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MOVING AVERAGES OF SUPERADDITIVE PROCESSES WITH RESPECT TO L_p-CONTRACTIONS, $1 < p < \infty$

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ABSTRACT. It is shown that the moving averages of superadditive processes with respect to positive L_p-contractions converge almost everywhere, $1 < p < \infty$. The dominated estimate for the moving averages of such superadditive processes is obtained from the dominated estimate for the moving averages of additive processes.

1. Introduction. When T is an operator induced by a measure preserving point transformation τ on a measure space X, Bellow, Jones and Rosenblatt proved the almost everywhere convergence of the ergodic moving averages along a class of sequences satisfying a "cone condition" [**4**]. In fact, using the Calderón transference principle, they showed that if the sequence satisfies the cone condition, then the associated maximal operator is weak type $(1,1)$ and strong type (p,p) for $1 < p < \infty$ which, in turn, was used to prove the almost everywhere convergence theorem. Later, Jones and Olsen [**8**] extended this strong type (p,p) maximal inequality to the operator setting, $1 < p < \infty$, and used it to generalize some of the results in [**4**]. After the work of Bellow, Jones and Rosenblatt, it was observed that the class of sequences that satisfy the cone condition are the same as the class of sequences later introduced by Akcoglu and Déniel [**1**], known as B-sequences (see the definition below).

In this paper our main aim is to extend the result of Jones and Olsen to the superadditive setting, namely to obtain the almost everywhere convergence of the "moving averages" of type

$$\frac{1}{r_n} T^{a_n} f_{r_n},$$

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where $\{(a_n, r_n)\}_{n=0}^{\infty}$ is a B-sequence, thereby generalizing the result of Ferrando [**6**] to the operator setting. The ordinary (nonmoving) averages correspond to the case where $a_n = 0$ for all n. Hence, our result also generalizes the theorem of Hachem [**7**] to the moving averages case. To prove the almost everywhere convergence, we will use the techniques implemented in [**7**] and [**3**]. We will also obtain the superadditive version of the strong type (p,p) maximal inequality for B-sequences; however, we will not need it in proving the almost everywhere convergence.

2. Preliminaries. Let (X, Σ, μ) be a σ-finite measure space, $T : L_p(X) \to L_p(X)$ be a positive linear contraction, with p fixed, $1 < p < \infty$. The average

$$\frac{1}{n}\sum_{k=0}^{n-1} T^k$$

will be denoted by $A_n(T)$. If $\mathbf{n} : \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ denotes a subsequence $\mathbf{n} = \{(a_n, r_n)\}_{n=1}^{\infty}$, then $A_{(a_n,r_n)}(T)$ will mean the *moving average*

$$A_{(a_n,r_n)}(T) = \frac{1}{r_n}\sum_{i=0}^{r_n-1} T^{a_n+i},$$

where $\mathbf{N}$ is the set of nonnegative integers.

A family $F = \{f_n\}_{n\geq 0}$ of measurable functions is called a T-*superadditive* process if

$$f_{n+m} \geq f_n + T^n f_m \quad \text{a.e. for all} \quad n, m \geq 0,$$

where T is an operator on the function space f_n's belong to. If the reverse inequality holds, the process is called T-*subadditive*, and if the equality holds it is called T-*additive*. When there is no confusion, we will drop T and call the process simply *superadditive* (or *subadditive*, or *additive*). If there exists a function $\varphi \in L_p^+$ satisfying $f_n \leq \sum_{i=0}^{n-1} T^i \varphi$, for all $n \geq 1$, then the process $F = \{f_n\}$ is called *dominated* and the function φ is called a *dominant* for F.

Remark 2.1. (i) If $\{f_n\}$ is a T-additive process, then $f_n = \sum_{i=0}^{n-1} T^i f_1$.

(ii) If $\{f_n\}$ is not necessarily a positive T-superadditive process, then, by the superadditivity,

$$f_n \geq \sum_{i=0}^{n-1} T^i f_1.$$

That is, any such superadditive process dominates the additive process $\{\sum_{i=0}^{n-1} T^i f_1\}$. Consequently, the process $\hat{F} = \{\hat{f}_n\}$, where $\hat{f}_n = f_n - \sum_{i=0}^{n-1} T^i f_1$ is a *nonnegative* T-superadditive process.

