

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for $0 < \beta < 1$ and $x - 1 + \beta > 0$. These inequalities were also investigated in [**3**]. Further, a result of Alzer [**2**] reads

$$
\begin{aligned}
&\frac{(x+\beta)^{x+\beta-1/2}}{x^{x-1/2}} \exp\left(-\beta + \frac{1}{12}(\Psi'(x+\beta) - \Psi'(x))\right) \\
&\qquad < Q(x,\beta) < \frac{(x+\beta)^{x+\beta-1/2}}{x^{x-1/2}} \\
&\qquad\quad \cdot \exp\left(-\beta + \frac{1}{12}(\Psi'(x+\beta+a) - \Psi'(x+a))\right),
\end{aligned} \tag{4}
$$

for $\beta \in (0,1)$, $x - 1 + \beta > 0$ and $a \geq 1/2$.

As we shall show in the subsequent sections, inequalities of this type can be obtained as a natural consequence of convexity of Schur-convexity of certain functions. This approach outlines a general method of producing and sharpening the inequalities of mentioned type. In [**11**], we applied a similar method based on convexity for obtaining bounds for Q in terms of elementary functions.

2. Convexity and bounds for the ratio Q. The results in this section are based on the following key theorem.

Theorem 1. *For $a > 0$, let*

$$
F_a(x) = \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log x - \frac{1}{12}\Psi'(x+a). \tag{5}
$$

Then

(i) *the function $x \mapsto F_0(x)$ is strictly concave on $x > 0$.*

(ii) *For $a \geq 1/2$, the function $x \mapsto F_a(x)$ is strictly convex on $x > 0$.*

Proof. Using the expressions for the second derivative of the function

$x \mapsto \log \Gamma(x)$ (see [**1**], for example), we get

$$\begin{aligned}
F_a''(x) &= \sum_{k=0}^{+\infty} \frac{1}{(x+k)^2} - \frac{1}{x} - \frac{1}{2x^2} - \sum_{k=0}^{+\infty} \frac{1}{2(x+k+a)^4} \\
&= \sum_{k=0}^{+\infty} \bigg(\frac{1}{(x+k)^2} - \frac{1}{x+k} + \frac{1}{x+k+1} \\
&\qquad - \frac{1}{2(x+k)^2} + \frac{1}{2(x+k+1)^2} - \frac{1}{2(x+k+a)^4} \bigg) \\
&= \sum_{k=0}^{+\infty} \left(\frac{1}{2(x+k)^2(x+k+1)^2} - \frac{1}{2(x+k+a)^4} \right).
\end{aligned}$$

Now from $(x+k)^2(x+k+1)^2 > (x+k)^4$ for $x > 0$ and $k \geq 0$, it follows that $F_0''(x) < 0$ for $x > 0$ and (i) is proved. To prove (ii), note that by log-concavity of the function $x \mapsto (x+k)^2$,

$$(x+k)^2(x+k+1)^2 < (x+k+1/2)^4, \quad x > 0, k \geq 0,$$

and therefore $F_a''(x) \geq F_{1/2}''(x) > 0$ for $a \geq 1/2$. $\square$

We will now show that both inequalities in (4) can be derived from Theorem 1, with an expanded domain.

Corollary 1. *Inequalities* (4) *hold for* $x > 0$, $\beta > 0$ *and* $a \geq 1/2$.

Proof. Firstly we note that, for each $a > 0$,

$$F_a'(x) = \Psi(x) - \log x + \frac{1}{2x} - 1 - \frac{1}{12}\Psi''(x+a)$$

and, using the asymptotic formula for Ψ and the expression for Ψ'' [**1**, Chapter 6], we find that

$$\lim_{x \to +\infty} F_a'(x) = -1.$$

By concavity of F_0, its first derivative is a decreasing function on $(0, +\infty)$ and so,

$$F_0'(x) > \lim_{x \to +\infty} F_0'(x) = 1, \quad x > 0.$$

By the mean value theorem,

$$\frac{F_0(x+\beta)-F_0(x)}{\beta} > -1, \quad x>0,\ \beta>0,$$

which is equivalent to the left inequality in (4).

