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COINCIDENCE PRINCIPLES AND FIXED POINT THEORY FOR MAPPINGS IN LOCALLY CONVEX SPACES

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ABSTRACT. We present a coincidence principle for concentrative maps. This leads to new fixed point theory for nonlinear operators.

1. Introduction. A general coincidence theory is presented for concentrative mappings between locally convex Hausdorff linear topological spaces in this paper. These general results are used to obtain a variety of new fixed point theorems for the sum of two operators, for example, an m-accretive plus a condensing operator, between Banach spaces (one could also obtain results for operators between locally convex Hausdorff linear topological spaces). The fixed point results were motivated from a variety of sources, in particular we mention the work of Browder [**4**], Daneš [**7**], Furi and Pera [**14**], Gatica and Kirk [**15**], Granas [**16**], Petryshyn [**25**], Precup [**26**], Reinermann [**27**] and Schöneberg [**28**]. Some applications of our results are also presented in this paper.

For the remainder of this section we gather together some definitions and some known facts. Let (E, d) be a pseudometric space [**18**] and M a subset of E. For $x \in M$, let $B(x, \varepsilon)$ denote the closed ε-ball with center x, i.e., $B(x, \varepsilon) = \{y \in E : d(x, y) \leq \varepsilon\}$. The *measure of noncompactness* of the set M is defined by

$$\alpha(M) = \inf Q(M); \qquad \inf(\varnothing) = \infty,$$

where

$$Q(M) = \{\varepsilon \in \mathbf{R} : \varepsilon > 0 \text{ and there is a finite } \varepsilon\text{-net for } M \text{ in } E \\ \text{i.e., } M \subseteq B(A, \varepsilon) \text{ for some finite subset } A \text{ of } E\}.$$

Note $B(A, \varepsilon) = \{x \in E : \inf\{d(x, y) : y \in A\} \leq \varepsilon\}$.

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Recall [**7, 8**] that if (E, d) is a pseudo metric space and M, N are subsets of E, then

(i) M is bounded if and only if $\alpha(M) < \infty$,

(ii) M is precompact (totally bounded) if and only if $\alpha(M) = 0$,

(iii) if $M \subseteq N$, then $\alpha(M) \leq \alpha(N)$,

(iv) $\alpha(M \cup N) = \max\{\alpha(M), \alpha(N)\}$.

In addition, if $(E, \|\cdot\|)$ is a pseudo normed (semi-normed) space and M is a subset of E, then $\alpha(\overline{\text{co}}(M)) = \alpha(M)$.

Now let E be a locally convex Hausdorff linear topological space, and let P be a defining system [**7, 8, 14, 29**] of semi-norms on E. Let C be a subset of E. A map $f : C \to E$ is said to be a P-concentrative mapping if f is continuous and if $p \in P$ and M is a bounded non-p-precompact subset of C, i.e., if M is not precompact in the pseudo normed space (E, p), then

$$\alpha_p(f(M)) < \alpha_p(M)$$

where $\alpha_p(\cdot)$ denotes the measure of noncompactness in the pseudo normed space (E, p). A map $f : C \to E$ is called a 1-mcL mapping [**8**] if f maps bounded sets into bounded sets and if $p \in P$, we have $\alpha_p(f(M)) \leq \alpha_p(M)$ for any subset M of C.

We now state Daneš fixed point theorem [**7, 8**].

Theorem 1.1. *Let E be a locally convex Hausdorff linear topological space and C a nonempty complete convex subset of E. Let P be a defining system of semi-norms and $f : C \to C$ a P-concentrative mapping. In addition, assume f is bounded, i.e., $f(C)$ is a subset of a bounded set in C. Then f has a fixed point.*

Let us now restrict the above discussion to the case when E is a Banach space. Let Ω_E be the bounded subsets of E. The *Kuratowskii measure of noncompactness* is the map $\alpha : \Omega_E \to [0, \infty)$ defined by

$$\alpha(X) = \inf \left\{ \varepsilon > 0 : X \subseteq \bigcup_{i=1}^{n} X_i, \ \text{diam } (X_i) \leq \varepsilon \right\};$$

here $X \in \Omega_E$. Of course, if $S, T \in \Omega_E$, then

(i) $\alpha(S) = 0$ if and only if $\overline{S}$ is compact.

(ii) $\alpha(\overline{S}) = \alpha(S)$.

(iii) If $S \subseteq T$, then $\alpha(S) \leq \alpha(T)$.

(iv) $\alpha(\mathrm{co}\,(S)) = \alpha(S)$.

(v) $\alpha(T + S) \leq \alpha(T) + \alpha(S)$.

Let B_1 and B_2 be two Banach spaces, and let $F : Y \subseteq B_1 \to B_2$ be continuous and map bounded sets into bounded sets. We call F an α-Lipschitzian map if F is continuous, bounded and there is a constant $k \geq 0$ with $\alpha(F(X)) \leq k\alpha(X)$ for all bounded sets $X \subseteq Y$. We call F a condensing map if F is α-Lipschitzian with $k = 1$ and $\alpha(F(X)) < \alpha(X)$ for all bounded sets $X \subseteq Y$ with $\alpha(X) \neq 0$.

We now state Sadovskii fixed point theorem [**1, 2, 23, 30**].

Theorem 1.2. *Let C be a closed convex subset of a Banach space B and $F : C \to C$ a condensing map. Then F has a fixed point.*

Next we recall some results [**21**] about the measure of noncompactness of subsets in $C([0,1], B)$ and $C^1([0,1], B)$; here B is a Banach space.

Theorem 1.3. *If $H \subseteq C([0,1], B)$ is a bounded and equicontinuous set, then $\alpha(H) = \alpha(H(I)) = \sup_{t\in I} \alpha(H(t))$; here $I = [0,1]$.*

Remark. Here $H(t) = \{\phi(t) : \phi \in H\}$ and $H(I) = \cup_{t\in I}\{\phi(t) : \phi \in H\}$.

Theorem 1.4. *Let A be a bounded subset of $C^1([0,1], B)$. Suppose $A' = \{\phi' : \phi \in A\}$ is an equicontinuous set. Then*

$$\alpha(A) = \max\Big\{\sup_{t\in I} \alpha(A(t)), \sup_{t\in I} \alpha(A'(t))\Big\}.$$

Remark. Here $A'(t) = \{\phi'(t) : \phi \in A\}$.

Theorem 1.5. *Let $A \subseteq C^1([0,1], B)$ be bounded. Then $\alpha(A) \geq$*

$\alpha(A(I))$.

Let B be a real Banach space, and let B^* denote the dual of B. Notice from the Hahn-Banach theorem that

$$\{x^* \in B^* : x^*(x) = \|x\|^2, \|x^*\| = \|x\|\} \neq \varnothing$$

for every $x \in B$. The mapping $F : B \to 2^{B^*}$ defined by

$$F(x) = \{x^* \in B^* : x^*(x) = \|x\|^2 = \|x^*\|^2\}$$

is called the *duality map* [**4, 10, 21**] of B. By means of F, the semi-inner product $(.,.)_+ : B \times B \to \mathbf{R}$, is defined by

$$(x, y)_+ = \sup\{y^*(x) : y^* \in F(y)\}.$$

Let $\Omega \subseteq B$. A mapping $T : \Omega \to B$ is said to be

(i) *strongly accretive* if, for some $c > 0$,

$$(T(x) - T(y), x - y)_+ \geq c\|x - y\|^2 \quad \text{for all } x, y \in \Omega; \tag{1.1}$$

(ii) *accretive* if

$$(T(x) - T(y), x - y)_+ \geq 0 \quad \text{for all } x, y \in \Omega$$

(iii) *m-accretive* if T is accretive and $I + \mu T$ is onto B for some (equivalently for all) $\mu > 0$.

Recall [**1, 11**] that, if $T : \Omega \to b$ is m-accretive, then the mapping $(I + \mu T)^{-1} : B \to \Omega$ is nonexpansive for each $\mu > 0$.

Theorem 1.6 [**1, 21**]. *Let B be a real Banach space and T a continuous, everywhere defined, accretive map. Then T is m-accretive.*

Theorem 1.7 [**10**]. *Let B be a real Banach space and $T : B \to B$ a continuous and strongly accretive map, i.e.,* (1.1) *holds for some $c > 0$. Then T is a homeomorphism from B onto B. Also $T^{-1} : B \to B$ is a Lipschitz map with Lipschitz constant $1/c$.*

Finally we state three standard results in functional analysis which will be used in Sections 2 and 3.

Theorem 1.8 [**20**]. *Every topological Hausdorff linear space is Tychonoff* ($T_{3\frac{1}{2}}$).

Theorem 1.9 [**13**]. *If* A *is a compact subset of a Tychonoff space* X*, then, for every closed set* D *disjoint from* A*, there exists a continuous function* $\mu : X \to [0,1]$ *such that* $\mu(x) = 1$ *for* $x \in A$ *and* $\mu(x) = 0$ *for* $x \in D$.

Theorem 1.10 [**3**]. *Let* B *be a uniformly convex Banach space whose dual* B^* *is also uniformly convex and* $T : B \to B$. *Suppose* $I - T$ *is a continuous accretive mapping defined on all of* B. *Then* $I - T$ *is demi-closed on* X.

Remark. A mapping $T : \Omega \subseteq B \to B$ is called demi-closed on Ω if, for every sequence $\{x_n\} \in \Omega$ with $x_n \rightharpoonup x$ and $T(x_n) \to y$ as $n \to \infty$, we have $x \in \Omega$ and $T(x) = y$; here $\rightharpoonup$ denotes weak convergence.

2. General coincidence theory. We formulate a coincidence theorem when E is a locally convex Hausdorff linear topological space. Let C be a complete convex subset of E, $X \subseteq C$ and $A \subseteq X$ with A closed in X and X closed in C. As in Section 1, let P be the defining system of semi-norms on E. Also $L : X \to C$ is a continuous operator. In the literature coincidence theory of the type formulated here, Granas, see [**17**], presented the case when E is a Banach space and the operator $F : X \to C$ is compact.

$K^0(X, C)$ will denote the set of all continuous, compact mappings $F : X \to C$. Associated with $K^0(X, C)$ we will assume that L satisfies the following condition:

(2.1) $L^{-1}(\Omega)$ is precompact in E for every compact subset Ω of C.