Throughout this article we will assume that all the processes under study will satisfy the condition

$$(*) \qquad \liminf_{n\to\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} (f_i - Tf_{i-1}) \right\|_p < \infty.$$

This condition was introduced in [**5**] and has been used in [**7**] to obtain the almost everywhere convergence of the superadditive processes with respect to L_p-contractions. The following result, which was obtained in [**7**] and will be adopted here without proof, states that any process satisfying the condition $(*)$ has a dominant:

Theorem A. *Let T be a positive L_p-contraction and F be a T-superadditive process satisfying the condition $(*)$. Then there exists a dominant $\varphi \in L_p^+$ for F with*

$$\|\varphi\|_p \leq \liminf_{n\to\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} (f_i - Tf_{i-1}) \right\|_p.$$

Remark 2.2. Under the conditions of Theorem A, if φ is a dominant for F, then

$$\left\| \sup_{n\geq 1} \frac{1}{n} f_n \right\|_p \leq \|\varphi\|_p \leq \liminf_{n\to\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} (f_i - Tf_{i-1}) \right\|_p.$$

A sequence $\mathbf{n} = \{(a_n, r_n)\}$ in $\mathbf{N} \times \mathbf{N}$ such that $r_n > 0$ for all n is called a *B-sequence* if there is a constant $B > 0$ satisfying

$$|\{k : \exists n, k + [a_n, a_n + r_n) \subset I\}| \leq B|I|$$

for every interval $I \subset \mathbf{N}$, where $|S|$ denotes the cardinality of a set $S \subset \mathbf{N}$ [**1**]. Given a B-sequence $\mathbf{n} = \{(a_n, r_n)\}$, we will define the average of a T-superadditive process along $\mathbf{n}$ as

$$\frac{1}{r_n} T^{a_n} f_{r_n},$$

which seems a natural way of defining the moving averages of superadditive processes. It should be noted here that, in the superadditive setting, it is possible to give alternative definitions of moving averages; however, such averages may fail to converge almost everywhere as shown by Ferrando [**6**].

In proving the almost everywhere convergence, the following result of Akcoglu and Sucheston [**2**] will be instrumental.

Theorem B. *Let T be a positive L_p-contraction, $1 < p < \infty$ fixed. Then there is a unique decomposition of X into sets E and E^c such that*

(i) *E is the support of a T-invariant function $h \in L_p$, and the support of each T-invariant function is contained in E.*

(ii) *The subspaces $L_p(E)$ and $L_p(E^c)$ are both invariant under T.*

Before stating the theorems we make some observations. If $X = E$, where E is as in Theorem B, then the measure $m = h^p \mu$ on X is finite (and is equivalent to μ). Hence, the operator defined as

$$Pf = h^{-1} T(fh), \quad \text{for all} \quad f \in L_p(m),$$

is a positive $L_p(m)$-contraction with $P1 = 1$ and $\int Pf \, dm = \int f \, dm$. So P can be extended to a Markovian operator on $L_1(m)$. Furthermore, if $F = \{f_n\}$ is a positive T-superadditive process, then the process $F' = \{h^{-1} f_n\}$ is a P-superadditive process. Now, if F satisfies condition $(*)$, then

$$\begin{aligned}\left\|\frac{1}{n}h^{-1}f_n\right\|_{L_1(m)} &= \int \frac{1}{n}f_n h^{p-1}\,d\mu \\ &\le \left\|\frac{1}{n}f_n\right\|_{L_p(\mu)} \|h^{p-1}\|_{L_q(\mu)} \\ &\le \left\|\sup_{n\ge 1}\frac{1}{n}f_n\right\|_{L_p(\mu)} \|h\|^{p-1}_{L_p(\mu)} \\ &\le \frac{p}{p-1}\liminf_{n\to\infty}\left\|\frac{1}{n}\sum_{i=1}^{n}(f_i - Tf_{i-1})\right\|_{L_p(\mu)} \|h\|^{p-1}_{L_p(\mu)} \\ &< \infty,\end{aligned}$$

where the last inequality follows from Remark 2.2. Thus F' is bounded with time constant $\gamma_{F'} = \sup_{n\ge 1}\int (1/n)h^{-1}f_n\,dm$. On the other hand, for any $f \in L_p(\mu)$, $\|h^{-1}f\|_{L_p(m)} = \|f\|_{L_p(\mu)}$, so $h^{-1}f \in L_p(m)$, and hence $h^{-1}f \in L_1(m)$. Also, the relation $P(h^{-1}f) = h^{-1}Tf$ implies that $h^{-1}A_n(T)f = A_n(P)(h^{-1}f)$, and consequently $A_{(a_n,r_n)}(P)(h^{-1}f) = h^{-1}A_{(a_n,r_n)}(T)f$, for any subsequence $\{(a_n,r_n)\}$. Therefore, $f^* = \lim_{n\to\infty} A_{(a_n,r_n)}(T)f$ exists μ-almost everywhere if and only if $h^{-1}f^* = \lim_{n\to\infty} A_{(a_n,r_n)}(P)(h^{-1}f)$ exists m-almost everywhere. In addition, since