Similarly, by convexity of F_a for $a \geq 1/2$ and the mean value theorem, we have that

$$\frac{F_a(x+\beta)-F_a(x)}{\beta} < -1, \quad x>0, \beta>0, a\geq 1/2,$$

which is the right inequality in (4). □

In the above proof we used only the fact that $F_a'(x)$ has an extremum at $x=+\infty$. Using Jensen's inequality for convex or concave functions, we can obtain new bounds as in the next corollary.

Corollary 2. *Let*

$$A(x,\beta) = \frac{(x-1+\beta)^{\beta(x+\beta-1/2)}(x+\beta)^{(1-\beta)(x+\beta-1/2)}}{x^{x-1/2}},$$
$$B(x,\beta) = \frac{(x+\beta)^{x+\beta-1/2}}{(x+1)^{\beta(x+1/2)}x^{(1-\beta)(x-1/2)-\beta}}.$$

Let $\beta\in(0,1)$ and $a\geq 1/2$. Then, for $x>1-\beta$, we have

$$\begin{aligned} &A(x,\beta)\exp\biggl(\frac{1}{12}(\beta\Psi'(x+a-1+\beta) \\ &\qquad + (1-\beta)\Psi'(x+a+\beta)-\Psi'(x+a))\biggr) < Q(x,\beta) \\ &\quad < A(x,\beta)\exp\biggl(\frac{1}{12}(\beta\Psi'(x-1+\beta) \\ &\qquad\qquad + (1-\beta)\Psi'(x+\beta)-\Psi'(x))\biggr). \end{aligned} \tag{6}$$

Further, for any $x > 0$, we have

$$\begin{aligned} B(x,\beta)\exp\left(\frac{1}{12}(\Psi'(x+\beta)-(1-\beta)\Psi'(x)-\beta\Psi'(x+1))\right)& \\ < Q(x,\beta)& \\ < B(x,\beta)\exp\left(\frac{1}{12}(\Psi'(x+a+\beta)-(1-\beta)\Psi'(x+a)\right.& \\ \left.-\beta\Psi'(x+a+1))\right).& \end{aligned} \tag{7}$$

For $\beta > 1$ all four inequalities hold for all $x > 0$, with $<$ and $>$ interchanged.

Proof. Let $\beta \in (0,1)$ and $x > 1-\beta$. Starting from

$$x = \beta(x-1+\beta)+(1-\beta)(x+\beta), \tag{8}$$

and applying Jensen's inequality with the convex function F_a, $a \geq 1/2$, we find that

$$F_a(x) < \beta F_a(x-1+\beta)+(1-\beta)F_a(x+\beta). \tag{9}$$

The left inequality in (6) now follows from (9), (5) and the recurrence relation $\Gamma(x-1+\beta) = \Gamma(x+\beta)/(x-1+\beta)$. The right inequality in (6) is obtained in the same way with the concave function F_0. Inequalities in (7) can be derived similarly, replacing (8) with

$$x+\beta = (1-\beta)x+\beta(x+1), \tag{10}$$

where now it suffices to assume that $x > 0$.

If $\beta > 1$, then we start from

$$x-1+\beta = \frac{1}{\beta}x+\frac{\beta-1}{\beta}(x+\beta), \quad x > 0$$

and, applying Jensen's inequality with F_a, we get

$$F_a(x) > \beta F_a(x-1+\beta)+(1-\beta)F_a(x+\beta),$$

which leads to the opposite left inequality in (6). Other inequalities can be similarly dealt with. □

Theorem 2 (Comparison of inequalities). *For $x > 0$ and $\beta \in (0,1)$, each inequality in* (7) *is sharper than the corresponding inequality in* (4).

Proof. The lower bound in (7) is greater than the lower bound in (4) if and only if

$$\left(\frac{x}{x+1}\right)^{\beta(x+1/2)} \exp\left(\beta\left(\frac{1}{12}(\Psi'(x) - \Psi'(x+1)) + 1\right)\right) > 1,$$

which is, by $\Psi'(x) - \Psi'(x+1) = 1/x^2$, equivalent to

$$\left(1 + \frac{1}{x}\right)^{x+1/2} < e^{1+1/(12x^2)}. \tag{11}$$

Let

$$f(x) = \left(x + \frac{1}{2}\right)\log\left(1 + \frac{1}{x}\right) - 1 - \frac{1}{12x^2}.$$

Then we have that

$$\lim_{x\to+\infty} f(x) = 0, \qquad \lim_{x\to+\infty} f'(x) = 0,$$
$$f''(x) = -\frac{2x+1}{2x^4(x+1)^2} < 0 \quad \text{for } x > 0,$$

and therefore $f'(x) < 0$ and finally $f(x) < 0$ for all $x > 0$, which proves (11).