Remark. Here L^{-1} denotes the inverse image.

$K^1(X, C)$ will denote the set of all (continuous) P-concentrative mappings $F : X \to C$ with $F(X)$ a subset of a bounded set in C. Associated with $K^1(X, C)$ we will assume that L satisfies the following condition:

$$(2.2) \qquad \begin{cases} L^{-1} \text{ takes bounded sets into bounded sets, and} \\ \text{if } W \text{ is a bounded non-}p\text{-precompact subset of } X, \\ \text{then } \alpha_p(L(W)) \geq \alpha_p(W). \end{cases}$$

Remark. If L is one-to-one and L^{-1} is a 1-mcL mapping, then (2.2) is satisfied since $\alpha_p(W) = \alpha_p(L^{-1}(L(W))) \leq \alpha_p(L(W))$.

Definition 2.1. We let $K_A^0(X, C; L)$, respectively $K_A^1(X, C; L)$, denote the set of all mappings $F \in K^0(X, C)$, respectively $F \in K^1(X, C)$, such that $L - F$ is *zero free* on A.

We call $N : X \times [0, 1] \to C$ a compact mapping if N is continuous and $N(X \times [0, 1])$ is relatively compact. We call $N : X \times [0, 1] \to C$ a P-concentrative mapping if N is continuous, $N(X \times [0, 1])$ is a subset of a bounded set in C, and if $p \in P$ and W is a bounded non-p-precompact subset of $X \times [0, 1]$, i.e., $\alpha_p(\pi W) \neq 0$, then

$$\alpha_p(N(W)) < \alpha_p(\pi W)$$

where $\pi : X \times [0, 1] \to X$ is the natural projection.

Remark. If W is a non-p-precompact subset (product topology $(E, p) \times$ real) of $X \times [0, 1]$, then πW is a non-p-precompact subset of X so $\alpha_p(\pi W) \neq 0$.

Definition 2.2. A map $F \in K_A^0(X, C; L)$, respectively $F \in K_A^1(X, C; L)$, is *L-essential* if, for every $G \in K_A^0(X, C; L)$, respectively $G \in K_A^1(X, C; L)$, which agrees with F on A, we have that $L - G$ has a zero in X. Otherwise, F is *L-inessential*, i.e., there exists a $G \in K_A^0(X, C; L)$, respectively $G \in K_A^1(X, C; L)$, which agrees with F on A and $L - G$ is zero free on X.

Definition 2.3. Two mappings $F, G \in K_A^0(X, C; L)$, respectively $F, G \in K_A^1(X, C; L)$, are *homotopic* in $K_A^0(X, C; L)$, respectively $K_A^1(X, C; L)$, written $F \cong G$ in $K_A^0(X, C; L)$, respectively $K_A^1(X, C; L)$ if there is a continuous, compact mapping, respectively a bounded P-concentrative mapping, $N : X \times [0,1] \to C$ with $N_t(u) = N(u,t) : X \to C$ belonging to $K_A^0(X, C; L)$, respectively $K_A^1(X, C; L)$, for each $t \in [0,1]$ and $N_0 = F$, $N_1 = G$.

Remark. Notice $\cong$ is an equivalence relation in $K_A^0(X, C; L)$, respectively $K_A^1(X, C; L)$.

Theorem 2.1. *Let C, X, A and E be as above.*

(a) *Assume $L : X \to C$ is continuous and satisfies* (2.1) *and that $F \in K_A^0(X, C; L)$. Then the following are equivalent*:

(i) *F is L-inessential.*

(ii) *There is a $G \in K_A^0(X, C; L)$ with $F \cong G$ in $K_A^0(X, C; L)$ and with $L - G$ zero free on X.*

(b) *Assume $L : X \to C$ is continuous and satisfies* (2.2) *and that $F \in K_A^1(X, C; L)$. Then the following are equivalent*:

(i) *F is L-inessential.*

(ii) *There is a $G \in K_A^1(X, C; L)$ with $F \cong G$ in $K_A^1(X, C; L)$ and with $L - G$ zero free on X.*

Proof. We first show (i) implies (ii), in both case (a) and (b). Let $G \in K_A^0(X, C; L)$, respectively $G \in K_A^1(X, C; L)$, with $G = F$ on A and $L - G$ zero free on X. Define $N : X \times [0,1] \to C$ by

$$N(x,t) = tG(x) + (1-t)F(x).$$

Clearly N is continuous.

Case (a). Suppose $F, G \in K_A^0(X, C; L)$.

Then clearly $N : X \times [0,1] \to C$ is a compact map. Also, since $F = G$ on A and $L - G$ is zero free on X, we have for $x \in A$ that

$$L(x) - N_t(x) = Lx - (tG(x) + (1-t)F(x)) = L(x) - G(x) \neq 0,$$

so $L-N_t$ is zero free on A for each $t\in[0,1]$. Also, clearly, $N_t \in K_A^0(X,C;L)$ for each $t\in[0,1]$. Finally, $N_0=F$ and $N_1=G$ so $F\cong G$ in $K_A^0(X,C;L)$.

Case (b). Suppose $F,G\in K_A^1(X,C;L)$.

We claim that $N: X\times[0,1]\to C$ is a bounded P-concentrative mapping. To see this, let $p\in P$ and let W be a bounded non-p-precompact subset of $X\times[0,1]$. Notice if $(x,t)\in W$, then $N(x,t)=tG(x)+(1-t)F(x)\subseteq \mathrm{co}\,(G(\pi W)\cup F(\pi W))$. Consequently,

$$N(W)\subseteq \mathrm{co}\,(G(\pi W)\cup F(\pi W))$$

and so

$$\begin{aligned}\alpha_p(N(W)) &\le \alpha_p(\overline{\mathrm{co}}\,(G(\pi W)\cup F(\pi W)))\\ &= \alpha_p(G(\pi W)\cup F(\pi W))\\ &= \max\{\alpha_p(G(\pi W)),\alpha_p(F(\pi W))\}\\ &< \max\{\alpha_p(\pi W),\alpha_p(\pi W)\}\\ &= \alpha_p(\pi W).\end{aligned}$$

Thus, N is a P-concentrative mapping. Essentially the same reasoning as in case (a) implies $L-N_t$ is zero free on A for each $t\in[0,1]$. It remains to show $N_t\in K_A^1(X,C;L)$ for each $t\in[0,1]$. Fix $t\in[0,1]$ and let Ω be a bounded non-p-precompact subset, i.e., $\alpha_p(\Omega)\neq 0$, of X. Then

$$\alpha_p(N_t(\Omega))=\alpha_p(N(\Omega\times\{t\}))<\alpha_p(\pi(\Omega\times\{t\}))=\alpha_p(\Omega)$$

since $\pi(\Omega\times\{t\})=\Omega$.

Thus for each $t\in[0,1]$, we have $N_t\in K_A^1(X,C;L)$ and so $F\cong G$ in $K_A^1(X,C;L)$.

We now show (ii) implies (i), in both case (a) and (b). Let $N: X\times[0,1]\to C$ be a continuous, compact map, respectively bounded P-concentrative map, from $G\in K_A^0(X,C;L)$, respectively $G\in K_A^1(X,C;L)$, to F with $N_0=G$ and $N_1=F$. In particular, $L-N_t$ is zero free on A for each $t\in[0,1]$. Let

$$B=\{x\in X: L(x)=N(x,t) \text{ for some } t\in[0,1]\}.$$

If $B = \varnothing$ then, in particular, $L - N_1 = L - F$ is zero free and so F is L-inessential. So it remains to consider the case when $B \neq \varnothing$. First note $B \cap A = \varnothing$. Also B is closed. To see this, let $(x_\alpha) \in B$ be a net of points of B, i.e., $L(x_\alpha) = N(x_\alpha, t_\alpha))$ converging to x. Without loss of generality, assume t_α converges to $t \in [0,1]$. By the continuity of N and L, we have $L(x) = N(x,t)$ so $x \in B$ and B is closed. Next we claim that B is *compact*.

Case (a). Suppose $F, G \in K_A^0(X, C; L)$.

Then, since

$$L(B) \subseteq N(B \times [0,1])$$

we have that $L(B)$ is relatively compact. In addition

$$B \subseteq L^{-1}(L(B)) \subseteq L^{-1}(\overline{L(B)})$$

together with (2.1) implies that B is precompact in E. Now since B is a closed precompact subset of $X \subseteq C$ and C is complete, then B is compact [**18**].

Case (b). Suppose $F, G \in K_A^1(X, C; L)$.

If B is a non-p-precompact subset, i.e., $\alpha_p(B) \neq 0$, of X, then

$$\alpha_p(L(B)) \leq \alpha_p(N(B \times [0,1])) < \alpha_p(\pi(B \times [0,1])) = \alpha_p(B) \tag{2.3}$$

since $\pi(B \times [0,1]) = B$.

Now (2.3) together with (2.2) yields a contradiction and so, for *each* $p \in P$, we have that B is precompact in the semi-normed space (E, p). Hence, B is precompact in E. Now, since B is a closed precompact subset of $X \subseteq C$ and C is complete, then B is compact [**18**].

So in both cases B is compact. Now Theorem 1.8 implies that E, and hence X, with subspace topology, is Tychonoff. In addition, since $A \cap B = \varnothing$, then Theorem 1.9 implies that there is a continuous (Urysohn) function $\mu : X \to [0,1]$ with $\mu(A) = 1$ and $\mu(B) = 0$. Define

$$J : X \to C \quad \text{by} \quad J(x) = N(x, \mu(x)).$$

Clearly J is continuous. We claim that $J : X \to C$ is a compact, respectively bounded P-concentrative, map with $J = F$ on A and

$L-J$ zero free on X. If this is true, then $J \in K_A^0(X,C;L)$, respectively $K_A^1(X,C;L)$, with $L-J$ zero free on X and $J=F$ on A. Consequently, F is L-inessential and we are finished.