$$\begin{aligned}\|A_{(a_n,r_n)}(P)(h^{-1}f)\|_{L_1(m)} &\le \|A_{(a_n,r_n)}(P)(h^{-1}f)\|_{L_p(m)} \\ &= \|A_{(a_n,r_n)}(T)f\|_{L_p(\mu)},\end{aligned}$$

if $A_{(a_n,r_n)}(T)f$ converge in $L_p(\mu)$-norm, $1 < p < \infty$, then $A_{(a_n,r_n)}(P)(h^{-1}f)$ converge in $L_p(m)$-norm and $L_1(m)$-norm.

3. Ergodic theorems. This section is devoted to the ergodic theorems for superadditive processes along B-sequences. First we observe that the dominated estimate for the averages of superadditive processes along B-sequences is an easy consequence of the dominated estimate obtained in [**8**] for the additive processes along the same sequences.

Proposition 3.1. *Let T be a positive L_p-contraction, $1 < p < \infty$ fixed, and F be a nonnegative T-superadditive process satisfying $(*)$. If*

$\mathbf{n} = \{(a_n, r_n)\}$ *is a* B*-sequence, then*

$$\left\| \sup_{n\geq 1} \frac{1}{r_n} T^{a_n} f_{r_n} \right\|_p \leq C(p)\|\varphi\|_p,$$

where $C(p)$ *is a constant independent of* $\mathbf{n}$ *and* F*, and* φ *is a dominant for* F*.*

Proof. By Theorem A there exists a dominant φ for the process. Clearly, along the B-sequence,

$$\frac{1}{r_n} T^{a_n} f_{r_n} \leq A_{(a_n, r_n)}(T)\varphi.$$

Since the additive process $\{\sum_{i=0}^{r_n-1} T^{a_n+i}\varphi\}$ admits a dominated estimate along the B-sequences, there exists a constant $C(p)$, independent of the process and the sequence, such that

$$\| \sup_{n\geq 1} A_{(a_n, r_n)}(T)\varphi\|_p \leq C(p)\|\varphi\|_p \quad [\mathbf{8}].$$

This, combined with the first inequality, proves the assertion. □

Now we state and prove the main theorem.

Theorem 3.2. *Let* T *be a positive linear* L_p*-contraction,* $1 < p < \infty$ *fixed, and* F *be a nonnegative* T*-superadditive process satisfying* $(*)$*. If* $\{(a_n, r_n)\}$ *is a* B*-sequence, then the moving averages*

$$\frac{1}{r_n} T^{a_n} f_{r_n} \quad \textit{converge a.e.}$$

Proof. Since moving averages of T-additive processes converge almost everywhere [**8**], by Remark 2.1 (ii), we can assume that F is nonnegative. Hence, by the same fact and by the existence of a dominant φ for F, we have

$$0 \leq \limsup_{n\to\infty} \frac{1}{r_n} T^{a_n} f_{r_n} \leq \lim_{n\to\infty} A_{(a_n, r_n)}(T)\varphi = \varphi^*.$$

Observing that

$$\sum_{n=0}^{\infty}\left\|\frac{1}{r_n}(T^{a_n+r_n}\varphi - T^{a_n}\varphi)\right\|_p^p \le 2\|\varphi\|_p^p \sum_{n=0}^{\infty}\frac{1}{r_n^p} < \infty,$$

we deduce that φ^* is T-invariant, and consequently, $\varphi^* = 0$ almost everywhere on E^c by Theorem B. This implies that $\lim(1/r_n)T^{a_n}f_{r_n} = 0$ almost everywhere on E^c. Therefore, for the rest of the proof we can assume that $X = E$.