The upper bound in (7) is smaller than the upper bound in (4) if and only if

$$\left(1 + \frac{1}{x}\right)^{x+1/2} > e^{1+1/(12(x+a)^2)}. \tag{12}$$

Obviously, it suffices to prove (12) for $a = 1/2$. If we define

$$g(x) = \left(x + \frac{1}{2}\right)\log\left(1 + \frac{1}{x}\right) - 1 - \frac{1}{12(x+1/2)^2},$$

we have that $\lim_{x\to+\infty} g(x) = 0$, $\lim_{x\to+\infty} g'(x) = 0$ and

$$g''(x) = \frac{8x^2+8x+1}{2x^2(x+1)^2(2x+1)^4} > 0 \quad \text{for } x > 0,$$

and we conclude that $g(x) > 0$ for all $x > 0$, which was to be proved. $\square$

In passing, let us note that the double inequality

$$e^{1+1/(12(x+1/2)^2)} < \left(1+\frac{1}{x}\right)^{x+1/2} < e^{1+1/(12x^2)}, \quad x > 0,$$

is an improvement of a result in [**12**, Section 3.63].

Numerical results show that, for $x > 1-\beta$, bounds in (6) are not comparable with bounds in (4).

An advantage of inequalities based on convexity is that they can be infinitely sharpened, making so a basis for Euler-like products. This can be done with all inequalities of Corollary 2, following the pattern indicated in the next theorem.

Theorem 3. *Let $A(x,\beta)$ be as defined in Corollary 2. Then, for any $x > 0$, $\beta \in (0,1)$ and $n = 1, 2, \dots$,*

$$\begin{aligned} Q(x,\beta) &< \frac{x(x+1)\cdots(x+n-1)A(x+n,\beta)}{(x+\beta)(x+\beta+1)\cdots(x+\beta+n-1)} \\ &\quad \cdot \exp\Bigg(\frac{1}{12}(\beta\Psi'(x+n-1+\beta) \\ &\qquad + (1-\beta)\Psi'(x+n+\beta) - \Psi'(x+n))\Bigg). \end{aligned} \tag{13}$$

The absolute and relative error in (13) *converges to zero as $n \to +\infty$.*

Proof. For any $x > 0$, $\beta \in (0,1)$ and $n = 1, 2, \dots$, we have that

$$x + n = \beta(x+n-1+\beta) + (1-\beta)(x+n+\beta).$$

An application of Jensen's inequality with the concave function F_0 yields

$$F_0(x+n) > \beta F_0(x+n-1+\beta) + (1-\beta)F_0(x+n+\beta). \tag{14}$$

Considering the difference between the right and left side in (14), we find that

$$\begin{aligned} r_n &= F_0(x+n) - \beta F_0(x+n-1+\beta) \\ &\quad - (1-\beta)F_0(x+n+\beta) \\ &= \beta(F_0(x+n) - F_0(x+n-1+\beta)) \\ &\quad + (1-\beta)(F_0(x+n) - F_0(x+n+\beta)) \\ &= \beta(1-\beta)F_0'(c_1) - \beta(1-\beta)F_0'(c_2) \\ &= \beta(1-\beta)(F_0'(c_1) - F_0'(c_2)) \\ &= \beta(1-\beta)(c_1-c_2)F_0''(c), \end{aligned}$$

where, by the mean value theorem, $c_1 \in (x+n-1+\beta, x+n)$, $c_2 \in (x+n, x+n+\beta)$ and $c \in (c_1, c_2)$. Since $|c_1 - c_2| < 1$, we have that $r_n \le \beta(1-\beta)|F_0''(c)|$. It is not difficult to see that the function $x \mapsto |F_0''(x)|$ is decreasing and $\lim_{x\to+\infty} F_0''(x) = 0$. Therefore,

$$\begin{gathered} 0 < r_n \le \beta(1-\beta)|F_0''(x+n-1+\beta)| \longrightarrow 0 \\ \text{as } n \to +\infty. \end{gathered} \tag{15}$$