It remains to prove the claim. $L-J$ is zero free since $L(x)-J(x)=0$ means $L(x)=N(x,\mu(x))$ which means $x \in B$ and so $\mu(x)=0$, i.e., $L(x)=N(x,0)$, a contradiction since $L(x)-N(x,0)=L(x)-G(x)$ is zero free. To see that $J=F$ on A, notice if $x \in A$, then $\mu(x)=1$ and so $J(x)=N(x,\mu(x))=N(x,1)=F(x)$. It remains to show $J:X \to C$ is a compact, respectively bounded P-concentrative, map.

Case (a). Suppose $F,G \in K_A^0(X,C;L)$.

Clearly J is a compact map.

Case (b). Suppose $F,G \in K_A^1(X,C;L)$.

Let Ω be a bounded non-p-precompact subset of X, and let $\Omega^* = \{(x,\mu(x)) : x \in \Omega\} \subseteq X \times [0,1]$. Then, since $J(\Omega)=N(\Omega^*)$ and $\pi(\Omega^*)=\Omega$, we have

$$\alpha_p(J(\Omega)) = \alpha_p(N(\Omega^*)) < \alpha_p(\pi(\Omega^*)) = \alpha_p(\Omega).$$

Remark. In the above inequality we used the fact that, if Ω is a non-p-precompact subset of X, then Ω^* is a non-p-precompact subset of $X \times [0,1]$.

Thus J is a bounded P-concentrative mapping. □

Theorem 2.2. *Let C, X, A and E be as above.*

(a) *Assume $L:X \to C$ is continuous and satisfies* (2.1). *Suppose F and G are two maps in $K_A^0(X,C;L)$ such that $F \cong G$ in $K_A^0(X,C;L)$. Then F is L-essential if and only if G is L-essential.*

(b) *Assume $L:X \to C$ is continuous and satisfies* (2.2). *Suppose F and G are two maps in $K_A^1(X,C;L)$ such that $F \cong G$ in $K_A^1(X,C;L)$. Then F is L-essential if and only if G is L-essential.*

Proof. If F is L-inessential, then Theorem 2.1 guarantees a map $T \in K_A^0(X,C;L)$, respectively $K_A^1(X,C;L)$, with $L-T$ zero free on

X and $F \cong T$ in $K_A^0(X, C; L)$, respectively $K_A^1(X, C; L)$. Thus $G \cong T$ in $K_A^0(X, C; L)$, respectively $K_A^1(X, C; L)$, and so G is L-inessential by Theorem 2.1. Symmetry will now imply that F is L-inessential if and only if G is L-inessential. □

Remark. If E is metrizable, then we can remove assumption (2.1) and (2.2) in Theorems 2.1 and 2.2. This is due to the fact that metric spaces are normal spaces so as to guarantee the existence of the Urysohn function in Theorem 2.1 we need only B to be closed.

The remainder of the paper concerns the case where E is a Banach space. For convenience, we will restate Theorem 2.2 in this setting. Let E be a Banach space and C a closed convex subset of E, $X \subseteq C$ and $A \subseteq X$ with A closed in X and X closed in C. Also, let $L : X \to C$ be a continuous operator. $K(X, C; L)$ denotes the set of all (bounded, continuous) condensing maps $F : X \to C$, and $K_A(X, C; L)$ denotes the set of all mappings $F \in K(X, C; L)$ such that $L - F$ is zero free on A. We call $N : X \times [0, 1] \to C$ a condensing map if N is continuous, bounded, $\alpha(N(W)) \leq \alpha(\pi W)$ for all bounded sets W of $X \times [0, 1]$ and $\alpha(N(\Omega)) < \alpha(\pi\Omega)$ for all bounded nonprecompact subsets Ω of $X \times [0, 1]$. A map $F \in K_A(X, C; L)$ is L-essential if, for every $G \in K_A(X, C; L)$ which agrees with F on A we have that $L - G$ has a zero in X. Two mappings $F, G \in K_A(X, C; L)$ are homotopic in $K_A(X, C; L)$, written $F \cong G$ in $K_A(X, C; L)$ if there is a condensing map $N : X \times [0, 1] \to C$ with $N_t(u) = N(u, t) : X \to C$ belonging to $K_A(X, C; L)$ for each $t \in [0, 1]$ and $N_0 = F$, $N_1 = G$.

Theorem 2.3. *Let E be a Banach space with C, X and A as above. Also assume $L : X \to C$ is continuous. Suppose F and G are two maps in $K_A(X, C; L)$ such that $F \cong G$ in $K_A(X, C; L)$. Then F is L-essential if and only if G is L-essential.*

Theorem 2.3 immediately yields the following nonlinear alternative of Leray-Schauder type.

Theorem 2.4. *Let U be an open subset of a closed convex set C in a Banach space E and $L : \overline{U} \to C$ a continuous map. In addition, assume* *the mapping* $G \in K_{\partial U}(\overline{U}, C; L)$ *is* L*-essential. Then every condensing map* $F : \overline{U} \to C$ *has at least one of the following properties*:

(A1) $L(x) = F(x)$ *for some* $x \in \overline{U}$ *or*

(A2) *there exists an* $x \in \partial U$ *and* $\lambda \in (0, 1)$ *with*

$$L(x) = \lambda F(x) + (1 - \lambda)G(x).$$

Remark. $\overline{U}$ and ∂U denote the closure of U and the boundary of U in C.

Proof. We can assume $L - F|_{\partial U}$ is zero free for otherwise (A1) is satisfied. Consider the homotopy $N : \overline{U} \times [0, 1] \to C$ joining G and F given by

$$N(u, t) = tF(u) + (1 - t)G(u).$$

Now N is a condensing map since, if W is a nonprecompact subset of $\overline{U} \times [0, 1]$, then

$$\begin{aligned} \alpha(N(W)) &\le \alpha(\overline{\text{co}}(F(\pi W) \cup G(\pi W))) \\ &\le \max\{\alpha(F(\pi W)), \alpha(G(\pi W))\} \\ &< \alpha(\pi W). \end{aligned}$$

Now either $L - N_t$ is zero free on ∂U for each $t \in [0, 1]$ or it is not. If $L - N_t$ is zero free on ∂U for each $t \in [0, 1]$, then Theorem 2.3 implies that $L - F$ has a zero in U so (A1) occurs. If $L - N_t$ is not zero free on ∂U for each $t \in [0, 1]$, then there exists $x \in \partial U$ and $\lambda \in [0, 1]$ with $L(x) - [\lambda F(x) + (1 - \lambda)G(x)] = 0$. Now $\lambda \ne 1$ since $L - F|_{\partial U}$ was assumed to be zero free and $\lambda \ne 0$ since $G \in K_{\partial U}(\overline{U}, C; L)$, in particular, $L - G$ is zero free on ∂U. Hence (A2) occurs. □

Remark. There is an analogue of Theorem 2.4 for the case when E is a locally convex Hausdorff linear topological space.

3. Fixed point theorems. In this section a variety of fixed point results are established. The fixed point results, Theorems 3.7–3.10, are all new. Also nonlinear alternatives of Leray-Schauder type are established in this section; our main results are Theorems 3.4–3.6 (plus remarks). Applications of these results will be given in Section 4. Our first results were motivated by work of Browder [**4**] and Precup [**26**].

Theorem 3.1. *Let U be an open set in a closed convex set C of a real Banach space E and $\Omega \supseteq \overline{U}$ a subset of E. Assume $p \in U$, $F_2(\overline{U})$ bounded and $F : \overline{U} \to E$ is given by $F = F_1 + F_2$ where $-F_1 : \Omega \to E$ (single valued) is m-accretive and $F_2 : \overline{U} \to E$ is a (bounded, continuous) condensing map. In addition, suppose $(I - \lambda F_1)^{-1}(\lambda F_2(\cdot) + (1-\lambda)p) : \overline{U} \to C$ for each $\lambda \in [0,1]$. Then, either*

(A1) *F has a fixed point in $\overline{U}$, or*

(A2) *there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u) + (1-\lambda)p$.*

Proof. Let $N - 1 = (I - F_1)^{-1} F_2$. Recall [**2, 11**], since $-F_1$ is m-accretive, then $(I - \mu F_1)^{-1} : E \to \Omega$ is nonexpansive for each $\mu > 0$. We can assume that $I - N_1|_{\partial U}$ is zero free for otherwise (A1) occurs. Let $G(x) = p$ for $x \in \overline{U}$. Consider the homotopy $N : \overline{U} \times [0,1] \to C$ joining G and $N_1 = (I - F_1)^{-1} F_2$ given by

$$N(u,\lambda) = (I - \lambda F_1)^{-1}(\lambda F_2(u) + (1-\lambda)p).$$

We claim

$$N : \overline{U} \times [0,1] \longrightarrow C \text{ is a condensing map.} \tag{3.1}$$

To see this, we first show $N(\cdot,\lambda) : \overline{U} \to C$ is a condensing map for each $\lambda \in [0,1]$. Fix $\lambda \in [0,1]$ and let W be a bounded subset of $\overline{U}$. Then

$$N_\lambda(W) \subseteq (I - \lambda F_1)^{-1} \mathrm{co}\,(F_2(W) \cup \{p\})$$

and since $(I - \mu F_1)^{-1} : E \to \Omega$ is nonexpansive for each $\mu > 0$, we have that $N_\lambda : \overline{U} \to C$ is a condensing map for each $\lambda \in [0,1]$.

Next we claim that

$$\{N(u,\cdot); u \in \overline{U}\} \quad \text{is equicontinuous for each} \quad t \in [0,1]. \tag{3.2}$$

Suppose for the moment that (3.2) is true. We will now show that (3.1) is true. Let W be a bounded nonprecompact subset of $\overline{U} \times [0,1]$. Let $\varepsilon(t) > 0$ be such that

$$\alpha(N_t(\pi W)) < \alpha(\pi W) - 2\varepsilon(t), \tag{3.3}$$

and let $V(t)$ be a neighborhood of t such that

$$\|N_t(u) - N_s(u)\| \le \varepsilon(t) \quad \text{for all} \quad s \in V(t) \text{ and } u \in \pi W. \tag{3.4}$$

Remark. In (3.3) we used the fact that, if W is a nonprecompact subset of $\overline{U} \times [0,1]$, then πW is a nonprecompact subset of $\overline{U}$.