Now let $m = h^p\mu$ be a new measure on X where h is a T-invariant function that is given by Theorem B, and consider the operator $Pf = h^{-1}T(fh)$, $f \in L_p(m)$. Hence, P is a positive $L_p(m)$-contraction with $P1 = 1$ and Markovian on $L_1(m)$. Also, let $F' = \{h^{-1}f_n\}$, which is a P-superadditive, bounded process with time constant $\gamma_{F'} = \sup_{n\ge 1}\int(1/n)h^{-1}f_n\, dm$. Then it is easy to see that $\gamma_{F'} = \lim_{n\to\infty}(1/n)\int h^{-1}f_n\, dm$. By the theorem of Akcoglu-Sucheston [**3**], F' has an exact dominant $\delta \in L_1(m)$,that is, δ is a dominant for F' with $\int \delta\, dm = \gamma_{F'}$.

On the other hand, if $s_k = (1/k)f_k$, for a fixed $k \ge 1$, then

$$\sum_{i=0}^{n-k-1} T^i s_k \le f_n \quad \text{for each } n \ge 1$$

(with the convention that sums over void sets are zero) [**3**]. Therefore, by the positivity of T,

$$\frac{1}{r_n}\sum_{i=a_n}^{a_n+r_n-k-1} T^i s_k \le \frac{1}{r_n}T^{a_n}f_{r_n}.$$

Theorem 3.1 of Jones and Olsen [**8**] implies that

$$s_k^* := \lim_{n\to\infty}\frac{1}{r_n}\sum_{i=a_n}^{a_n+r_n-k-1} T^i s_k = \lim_{n\to\infty}\frac{1}{r_n}\sum_{i=a_n}^{a_n+r_n-1} T^i s_k$$

exists almost everywhere. Since the family $\{\sum_{i=0}^{n-1} T^i s_k\}_{n\ge 1}$ is weakly sequentially compact, this convergence is weak, and hence, is strong as well. Therefore, for the P-additive process $\{\sum_{i=0}^{n-1} P^i(h^{-1}s_k)\}_{n\ge 1}$,

$$h^{-1}s_k^* := \lim_{n\to\infty} A_{(a_n,r_n)}(P)(h^{-1}s_k)$$
$$\text{exists a.e. and in} \quad L_1(m).$$

Since $\varphi \in L_p(\mu)$ is a dominant for F, F' is dominated by the P-additive process $\{\sum_{i=0}^{n-1} P^i(h^{-1}\varphi)\}_{n\geq 1}$. So $\sum_{i=0}^{n-1} P^i\delta \leq \sum_{i=0}^{n-1} P^i(h^{-1}\varphi)$ must be true for all $n \geq 1$. In particular, $\delta \leq h^{-1}\varphi$, which implies that $\delta \in L_p(m)$. Hence,

$$\delta^* := \lim_{n\to\infty} A_{(a_n,r_n)}(P)\delta \quad \text{exists a.e. and in} \quad L_1(m).$$

On the other hand, since P is Markovian,

$$\int s_k h^{-1}\, dm = \int s_k^* h^{-1}\, dm \quad \text{and} \quad \int \delta\, dm = \int \delta^*\, dm. \tag{1}$$

Also

$$\lim_{k\to\infty} \int s_k h^{-1}\, dm = \lim_{k\to\infty} \frac{1}{k} \int f_k h^{-1}\, dm = \gamma_{F'} = \int \delta\, dm. \tag{2}$$

Now, if

$$\underline{f} = \liminf_{n\to\infty} \frac{1}{r_n} P^{a_n}(h^{-1} f_{r_n})$$

and

$$\bar{f} = \limsup_{n\to\infty} \frac{1}{r_n} P^{a_n}(h^{-1} f_{r_n}),$$

then

$$\lim_{n\to\infty} \frac{1}{r_n} \sum_{i=a_n}^{a_n+r_n} P^i(h^{-1}s_k) \leq \underline{f} \leq \bar{f} \leq \lim_{n\to\infty} \frac{1}{r_n} \sum_{i=a_n}^{a_n+r_n} P^i\delta,$$

and, hence,

$$\int h^{-1} s_k^*\, dm \leq \int \underline{f}\, dm \leq \int \bar{f}\, dm \leq \int \delta^*\, dm.$$

Taking the limit as $k \to \infty$, and using (1) and (2), we obtain that $\underline{f} = \bar{f}$ almost everywhere which implies that

$$\lim_{n\to\infty} \frac{1}{r_n} T^{a_n} f_{r_n} \quad \text{exists a.e.} \qquad \square$$

Remark 3.3. The proof of the above theorem works for the case of block sequences as well (see [**8**] for definitions). By defining the averages of superadditive processes along block sequences similarly, we see that averages of superadditive processes along block sequences converge almost everywhere.

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