Now the inequality (13) can be obtained from (14) after successive applications of the recurrence relation for the gamma function. For a pair (x, β) being fixed, let R_n denote the righthand side of (13). Then $Q = R_n \cdot e^{-r_n}$, which implies that $\lim_{n\to+\infty} \log R_n = \log Q$ and further,

$$\lim_{n\to+\infty} (R_n - Q) = 0 \quad \text{and} \quad \lim_{n\to+\infty} \frac{R_n - Q}{Q} = 0. \qquad \square$$

Remark. From the proof of Theorem 1 it follows that

$$\begin{aligned} F_a''(x) &= \sum_{k=0}^{+\infty} \left(\frac{1}{2(x+k)^2(x+k+1)^2} - \frac{1}{2(x+k+a)^4} \right) \\ &= \sum_{k=0}^{+\infty} \frac{(x+k)^3(4a-2) + (x+k)^2(6a^2-1) + 4a^3(x+k) + a^4}{2(x+k)^2(x+k+1)^2(x+k+a)^4}. \end{aligned}$$

From this expression we see that $F_a''(x) < 0$ for each $a \in (0, 1/2)$ and x large enough, therefore F_a is strictly concave on $x > x_a$, where x_a depends on $a \in (0, 1/2)$ and can be evaluated numerically. Consequently, we may apply presented methods to obtain further bounds for $Q(x, \beta)$ when x is large.

As a prototype of such results, we note that using the method of proof of Corollary 1, F_0 replaced by F_a for some $a \in (0, 1/2)$, we get the inequality

$$\begin{aligned}(16)\quad \frac{(x+\beta)^{x+\beta-1/2}}{x^{x-1/2}} \exp\bigg(-\beta + \frac{1}{12}(\Psi'(x+\beta+a) \\ - \Psi'(x+a))\bigg) < Q(x, \beta),\end{aligned}$$

where $a \in (0, 1/2)$ and $x > x_a$. Since $F_a''(x) > F_0''(x)$ for $a < 1/2$, it follows from the arguments similar to the proof of Theorem 3 (see also the proof of Theorem 2 in [**11**]) that on the domain $x > x_a$, the inequality (16) is sharper than the left inequality in (4).

A similar idea (sharper inequalities for large x) has been exploited in [**8**] for bounds for Q in terms of elementary functions.

3. Schur-convexity and further bounds for Q. Let F be a function of n arguments, defined on I^n, where I is an interval. We say that F is Schur-convex on I^n if

$$(17)\qquad F(x_1, \dots, x_n) \le F(y_1, \dots, y_n)$$

for each two n-tuples $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ in I^n, such that

$$(18)\qquad \begin{aligned} &\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, \quad k = 1, \dots, n-1, \\ &\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \end{aligned}$$

where $x_{[i]}$ denotes the ith largest component in x. If (18) holds, we say that x is majorized by y and write $x \prec y$. We say that F is strictly

Schur-convex on I if a strict inequality holds in (17) whenever $x \prec y$ and x is not a permutation of y. For $n = 2$, a function $(x_1, x_2) \mapsto F(x_1, x_2)$ is Schur-convex on I^2 if and only if

$$F(x_1, x_2) \le F(x_1 - \varepsilon, x_2 + \varepsilon)$$

for each $(x_1, x_2) \in I^2$, $x_1 \le x_2$ and each $\varepsilon > 0$ such that $(x_1 - \varepsilon, x_2 + \varepsilon) \in I^2$.

A function F is said to be (strictly) Schur-concave on I^n if the function $-F$ is (strictly) Schur-convex on I^n.

For further details on Schur-convexity, see [**10**].

The results of this section are based on the following theorem.

Theorem 4. *The function $(x, y) \mapsto F(x, y)$ defined by*

$$F(x, y) = \frac{\log\Gamma(x) - \log\Gamma(y)}{x - y}, \quad x \ne y, \qquad F(x, x) = \Psi(x) \tag{19}$$

is strictly Schur-concave on $x > 0$, $y > 0$.