Also,if $s, s_1 \in V(t)$ and $u, u_1 \in \pi W$, we have

$$\begin{aligned} N(u,s) - N(u_1,s_1) = {} & [N(u,s) - N(u,t)] + [N(u_1,t) - N(u_1,s_1)] \\ & + [N_t(u) - N_t(u_1)] \end{aligned}$$

and so (3.3) and (3.4) imply

$$\alpha(N(\pi W \times V(t))) < \alpha(\pi W). \tag{3.5}$$

Now $\{V(t), t \in [0,1]\}$ is an open cover of $[0,1]$ and, since $[0,1]$ is compact, we suppose

$$\{V(t_i), i = 1, \dots, n\} \text{ is a finite covering of } [0,1].$$

Now (3.5) together with the properties of α imply

$$\begin{aligned} \alpha(N(W)) & \le \alpha(N(\pi W \times [0,1])) \\ & \le \max\{\alpha(N(\pi W \times V(t_i))), i = 1, \dots, n\} \\ & < \alpha(\pi W) \end{aligned}$$

so (3.1) is true. It remains to prove (3.2). Let $t = 0$ to begin with. Since $(I - \mu F_1)^{-1} : E \to \Omega$ is nonexpansive, we have for $\lambda \in [0,1]$ and $u \in \overline{U}$ that

$$\begin{aligned} \|N(u,\lambda) - p\| \le {} & \|(I - \lambda F_1)^{-1}(\lambda F_2(u) + (1-\lambda)p) - (I - \lambda F_1)^{-1}(p)\| \\ & + \|(I - \lambda F_1)^{-1}(p) - p\| \\ \le {} & \lambda \|F_2(u) - p\| + \|(I - \lambda F_1)^{-1}(p) - p\|. \end{aligned}$$

Now, since $F_2(\overline{U})$ is bounded and $(I - \lambda F_1)^{-1}(p) \to p$ as $\lambda \to 0$ (note since $(I - \lambda F_1)^{-1}$ is nonexpansive $\|(I - \lambda F_1)^{-1}(p) - p\| \le$

$\|p - (I - \lambda F_1)p\| = \lambda\|F_1(p)\|$), then (3.2) holds when $t = 0$. Next fix $t \in (0, 1]$. For $\lambda \in (0, 1]$ and $u \in \overline{U}$, we have

$$\begin{aligned}\|N(u,\lambda) - N(u,t)\| &\le \|(I-\lambda F_1)^{-1}(\lambda F_2(u) + (1-\lambda)p) \\ &\qquad - (I - tF_1)^{-1}(\lambda F_2(u) + (1-\lambda)p)\| \\ &\quad + \|(I - tF_1)^{-1}(\lambda F_2(u) + (1-\lambda)p) \\ &\qquad - (I - tF_1)^{-1}(tF_2(u) + (1-t)p)\| \\ &\le \|(I-\lambda F_1)^{-1}(\lambda F_2(u) + (1-\lambda)p) \\ &\qquad - (I - tF_1)^{-1}(\lambda F_2(u) + (1-\lambda)p)\| \\ &\quad + |\lambda - t|\|F_2(u) - p\|.\end{aligned}$$

Let $y \in \operatorname{co}(F_2(\overline{U}) \cup \{p\})$. We claim that, if $\lambda \in (0, 1]$ and t is as above, then

$$\|(I-\lambda F_1)^{-1}(y) - (I-tF_1)^{-1}(y)\| \le \frac{|\lambda - t|}{t}\|y - (I-tF_1)^{-1}(y)\|. \tag{3.6}$$

If this is true, then (3.6), together with the second last inequality, implies, since $F_2(\overline{U})$ and $(I - tF_1)^{-1}(\operatorname{co}(F_2(\overline{U}) \cup \{p\}))$ are bounded, that $\{N(u,\cdot) : u \in \overline{U}\}$ is equicontinuous at t, and we are finished. It remains to prove (3.6). To see this, notice

$$\begin{aligned}\frac{1}{\lambda}(I - (I-\lambda F_1)^{-1}) &= \frac{1}{\lambda}((I-\lambda F_1) - I)(I - \lambda F_1)^{-1}) \\ &= -F_1(I-\lambda F_1)^{-1}.\end{aligned}$$

Now, since $-F_1$ is accretive, we have for $y \in \operatorname{co}(F_2(\overline{U}) \cup \{p\})$ that

$$\begin{aligned}0 &\le \Big(\frac{1}{\lambda}(y - (I-\lambda F_1)^{-1}(y)) - \frac{1}{t}(y - (I-tF_1)^{-1}(y)), \\ &\qquad\qquad (I-\lambda F_1)^{-1}(y) - (I-tF_1)^{-1}(y)\Big)_+ \\ &= \Big(\Big(\frac{1}{\lambda} - \frac{1}{t}\Big)(y - (I-tF_1)^{-1}(y)) \\ &\qquad - \frac{1}{\lambda}((I-\lambda F_1)^{-1}(y) - (I-tF_1)^{-1}(y)), \\ &\qquad\qquad (I-\lambda F_1)^{-1}(y) - (I-tF_1)^{-1}(y)\Big)_+\end{aligned}$$

$$= \left(\left(\frac{1}{\lambda} - \frac{1}{t} \right) (y - (I - tF_1)^{-1}(y)), \right.$$
$$\left. (I - \lambda F_1)^{-1}(y) - (I - tF_1)^{-1}(y) \right)_+$$
$$- \frac{1}{\lambda} \|(I - \lambda F_1)^{-1}(y) - (I - tF_1)^{-1}(y)\|^2$$

since $(z_1 + \alpha z_2, z_2)_+ = (z_1, z_2)_+ + \alpha \|z_2\|^2$ for all $\alpha \in \mathbf{R}$. Now (3.6) follows immediately. Thus (3.1) is true. Also, $N : \overline{U} \times [0,1] \to C$ is continuous. To see this, suppose $(u_n, \lambda_n) \to (u, \lambda)$. Then

$$\begin{aligned}
\|N(u_n, \lambda_n) - N(u, \lambda)\| &= \|(I - \lambda_n F_1)^{-1}(\lambda_n F_2(u_n) + (I - \lambda_n)p) \\
&\qquad - (I - \lambda F_1)^{-1}(\lambda F_2(u) + (1 - \lambda)p)\| \\
&\leq \|(I - \lambda_n F_1)^{-1}(\lambda_n F_2(u_n) + (1 - \lambda_n)p) \\
&\qquad - (I - \lambda_n F_1)^{-1}(\lambda_n F_2(u) + (1 - \lambda_n)p)\| \\
&\quad + \|(I - \lambda_n F_1)^{-1}(\lambda_n F_2(u) + (1 - \lambda_n)p) \\
&\qquad - (I - \lambda F_1)^{-1}(\lambda F_2(u) + (1 - \lambda)p)\| \\
&\leq |\lambda_n| \|F_2(u_n) - F_2(u)\| + \|N(u, \lambda_n) - N(u, \lambda)\|.
\end{aligned}$$

Next $G \in K_{\partial U}(\overline{U}, C; I)$ since $p \in U$ and we claim G is I-essential. To see this, take any $H \in K_{\partial U}(\overline{U}, C; I)$ with $H(x) = p$ for $x \in \partial U$. We must show $I - H$ has a zero in U. Consider

$$J(x) = \begin{cases} H(x) & x \in \overline{U}, \\ p & x \in C/\overline{U}. \end{cases}$$

It is easy to see that $J : C \to C$ is continuous and J is a (bounded, continuous) condensing map. Sadovskii's fixed point theorem, Theorem 1.2, implies that J has a fixed point $u \in C$. In addition, since $J(x) = p \in U$ for $x \in C/\overline{U}$ we have $u \in \overline{U}$. Thus $u = J(u) = H(u)$ and since $H(x) = p$ for $x \in \partial U$ we have $u \in U$. Hence H has a fixed point in U, i.e., $I - H$ has a zero in U so G is I-essential.

Now either $I - N_t$ is zero free on ∂U for each $t \in [0,1]$ or it is not. If $I - N_t$ is zero free on ∂U for each $t \in [0,1]$, then Theorem 2.3 implies that $I - N_1$ has a zero in U so (A1) follows. If $I - N_t$ is not zero free on ∂U for each $t \in [0,1]$, then there exist $x \in \partial U$ and $\lambda \in [0,1]$ with $x = (I - \lambda F_1)^{-1}(\lambda F_2(x) + (1 - \lambda)p)$. Now $\lambda \neq 1$ since $I - N_1|_{\partial U}$ was assumed to be zero free and $\lambda \neq 0$ since $p \in U$. Hence, (A2) occurs. □

Remark. There is an analogue of Theorem 3.1 if E is a locally convex Hausdorff linear topological space. In this situation let $F_1 = 0$ and $F_2 : \overline{U} \to E$ be a bounded P-concentrative map. The reasoning is essentially the same (since $N : \overline{U} \times [0,1] \to C$, given by $N(u,\lambda) = \lambda F_2(u) + (1-\lambda)p$, is easily seen to be a bounded P-concentrative mapping, the only difference is that Daneš's fixed point theorem replaces Sadovskii's fixed point theorem.

Theorem 3.2. *Let U be an open set in a closed convex set C of a real Banach space E. Assume $p \in U$, $F_2(\overline{U})$ bounded and $F : \overline{U} \to E$ is given by $F = F_1 + F_2$ where $-F_1 : E \to E$ (single valued) is continuous and accretive and $F_2 : \overline{U} \to E$ is a (bounded, continuous) condensing map. In addition, suppose $(I - \lambda F_1)^{-1}(\lambda F_2(\cdot) + (1-\lambda)p) : \overline{U} \to C$ for each $\lambda \in [0,1]$. Then either*

(A1) *F has a fixed point in $\overline{U}$ or*

(A2) *There exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u) + (1-\lambda)p$.*

Proof. The result follows immediately from Theorem 3.1 since $-F_1$ is m-accretive [**2**, p. 159], [**21**, p. 124]. □

Remark. See [**4**, p. 139] for other examples of m-accretive maps.

Theorem 3.3. *Let U be an open set in a real Banach space E and $\Omega \supseteq \overline{U}$ a subset of E. Assume $0 \in U$ with $F_2(\overline{U})$ and $F(\overline{U})$ bounded and $F : \overline{U} \to E$ is given by $F = F_1 + F_2$ where $-F_1 := \Omega \to E$ is m-accretive and $F_2 : \overline{U} \to E$ is a bounded α-Lipschitzian map with $k = 1$. Also suppose $(I - F)(\overline{U})$ is closed. Then either*

(A1) *F has a fixed point in $\overline{U}$ or*

(A2) *there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Proof. Assume that (A2) does not hold. Consider, for each $n \in$

$\{2, 3, \dots\}$ the mapping

$$S_n = \left(1 - \frac{1}{n}\right) F : \overline{U} \to E. \tag{3.7}$$

Notice $(1 - 1/n)F_2 : \overline{U} \to E$ is condensing and $-(1 - 1/n)F_1 : \Omega \to E$ is accretive since, for $x, y \in \Omega$, we have

$$\begin{aligned}\left(-\left(1 - \frac{1}{n}\right) F_1(x) + \left(1 - \frac{1}{n}\right) F_1(y), x - y \right)_+ \\ &= \left(\left(1 - \frac{1}{n}\right)[-F_1(x) + F_1(y))], x - y \right)_+ \\ &= \left(1 - \frac{1}{n}\right)(-F_1(x) + F_1(y), x - y)_+ \geq 0\end{aligned}$$

since $(\alpha z_1, \beta z_2)_+ = \alpha\beta(z_1, z_2)_+$ for $z_1, z_2 \in E$ and $\alpha, \beta \in \mathbf{R}$ with $\alpha\beta \geq 0$. Also, since $I - \mu F_1$ is onto E for all $\mu > 0$, we have in particular that $I - (I - 1/n)F_1$ is onto E. Thus, $-(I - 1/n)F_1 : \Omega \to E$ is m-accretive. Apply Theorem 3.1 to S_n. If there exists $\lambda \in (0, 1)$ and $u \in \partial U$ with $u = \lambda S_n(u)$, then

$$u = \lambda \left(1 - \frac{1}{n}\right) F(u) = \eta F(u) \quad \text{where} \quad 0 < \eta = \lambda \left(1 - \frac{1}{n}\right) < 1,$$

which is a contradiction since (A2) was assumed not to hold. Consequently, for each $n \in \{2, 3, \dots\}$ we have that S_n has a fixed point $u_n \in \overline{U}$. Notice also, since $u_n = (1 - 1/n)F(u_n)$, we have that $u_n - F(u_n) = -(1/n)F(u_n)$ and so $u_n - F(u_n) \to 0$ as $n \to \infty$ (since $F(\overline{U})$ is bounded). Consequently $0 \in (I - F)(\overline{U})$ since $(I - F)(\overline{U})$ is closed. Thus there exists $u \in \overline{U}$ with $0 = (I - F)(u)$. $\square$

Theorem 3.4. *Let U be an open set in a closed convex set C of a real Banach space E. Assume $p \in U$, $F_2(\overline{U})$ bounded and $F : \overline{U} \to E$ is given by $F = F_1 + F_2$ where $I - F_1 : E \to E$ (single valued) is continuous and strongly accretive and $F_2 : \overline{U} \to E$ is a continuous, completely continuous, i.e., α-Lipschitzian with $k = 0$, map. In addition, suppose $(I - \lambda F_1)^{-1}(\lambda F_2(\cdot) + (1 - \lambda)p) : \overline{U} \to C$ for each $\lambda \in [0, 1]$. Then either*

(A1) *F has a fixed point in $\overline{U}$ or*

(A2) *there exist* $u \in \partial U$ *and* $\lambda \in (0,1)$ *with* $u = \lambda F(u) + (1-\lambda)p$.

Proof. Now there exists $c > 0$ with

(3.8) $((I-F_1)(x)-(I-F_1)(y), x-y)_+ \geq c\|x-y\|^2 \quad \text{for all} \quad x, y \in E.$

Also, from Theorem 1.7, $I - F_1$ is a homeomorphism from E onto E with $(I-F_1)^{-1} : E \to E$ a Lipschitz map with Lipschitz constant $1/c$.

Let $N_1 = (I-F_1)^{-1}F_2$. We can assume $I - N_1|_{\partial U}$ is zero free for otherwise (A1) occurs. Let $G(x) = p$ for $x \in \overline{U}$. As in Theorem 3.1, we have that $G \in K_{\partial U}(\overline{U}, C; I)$ and G is I-essential. We first claim that $I - \lambda F_1 : E \to E$ is strongly accretive; here $0 \leq \lambda \leq 1$. This is immediate since, for $x, y \in E$, we have

$$
\begin{aligned}
&((I-\lambda F_1)(x) - (I-\lambda F_1)(y), x-y)_+ \\
&\qquad = (\lambda[(I-F_1)(x) - (I-F_1)(y)] + (1-\lambda)(x-y), x-y)_+ \\
&\qquad = \lambda((I-F_1)x - (I-F_1)y, x-y)_+ + (1-\lambda)\|x-y\|^2 \\
&\qquad \geq (\lambda c + (1-\lambda))\|x-y\|^2
\end{aligned}
$$

since $(z_1 + \alpha z_2, z_2)_+ = (z_1, z_2)_+ + \alpha|z_2|^2$ for $z_1, z_2 \in E$ and α a scalar. Thus, Theorem 1.7 implies that $(I-\lambda F_1)^{-1} : E \to E$ a Lipschitz map with Lipschitz constant $1/c_\lambda$; here, $c_\lambda = [\lambda c + (1-\lambda)]$.

Remark. Notice $1/c_\lambda \leq 1/\min\{1, c\}$.

Consider the homotopy $N : \overline{U} \times [0,1] \to C$ joining G and N_1 given by

$$N(u, \lambda) = (I-\lambda F_1)^{-1}(\lambda F_2(u) + (1-\lambda)p).$$

We now check that $N : \overline{U} \times [0,1] \to C$ is a completely continuous map. Fix $\lambda \in [0,1]$ and let W be a bounded subset of $\overline{U}$. As in Theorem 3.1,

$$\alpha(N_\lambda(W)) \leq \alpha((I-\lambda F_1)^{-1}\mathrm{co}\,(F_2(W) \cup \{p\})) \leq \frac{1}{c_\lambda}\alpha(F_2(W)) = 0$$

so $N_\lambda : \overline{U} \to C$ is a completely continuous map for each $\lambda \in [0,1]$. Next we claim that $\{N(u, \cdot); u \in \overline{U}\}$ is equicontinuous for each $t \in [0,1]$. Fix $t = 0$. For $\lambda \in [0,1]$ and $u \in \overline{U}$, we have

$$\|N(u,\lambda) - p\| \leq \frac{\lambda}{\min\{1, c\}}\|F_2(u) - p\| + \|(I-\lambda F_1)^{-1}(p) - p\|.$$

Notice $(I-\lambda F_1)^{-1}(p) \to p$ as $\lambda \to 0$ since $\|(I-\lambda F_1)^{-1}(p)-p\| \leq (\lambda/\min\{1,c\})\|F_1(p)\|$. Next fix $t \in (0,1]$. Also, for $\lambda \in (0,1]$ and $u \in \overline{U}$, we have, as in Theorem 3.1, that

$$\begin{aligned}\|N(u,\lambda)-N(u,t)\| \leq \|(I-\lambda F_1)^{-1}(\lambda F_2(u)+(1-\lambda)p) \\ -(I-tF_1)^{-1}(\lambda F_2(u)+(1-\lambda)p)\| \\ +\frac{|\lambda-t|}{\min\{1,c\}}\|F_2(u)-p\|,\end{aligned}$$

and so $\{N(u,\cdot); u \in \overline{U}\}$ is equicontinuous for each $t \in (0,1]$ if we show that, for $\lambda \in (0,1]$ and $y \in \operatorname{co}(F_2(\overline{U}) \cup \{p\})$ we have

$$\|(I-\lambda F_1)^{-1}(y)-(I-tF_1)^{-1}(y)\| \leq \frac{2|\lambda-t|}{ct^2}\|y-(I-tF_1)^{-1}(y)\| \quad \text{for} \quad \lambda > \frac{t}{2}.$$

To see this, notice

$$\begin{aligned}\frac{1}{\lambda}I+\left(1-\frac{1}{\lambda}\right)(I-\lambda F_1)^{-1} &= (I-\lambda F_1)^{-1}+\frac{1}{\lambda}(I-(I-\lambda F_1)^{-1}) \\ &= (I-F_1)(I-\lambda F_1)^{-1}.\end{aligned}$$

Now since $I-F_1$ is strongly accretive we have, for $y \in \operatorname{co}(F_2(\overline{U}) \cup \{p\})$, that

$$\begin{aligned}c\|(I-\lambda F_1)^{-1}(y)&-(I-tF_1)^{-1}(y)\|^2 \\ &\leq \left(\frac{1}{\lambda}y+\left(1-\frac{1}{\lambda}\right)(I-\lambda F_1)^{-1}(y)\right. \\ &\qquad -\frac{1}{t}y-\left(1-\frac{1}{t}\right)(I-tF_1)^{-1}(y), \\ &\qquad \left.(I-\lambda F_1)^{-1}(y)-(I-tF_1)^{-1}(y)\right)_+ \\ &= \left(\left(\frac{1}{\lambda}-\frac{1}{t}\right)(y-(I-tF_1)^{-1}(y)),\right. \\ &\qquad \left.(I-\lambda F_1)^{-1}(y)-(I-tF_1)^{-1}(y)\right)_+\end{aligned}$$

$$- \left(\frac{1}{\lambda} - 1\right) \|(I - \lambda F_1)^{-1}(y) - (I - tF_1)^{-1}(y)\|^2$$
$$\le \left(\left(\frac{1}{\lambda} - \frac{1}{t}\right)(y - (I - tF_1)^{-1}(y)), \right.$$
$$\left. (I - \lambda F_1)^{-1}(y) - (I - tF_1)^{-1}(y) \right)_+$$

so the above inequality holds. Let W be a bounded subset of $\overline{U} \times [0, 1]$. Let $\varepsilon > 0$ be given and let $V(t)$ be a neighborhood of t such that

$$\|N_t(u) - N_s(u)\| \le \varepsilon \quad \text{for all} \quad s \in V(t) \text{ and } u \in \pi W.$$

Also

$$\alpha(N_t(\pi W)) = 0,$$

and, as in Theorem 3.1,

$$\alpha(N(\pi W \times V(t))) \le 2\varepsilon \quad \text{and} \quad \alpha(N(W)) \le 2\varepsilon.$$

Consequently, $N : \overline{U} \times [0, 1] \to C$ is completely continuous. Essentially the same reasoning as in the last paragraph of Theorem 3.1 establishes the result. □

Remark. If, for example, $c > 1$ in (3.8), then we could take $F_2 : \overline{U} \to E$ to be a condensing map in the statement of Theorem 3.4. More generally, there is an analogue of Theorem 3.4 if $I - F_1 : E \to E$ satisfies (3.8) and $F_2 : \overline{U} \to E$ is an α-Lipschitzian map with $k = \min\{1, c\}$ and $\alpha(F_2(W)) < \min\{1, c\}\alpha(W)$ for all bounded nonprecompact subsets W of $\overline{U}$.