Proof. Since F is a continuously differentiable function, by [**10**, Chapter 3.A], it suffices to show that

$$\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right)(y - x) < 0 \quad \text{for all } x > 0, y > 0, x \ne y,$$

which is, in our case, equivalent to

$$\frac{\Psi(x) + \Psi(y)}{2} < \frac{\log\Gamma(y) - \log\Gamma(x)}{y - x}, \quad 0 < x < y. \tag{20}$$

For a fixed $x > 0$, let

$$\varphi(y) = \log\Gamma(y) - \log\Gamma(x) - (y - x)\frac{\Psi(y) + \Psi(x)}{2}, \quad y \ge x.$$

Then we have

$$\begin{aligned}\varphi'(y) &= \Psi(y) - \frac{\Psi(y) + \Psi(x)}{2} - \frac{y - x}{2}\Psi'(y)\\ &= \frac{y - x}{2}\left(\frac{\Psi(y) - \Psi(x)}{y - x} - \Psi'(y)\right).\end{aligned}$$

By the mean value theorem, we have that $(\Psi(y)-\Psi(x))/(y-x) = \Psi'(c)$, for some $c \in (x,y)$. Since Ψ' is decreasing, then $\Psi'(c) > \Psi'(y)$ and it follows that $\varphi'(y) > 0$ for all $y > x$. Therefore, $\varphi(y) > \varphi(x) = 0$ and (20) is proved. $\square$

Corollary 3. *For $x > 0$ and $\beta > 0$, we have*

$$\exp\left(\beta\frac{\Psi(x+\beta)+\Psi(x)}{2}\right) < Q(x,\beta) < \exp(\beta\Psi(x+\beta/2)). \tag{21}$$

Proof. The left inequality in (21) follows directly from (20) by replacing y with $x+\beta$. From

$$(x+\beta, x) \succ (x+\beta/2, x+\beta/2),$$

and Schur-convexity of the function F, we get the right inequality in (21). $\square$

Theorem 5. (Comparison of inequalities). *For $\beta \in (0,1)$ and $x > 1-\beta$, the left inequality in* (21) *is sharper than the left inequality in* (3), *i.e.,*

$$\frac{\Psi(x+\beta)+\Psi(x)}{2} > \Psi(x+\beta-1+\sqrt{1-\beta}). \tag{22}$$

Proof. Since Ψ is an increasing function, it suffices to show that

$$\frac{\Psi(x+\beta)+\Psi(x)}{2} > \Psi(x+\sqrt{1-\beta}), \quad x > 0, \beta > 0. \tag{23}$$

For a fixed $x > 0$, define

$$\varphi(\beta) = \Psi(x+\beta) + \Psi(x) - 2\Psi(x+\sqrt{1-\beta}).$$

Then $\varphi(0) = 0$, $\varphi'(\beta) > 0$ and therefore $\varphi(\beta) > 0$ for $\beta > 0$, which ends the proof. $\square$

Let us note that, from Theorem 4, we can produce various inequalities for the ratio of gamma functions. For example, from

$$(y,x) \succ (y-\varepsilon, x+\varepsilon), \quad x<y, 0<\varepsilon<(y-x)/2$$

we obtain

$$\left(\frac{\Gamma(y)}{\Gamma(x)}\right)^{1/(y-x)} < \left(\frac{\Gamma(y-\varepsilon)}{\Gamma(x+\varepsilon)}\right)^{1/(y-x-2\varepsilon)}, \quad 0<x<y,\ 0<\varepsilon<(y-x)/2.$$

4. Some inequalities for $Q(x+\lambda,\beta)/Q(x,\beta)$ and inequalities of Gurland's type. In this section we consider the ratio $Q(x+\lambda,\beta)/Q(x,\beta)$ for $\lambda\in(0,1)$.