Theorem 3.5. *Let U be an open set in a real Banach space, E. Assume $0 \in U$ with $F_2(\overline{U})$ and $F(\overline{U})$ bounded and $F : \overline{U} \to E$ is given by $F = F_1 + F_2$ where $I - F_1 : E \to E$ is continuous and* (3.8) *holds with $c > 1$ and $F_2 : \overline{U} \to E$ is a bounded α-Lipschitzian map with $k = 1$. Also, suppose $(I - F)(\overline{U})$ is closed. Then either*

(A1) *F has a fixed point in $\overline{U}$ or*

(A2) *there exist $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.*

Proof. Assume (A2) does not hold. Consider, for each $n \in \{2, 3, \dots\}$ the mapping S_n given by (3.7). Now $(1-1/n)F_2 : \overline{U} \to E$ is condensing and $I - (1 - 1/n)F_1 : E \to E$ is strongly accretive since, for $x, y \in E$ we have

$$\begin{aligned}\left(\left(I-\left(1-\frac{1}{n}\right)F_1\right)(x)-\left(I-\left(1-\frac{1}{n}\right)F_1\right)(y), x-y\right)_+ \\ &= \left(\left(1-\frac{1}{n}\right)[(I-F_1)(x)-(I-F_1)(y)] \right. \\ &\qquad \left. + \left(\frac{1}{n}\right)(x-y), x-y\right)_+ \\ &\geq \left(c\left(1-\frac{1}{n}\right)+\frac{1}{n}\right)\|x-y\|^2 \\ &\equiv c_n\|x-y\|^2.\end{aligned}$$

Remark. Notice $c_n = c(1 - 1/n) + 1/n > 1$.

Essentially the same reasoning as in Theorem 3.3 (except we use Theorem 3.4 and its remark) implies that S_n has a fixed point $u_n \in \overline{U}$. Also, as in Theorem 3.3, we have $0 \in (I - F)(\overline{U})$ so there exists $u \in \overline{U}$ with $0 = (I - F)(u)$. □

Remarks. (i) There is an analogue of Theorem 3.5 if $I - F_1 : E \to E$ satisfies (3.8) and $F_2 : \overline{U} \to E$ is an α-Lipschitzian map with $k = \min\{1, c\}$.

(ii) There is also an analogue of Theorem 3.5 if $I - F_1 : E \to E$ is accretive and $F_2 : \overline{U} \to E$ is completely continuous.

Theorem 3.6. *Let U be a bounded open convex set in a uniformly convex Banach space E. Also assume E^* is a uniformly convex Banach space, $0 \in U$, $F(\overline{U})$ bounded and $F : \overline{U} \to E$ is given by $F = F_1 + F_2$ where $I - F_1 : E \to E$ is continuous and accretive and $F_2 : \overline{U} \to E$ is a continuous compact map. In addition, suppose $F_2 : \overline{U} \to E$ is strongly continuous. Then either*

(A1) *F has a fixed point in $\overline{U}$ or*

(A2) *there exist* $u \in \partial U$ *and* $\lambda \in (0,1)$ *with* $u = \lambda F(u)$.

Remark. $F_2 : \overline{U} \to E$ is said to be strongly continuous [**27, 30**] if $x_n \rightharpoonup x$ implies $F_2(x_n) \to F_2(x)$; here $x_n, x \in \overline{U}$.

Proof. Assume (A2) does not hold. Consider for each $n \in \{2, 3, \dots\}$, the mapping S_n given by (3.7). Now $(1 - 1/n)F_2 : \overline{U} \to E$ is compact and $I - (1 - 1/n)F_1 : E \to E$ is strongly accretive since, for $x, y \in E$, we have

$$\left(\left(I - \left(1 - \frac{1}{n}\right)F_1\right)(x) - \left(I - \left(1 - \frac{1}{n}\right)F_1\right)(y), x - y\right)_+ \geq \frac{1}{n}\|x - y\|^2.$$

Essentially the same reasoning as in Theorem 3.3, except we use Theorem 3.4, implies that S_n has a fixed point $u_n \in \overline{U}$.

A standard result in functional analysis (if E is a reflexive Banach space, then any norm bounded sequence in E has a weakly convergent subsequence) implies, since $\overline{U}$ is bounded, that there exists a subsequence S of integers and a $u \in \overline{U}$ (notice $\overline{U}$ is strongly closed and convex so weakly closed) with

$$u_n \rightharpoonup u \quad \text{as } n \longrightarrow \infty \quad \text{in } S.$$

Notice, since $u_n = (1 - 1/n)F_1(u_n) + (1 - 1/n)F_2(u_n)$, then

$$\begin{aligned} \|(I - F_1)(u_n) - F_2(u)\| &= \left\| -\frac{1}{n}F_1(u_n) + \left(1 - \frac{1}{n}\right)F_2(u_n) - F_2(u)\right\| \\ &\leq \frac{1}{n}\|F(u_n)\| + \|F_2(u_n) - F_2(u)\| \end{aligned}$$

so since F_2 is strongly continuous and $F(\overline{U})$ is bounded, we have $(I - F_1)(u_n) \to F_2(u)$. Theorem 1.10, i.e., $I - F_1$ is demi-closed on E, now implies that $(I - F_1)(u) \to F_2(u)$. $\square$

Theorem 3.7. *Let Q be a closed, convex subset of a Banach space E with $0 \in \operatorname{int}(Q)$. Also let $\Omega \supseteq Q$ be a subset of E with*

$U_i = \{x \in E : d(x,Q) < 1/i\} \subseteq \Omega$ for i sufficiently large; here d denotes the metric induced by the norm. Assume $F_2(Q)$ is bounded, and $F : Q \to E$ is given by $F = F_1 + F_2$ where $-F_1 : E \to E$ is m-accretive and $F_2 : Q \to E$ is a bounded condensing map. In addition, suppose

$$\text{(3.9)}\quad \begin{cases} \text{if } \{(x_j,\lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial Q \times [0,1] \\ \text{converging to } (x,\lambda) \text{ with } x = \lambda F(x) \text{ and } 0 < \lambda < 1, \text{ and} \\ \text{if } \{z_j\} \text{ is a sequence in } U_m,\ m \text{ sufficiently large,} \\ \text{with } z_j \in \partial U_j \text{ for } j = m+1, m+2, \dots \text{ and } z_j \to x, \text{ then} \\ \lambda_j[F_1(z_j) + F_2(x_j)] \in Q \text{ for } j \text{ sufficiently large,} \end{cases}$$

holds. Then F has a fixed point.

Remark. Theorem 3.7 was proved by Furi and Pera [**14**] by a different method when $F_1 = 0$ and F_2 is a compact map. We also remark that $0 \in \operatorname{int}(Q)$ may be replaced by $0 \in Q$ if E is a Hilbert space (in the Proof take r to be the nearest point projection on Q).

Proof. Let $r : E \to Q$ be the continuous retraction given by

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for } x \in E$$

where μ is the Minkowski functional [**9**] on Q, i.e., $\mu(x) = \inf\{\alpha > 0; x \in \alpha Q\}$. Consider

$$B = \{x \in E : x = (I - F_1)^{-1} F_2 r(x)\}.$$

We claim $B \neq \varnothing$. To see this, we look at $r(I-F_1)^{-1}F_2$. Notice $r(I-F_1)^{-1}F_2 : Q \to Q$. Also $r(I-F_1)^{-1}F_2$ is a bounded condensing map. To see this, let W be a bounded nonprecompact subset of E. First notice that

$$\text{(3.10)}\qquad r(W) \subseteq \operatorname{co}(W \cup \{0\}).$$

To show (3.10), fix $x \in A$. If $x \in Q$, then $r(x) = x$ so (3.10) is true whereas if $x \notin Q$ then $r(x) = \lambda x + (1-\lambda)0$ where $\lambda = 1/\mu(x) < 1$

so (3.10) is again true. Consequently, since $(I - F_1)^{-1} : E \to \Omega$ is nonexpansive, we have

$$\begin{aligned}\alpha(r(I - F_1)^{-1}F_2(W)) &\le \alpha(\mathrm{co}\,((I - F_1)^{-1}F_2(W) \cup \{0\}))\\ &\le \alpha(F_2(W)) < \alpha(W).\end{aligned}$$

Thus $r(I - F_1)^{-1}F_2 : Q \to Q$ is a condensing map. Sadovskii's fixed point theorem, Theorem 1.2, implies that there exists a $y \in Q$ with $y = r(I - F_1)^{-1}F_2(y)$. Hence $z = (I - F_1)^{-1}F_2r(z)$ with $z = (I - F_1)^{-1}F_2(y)$ so $B \neq \varnothing$. In addition, the continuity of $(I - F_1)^{-1}F_2r$ implies that B is closed. We next claim that B is compact. To see this, first notice

$$B \subseteq (I - F_1)^{-1}F_2r(B). \tag{3.11}$$

Remark. Notice that B is bounded since $F_2(Q)$ is bounded and $(I - F_1)^{-1} : E \to \Omega$ is nonexpansive.

If $r(B)$ is a nonprecompact subset of E, then

$$\begin{aligned}\alpha(B) &\le \alpha((I - F_1)^{-1}F_2r(B)) \le \alpha(F_2r(B))\\ &< \alpha(r(B)) \le \alpha(\mathrm{co}\,(B \cup \{0\})) \le \alpha(B),\end{aligned}$$

a contradiction. Thus, $\alpha(r(B)) = 0$ and so

$$\alpha(B) \le \alpha((I - F_1)^{-1}F_2r(B)) \le \alpha(F_2r(B)) \le \alpha(r(B)) = 0.$$

Hence B is compact.

We now show $B \cap Q \neq \varnothing$. To do this, we argue by contradiction. Suppose $B \cap Q = \varnothing$. Then, since B is compact and Q is closed, there exists $\delta > 0$ with $\mathrm{dist}\,(B, Q) > \delta$. Define

$$U_i = \left\{x \in E : d(x, Q) < \frac{1}{i}\right\} \quad \text{for } i \in \{N, N+1, \dots\}.$$

Here $N \in \{1, 2, \dots\}$ is chosen so that $1 < \delta N$ and $\overline{U}_i \subseteq \Omega$ for $i \ge N$. Fix $i \in \{N, N+1, \dots\}$. Notice U_i is open, and since $\mathrm{dist}\,(B, Q) > \delta$, then $B \cap \overline{U}_i = \varnothing$. Also $F_2 r : \overline{U}_i \to E$ is a bounded condensing map since, if W is a bounded nonprecompact of $\overline{U}_i$, then if $\alpha(r(W)) > 0$, we have

$$\alpha(F_2(r(W))) < \alpha(r(W)) \le \alpha(\mathrm{co}\,(W \cup \{0\})) = \alpha(W)$$

whereas if $\alpha(r(W)) = 0$, then

$$\alpha(F_2(r(W))) \le \alpha(r(W)) = 0 < \alpha(W).$$

Also $\Omega \supseteq \overline{U}_i$ and $-F_1 : \Omega \to E$ is m-accretive. Now Theorem 3.1 (with $F_1 + F_2 r$) implies (since $B \cap \overline{U}_i = \varnothing$) that there exists $(y_i, \lambda_i) \in \partial U_i \times (0,1)$ with $y_i = \lambda_i[F_1(y_i) + F_2 r(y_i)]$, i.e., $y = (I - \lambda_i F_1)^{-1}(\lambda_i F_2 r(y_i))$.

Consequently, for each $j \in \{N, N+1, \dots\}$, there exists $(y_j, \lambda_j) \in \partial U_j \times (0,1)$ with $y_j = \lambda_j[F_1(y_j) + F_2 r(y_j)]$. Notice, in particular, since $y_j \in \partial U_j$, that

$$\lambda_j[F_1(y_j) + F_2 r(y_j)] \notin Q \quad \text{for } j \in \{N, N+1, \dots\}. \tag{3.12}$$

We now claim that

$$D = \{x \in E : x = (I - \lambda F_1)^{-1}(\lambda F_2 r(x)) \text{ for some } \lambda \in [0,1]\}$$

is compact, so sequentially compact. Clearly D is closed. Now $D \subseteq N(r(D) \times [0,1])$ where N is defined in Theorem 3.1, with $p = 0$.

Remark. Notice D is bounded since $F_2(Q)$ is bounded, $(I - \lambda F_1)^{-1} : E \to \Omega$ is nonexpansive and $\|(I - \lambda F_1)^{-1}(0)\| = \|(I - \lambda F_1)^{-1}(0) - (I - \lambda F_1)^{-1}(I - \lambda F_1)(0)\| \le \|\lambda F_1(0)\| \le \|F_1(0)\|$.

Suppose $\alpha(D) > 0$. If $\alpha(r(D)) > 0$, then

$$\alpha(D) \le \alpha(N(r(D) \times [0,1])) < \alpha(r(D)) \le \alpha(\mathrm{co}\,(D \cup \{0\})) = \alpha(D),$$

a contradiction. Thus, $\alpha(r(D)) = 0$ and

$$\alpha(D) \le \alpha(N(r(D) \times [0,1])) \le \alpha(r(D)) = 0,$$

a contradiction. Thus $\alpha(D) = 0$ so D is compact.

This together with $d(y_j, Q) = 1/j$, $|\lambda_j| \leq 1$, for $j \in \{N, N+1, \dots\}$, implies that we may assume without loss of generality that $\lambda_j \to \lambda^*$ and $y_j \to y^* \in \partial Q$; also

$$y_j = (I - \lambda_j F_1)^{-1}(\lambda_j F_2 r(y_j)) \longrightarrow (I - \lambda^* F_1)^{-1}(\lambda^* F_2 r(y^*))$$

so $y^* = (I - \lambda^* F_1)^{-1}(\lambda^* F_2 r(y^*))$, i.e., $y^* = \lambda^*[F_1(y^*) + F_2 r(y^*)] = \lambda^*[F_1(y^*) + F_2(y^*)] = \lambda^* F(y^*)$. If $\lambda^* = 1$, then $y^* = (I - F_1)^{-1}(F_2 r(y^*))$ which contradicts $B \cap Q = \varnothing$. If $\lambda^* = 0$, then $y^* = 0$ which contradicts $y^* \in \partial Q$. Hence we may assume $0 < \lambda^* < 1$. But in this case (3.9) with $x_j = r(y_j) \in \partial Q$, $x = y^* = r(y^*)$ and $z_j = y_j$, implies $\lambda_j[F_1(y_j) + F_2 r(y_j)] \in Q$ for j sufficiently large. This contradicts (3.12). Thus $B \cap Q \neq \varnothing$, so there exists $x \in Q$ with $x = (I - F_1)^{-1} F_2 r(x)$, i.e., $x = F(x)$. □

Theorem 3.8. *Let Q be a closed, convex subset of a Banach space E with $0 \in \operatorname{int}(Q)$. Also let $\Omega \supseteq Q$ be a subset of E with $U_i = \{x \in E : d(x, Q) < 1/i\} \subseteq \Omega$ for i sufficiently large. Assume $F_2(Q)$ and $F(Q)$ are bounded and $F : Q \to E$ is given by $F = F_1 + F_2$ where $-F_1 : \Omega \to E$ is m-accretive and $F_2 : Q \to E$ is a bounded α-Lipschitzian map with $k = 1$. Also suppose $(I - F)(Q)$ is closed and* (3.9) *holds. Then F has a fixed point.*

Remark. $0 \in \operatorname{int}(Q)$ may be replaced by $0 \in Q$ if E is a Hilbert space.

Proof. Consider, for each $n \in \{2, 3, \dots\}$, the mapping

$$S_n = \left(1 - \frac{1}{n}\right) F : Q \to E. \tag{3.13}$$

As in Theorem 3.3, $(1 - 1/n)F_2 : Q \to E$ is condensing and $-(1 - 1/n)F_1 : \Omega \to E$ is m-accretive. We will apply Theorem 3.7. Let $\{(x_j, \lambda_j)\}_{j=1}^{\infty}$ be a sequence in $\partial Q \times [0, 1]$ converging to (x, λ) with $x = \lambda S_n(x)$ and $0 < \lambda < 1$. Also let $\{z_j\}$ be a sequence in U_m, m sufficiently large, with $z_j \in \partial U_j$ for $j = m+1, m+2, \dots$ and $z_j \to x$. Then

$$\lambda_j \left(1 - \frac{1}{n}\right) F_1(z_j) + \lambda_j \left(1 - \frac{1}{n}\right) F_2(x_j)$$
$$= \mu_j F_1(z_j) + \mu_j F_2(x_j) \in Q \quad \text{for } j \text{ sufficiently large}$$

since (3.9) is satisfied (note $\mu_j = \lambda_j(1-1/n)$ is a sequence in $[0,1]$ with $\mu_j \to \lambda(1-1/n) = \mu$, $0 < \mu < 1$ and $x = \lambda S_n(x) = \lambda(1-1/n)F(x) = \mu F(x)$). Apply Theorem 3.7 to S_n to deduce that S_n has a fixed point $u_n \in Q$. Now since $u_n - F(u_n) = -(1/n)F(u_n)$, we have $0 \in \overline{(I-F)(Q)}$ since $(I-F)(Q)$ is closed. Thus there exists $u \in Q$ with $0 = (I-F)(u)$. □

Theorem 3.9. *Let Q be a closed, convex subset of a real Banach space E with $0 \in \operatorname{int}(Q)$. Assume $F_2(Q)$ is bounded and $F : Q \to E$ is given by $F = F_1 + F_2$ where $I - F_1 : E \to E$ is continuous and strongly accretive, i.e.,* (3.8) *is satisfied, and $F_2 : Q \to E$ is a continuous and completely continuous map. In addition, suppose* (3.9) *(here $U_i = \{x \in E : d(x, Q) < 1/i\}$) holds. Then F has a fixed point.*

Remark. $0 \in \operatorname{int}(Q)$ may be replaced by $0 \in Q$ if E is a Hilbert space.

Proof. Essentially the same reasoning as in Theorem 3.7, except we use Theorem 3.4, establishes the result. □

Remark. There is an analogue of Theorem 3.9 if $I - F_1 : E \to E$ satisfies (3.8) and $F_2 : Q \to E$ is an α-Lipschitzian map with $k = \min\{1, c\}$ and $\alpha(F_2(W)) < \min\{1, c\}\alpha(W)$ for all bounded nonprecompact subsets W of Q.

Theorem 3.10. *Let Q be a closed, convex subset of a real Banach space E with $0 \in \operatorname{int}(Q)$. Assume $F_2(Q)$ and $F(Q)$ are bounded and $F : Q \to E$ is given by $F = F_1 + F_2$ where $I - F_1 : E \to E$ is continuous and* (3.8) *holds with $c > 1$ and $F_2 : Q \to E$ is a bounded α-Lipschitzian map with $k = 1$. Also suppose $(I-F)(Q)$ is closed and* (3.9) *holds. Then F has a fixed point.*

Remark. $0 \in \operatorname{int}(Q)$ may be replaced by $0 \in Q$ if E is a Hilbert space.