Theorem 6. *Let $I=(a,+\infty)$, $a\ge 0$. If for $\beta>0$ the function*

$$F(x)=\frac{\Gamma(x)}{\Gamma(x+\beta)}\cdot G(x,\beta) \tag{24}$$

is strictly log-convex with respect to $x\in I$, then for any $\lambda\in(0,1)$ and $x>a+1-\lambda$ the following holds:

$$\left(\frac{x+\beta}{x}\right)^{\lambda}\frac{G(x+\lambda,\beta)}{G^{\lambda}(x+1,\beta)G^{1-\lambda}(x,\beta)} < \frac{Q(x+\lambda,\beta)}{Q(x,\beta)} < \left(\frac{x-1+\lambda+\beta}{x-1+\lambda}\right)^{\lambda}\frac{G^{\lambda}(x-1+\lambda,\beta)G^{1-\lambda}(x+\lambda,\beta)}{G(x-1+\lambda,\beta)}. \tag{25}$$

Proof. By Jensen's inequality applied to the function $x\mapsto \log F(x)$, we have that

$$F(x+\lambda)<F^{1-\lambda}(x)F^{\lambda}(x+1),$$

i.e.,

$$\begin{aligned}\frac{\Gamma(x+\lambda)}{\Gamma(x+\lambda+\beta)}G(x+\lambda,\beta) &< \frac{\Gamma^{1-\lambda}(x)\Gamma^{\lambda}(x+1)}{\Gamma^{1-\lambda}(x+\beta)\Gamma^{\lambda}(x+\beta+1)}G^{1-\lambda}(x,\beta)G^{\lambda}(x+1,\beta)\\ &= \frac{\Gamma(x)}{\Gamma(x+\beta)}\left(\frac{x}{x+\beta}\right)^{\lambda}G^{1-\lambda}(x,\beta)G^{\lambda}(x+1,\beta),\end{aligned}$$

and the left inequality in (25) is proved. The right inequality follows similarly from

$$F(x) < F^{\lambda}(x-1+\lambda)F^{1-\lambda}(x+\lambda). \qquad \square$$

In [**2, 3**] and [**7**], it is proved that several functions of the form (24) are completely monotone on x for each fixed $\beta \in (0,1)$. Since each completely monotone function is log-convex (see [**4**], for example), Theorem 6 can be applied with these functions. For example, by [**3**], the function

$$F(x) = \frac{\Gamma(x)}{\Gamma(x+\beta)} \exp(\beta\Psi(x+\beta/2))$$

is strictly log-convex on $x > 0$ for each $\beta \in (0,1)$ (in fact, the result in [**3**] is stated for $x > 1-\beta$, but by an inspection of the proof it is easy to see that it holds for $x > 0$). For $x > 1-\lambda$, Theorem 6 then yields

$$\begin{aligned}\left(\frac{x+\beta}{x}\right)^{\lambda} &\exp\left(\beta\left(\Psi(x+\lambda+\beta/2) - \Psi(x+\beta/2) - \frac{\lambda}{x+\beta/2}\right)\right)\\ &< \frac{Q(x+\lambda,\beta)}{Q(x,\beta)} < \left(\frac{x-1+\lambda+\beta}{x-1+\lambda}\right)^{\lambda}\\ &\quad \cdot \exp\left(\beta\left(\Psi(x+\lambda+\beta/2)\right.\right.\\ &\qquad \left.\left. - \Psi(x+\beta/2) - \frac{\lambda}{x-1+\lambda+\beta/2}\right)\right),\end{aligned}$$

where the recurrence relation $\Psi(x+1) = \Psi(x) + 1/x$ was used.

In a special case when $\lambda = \beta$, we have that

$$\frac{Q(x+\lambda,\beta)}{Q(x,\beta)} = \frac{\Gamma(x+2\beta)\Gamma(x)}{\Gamma^2(x+\beta)}.$$

This ratio was investigated by Gurland [**6**] in 1956 and later by many authors, but it seems that bounds related to the digamma function have not been observed in the literature so far. Some bounds of this type can also be obtained from our Theorem 1. For example, from

Jensen's inequality applied to concave function F_0, we find that

$$\frac{\Gamma(x+2\beta)\Gamma(x)}{\Gamma^2(x+\beta)} < \frac{x^{x-1/2}(x+2\beta)^{x+2\beta-1/2}}{(x+\beta)^{2x+2\beta-1}} \cdot \exp\left(\frac{1}{12}(\Psi'(x+2\beta)+\Psi'(x)-2\Psi'(x+\beta))\right),$$

for $x > 0$ and $\beta \in (0,1)$.

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