Proof. Consider for each $n \in \{2, 3, \dots\}$ the mapping S_n given in (3.13). As in Theorem 3.5, $I - (1 - 1/n)F_1 : E \to E$ is strongly accretive with

$$\left(\left(I-\left(1-\frac{1}{n}\right)\right)F_1(x)-\left(I-\left(1-\frac{1}{n}\right)\right)F_1(y),x-y\right)_+ \geq \left(c\left(1-\frac{1}{n}\right)+\frac{1}{n}\right)\|x-y\|^2$$

for $x, y \in E$ and $(1-1/n)F_2 : Q \to E$ is a condensing map. Essentially the same reasoning as in Theorem 3.8, except we use Theorem 3.9 and its remark, implies that S_n has a fixed point $u_n \in Q$. Also $0 \in (I-F)(Q)$ since $u_n - F(u_n) = -(1/n)F(u_n)$. □

Remarks. (i) There is an analogue of Theorem 3.10 if $I - F_1 : E \to E$ satisfies (3.8) and $F_2 : Q \to E$ is an α-Lipschitzian map with $k = \min\{1, c\}$.

(ii) There is an analogue of Theorem 3.10 if $I - F_1 : E \to E$ is accretive and $F_2 : Q \to E$ is a continuous and completely continuous map.

4. Application. The fixed point theory in this paper can be used to establish existence principles for the second order boundary value problems in abstract spaces. In particular, we examine

$$\begin{cases} y'' + f(t, y, y') = 0, & 0 \leq t \leq 1 \\ y(0) = y(1) = 0 \end{cases} \tag{4.1}$$

where $f : [0,1] \times H \times H \to H$ is continuous; here $H = (H, |\cdot|)$ is a real Hilbert space. Problems of the above form have been discussed extensively in the literature. By a solution to (4.1) we mean a function $y \in C^2([0,1], H)$ with y satisfying the differential equation on $[0,1]$ and the stated boundary conditions.

Consider the problem

$$\begin{cases} w'(t) + f(t, \int_0^t w(x)\,dx, w(t)) = 0, & 0 \leq t \leq 1 \\ \int_0^1 w(x)\,dx = 0. \end{cases} \tag{4.2}$$

By a solution to (4.2) we mean a function $w \in C^1([0,1], H)$ with $w' = -f(t, \int_0^t w\,dx, w)$ on $[0,1]$ and $\int_0^1 w(x)\,dx = 0$. Notice y is a solution of (4.1) if and only if $w = y'$ is a solution of (4.2). For notational purposes, let

$$L_0^2([0,1],H) = \left\{u \in L^2([0,1],H) : \int_0^1 u(x)\,dx = 0\right\}.$$

Notice $L_0^2([0,1],H)$ is a closed subspace of $L^2([0,1],H)$ and, consequently, $L_0^2([0,1],H)$ is a Hilbert space.

Before we prove our two existence results, we gather together some information on the Sturm Liouville problem

$$\text{(4.3)} \qquad \begin{cases} y'' + \lambda q(t) y = 0, & 0 \le t \le 1 \\ y(0) = y(1) = 0, \end{cases}$$

where $y : [0,1] \to \mathbf{R}$ and $q \in C[0,1]$ with $q > 0$ on $(0,1)$. Let λ_1 be the first eigenvalue of (4.3). It is well known, Rayleigh-Riez minimization theorem, that

$$\lambda_1 \int_0^1 q u^2\,dt \le \int_0^1 [u']^2\,dt$$

for all functions $u : [0,1] \to \mathbf{R}$ with u' absolutely continuous and $u(0) = u(1) = 0$. This result together with the ideas used to prove Theorem 1.4 in [**22**] immediately yields

Theorem 4.1. *Let H be a real Hilbert space, $q \in C[0,1]$, with $q > 0$ on $(0,1)$ and $u : [0,1] \to H$ with u' absolutely continuous and $u(0) = u(1) = 0$. Then*

$$\text{(4.4)} \qquad \lambda_1 \int_0^1 q u^2\,dt \le \int_0^1 [u']^2\,dt.$$

Remark. If $q = 1$, then $\lambda_1 = \pi^2$ and (4.4) is Wirtinger's inequality.

Theorem 4.2. *Suppose $f : [0,1] \times H \times H \to H$ is continuous. Also assume the following conditions are satisfied*:

$$\text{(4.5)} \qquad \begin{cases} \textit{for each } r > 0 \textit{ there exists } h_r \in L^1[0,1] \\ \textit{with } |f(t,u,v)| \le h_r(t) \textit{ for all} \\ t \in [0,1], |u| \le r \textit{ and } v \in R \end{cases}$$

and

$$(4.6)\qquad \begin{cases} \textit{there exists } a_0 \geq 0 \textit{ with } a_0 < \lambda_1 \textit{ such that} \\ \langle f(x,u_0,v_0) - f(x,u_1,v_1), u_0 - u_1\rangle \leq a_0 q(x)|u_0-u_1|^2 \\ \textit{for all } t \in [0,1] \textit{ and } (u_0,v_0),(u_1,v_1) \in \mathbf{R}^2; \\ \textit{here } q \in C[0,1] \textit{ with } q > 0 \textit{ on } (0,1). \end{cases}$$

Then (4.2) *has a solution.*

Proof. Let $E = C = L^2_0([0,1],H)$. Define $F_1 : E \to E$ by

$$(4.7)\qquad \begin{aligned} F_1 w(t) = \int_0^1 \int_0^z f\biggl(s, \int_0^s w(x)\,dx, w(s)\biggr)\,ds\,dz \\ - \int_0^t f\biggl(s, \int_0^s w(x)\,dx, w(s)\biggr)\,ds. \end{aligned}$$

Notice that F_1 is well defined because of assumption (4.5). Also $F_1 : E \to E$ is continuous. This follows from (4.5), the Lebesgue dominated convergence theorem (version where convergence almost everywhere is replaced by convergence in measure), and a result of Nemytskii, if a sequence of functions $w_n(s)$, $s \in [0,1]$ converges in mean (so therefore in measure), then the sequence of functions $f(s, \int_0^s w_n(x)\,dx, w_n(s))$ also converges in measure.

We will show $I - F_1 : E \to E$ is strongly accretive if $0 \leq a_0 < \lambda_1$, here a_0 is as in (4.6), in fact, if $a_0 = 0$, $-F_1 : E \to E$ is accretive. Let $u, v \in E$. Then

(4.8)

$$\begin{aligned} \langle F_1 u - F_1 v, u - v\rangle = -\int_0^1 \biggl\langle \int_0^t \biggl[f\biggl(s, \int_0^s u\,dx, u(s)\biggr) \\ - f\biggl(s, \int_0^s fv\,dx, v(s)\biggr)\biggr]\,ds, u(t) - v(t)\biggr\rangle\,dt \end{aligned}$$

since $\int_0^1 [u(t) - v(t)]\,dt = 0$ implies

$$\begin{aligned} \int_0^1 \biggl\langle \int_0^1 \int_0^x \biggl[f\biggl(s, \int_0^s u(z)dz, u(s)\biggr) - f\biggl(s, \int_0^s v(z)\,dz, v(s)\biggr)\biggr]\,ds\,dx, \\ u(t) - v(t)\biggr\rangle\,dt = 0. \end{aligned}$$

Interchange the order of integration in (4.8) to obtain

$$\begin{aligned}\langle F_1u - F_1v, u-v\rangle \\ &= -\int_0^1 \biggl\langle \biggl[f\biggl(s, \int_0^s u\,dx, u(s)\biggr) - f\biggl(s, \int_0^s v\,dx, v(s)\biggr)\biggr], \\ &\qquad\qquad \int_s^1 [u(t)-v(t)]\,dt\biggr\rangle\,ds \\ &= \int_0^1 \biggl\langle \biggl[f\biggl(s, \int_0^s u\,dx, u(s)\biggr) - f\biggl(s, \int_0^s v\,dx, v(s)\biggr)\biggr], \\ &\qquad\qquad \int_0^s u(t)\,dt - \int_0^s v(t)\,dt\biggr\rangle\,ds,\end{aligned}$$

since $\int_0^1 [u(t)-v(t)]\,dt = 0$. This, together with condition (4.7) and Theorem 4.1 will give

$$\begin{aligned}\langle F_1u - F_1v, u-v\rangle \le a_0 \int_0^1 q(s)\biggl|\int_0^s u(t)\,dt - \int_0^s v(t)\,dt\biggr|^2\,ds \\ \le \frac{a_0}{\lambda_1}\|u-v\|_{L^2}^2.\end{aligned}$$

Consequently, for $u, v \in \overline{U}$, we have

$$\langle (I-F_1)u - (I-F_1)v, u-v\rangle \ge \biggl(1 - \frac{a_0}{\lambda_1}\biggr)\|u-v\|_{L^2}^2, \tag{4.9}$$

so $I - F_1 : E \to E$ is strongly accretive.

Let $F_2 = 0$ so $F = F_1 + F_2$ and notice $(I - \lambda F_1)^{-1}(\lambda F_2(\cdot)) = (I - \lambda F_1)^{-1}(0) : E \to E$ for any $\lambda \in [0,1]$. To apply Theorem 3.4 we construct an open set U such that (A2) does not occur. Let w be a solution to

$$(4.10)_\lambda \qquad \begin{cases} w'(t) + \lambda f(t, \int_0^t w(x)\,dx, w(t)) = 0, & 0 \le t \le 1 \\ \int_0^1 w(x)\,dx = 0, \end{cases}$$

for any $\lambda \in (0,1)$. Then

$$-w'(t) = \lambda t\biggl(t, \int_0^t w(x)\,dx, w(t)\biggr), \quad 0 \le t \le 1. \tag{4.11}$$

Take the inner product of (4.11) with $\int_0^t w\,ds$ and integrate from 0 to 1 using integration by parts, (4.6), Theorem 4.1 and Hölder's inequality, to obtain

$$\begin{aligned}\|w\|_{L^2}^2 &= \lambda \int_0^1 \left\langle f\left(t, \int_0^t w(x)\,dx, w(t)\right), \int_0^t w(x)\,dx \right\rangle dt \\ &= \lambda \int_0^1 \left\langle \left[f\left(t, \int_0^t w(x)\,dx, w(t)\right) - f(t,0,0)\right] \right. \\ &\qquad\qquad\qquad \left. + f(t,0,0), \int_0^t w(x)\,dx \right\rangle dt \\ &\le a_0 \int_0^1 q(t) \left| \int_0^t w(x)\,dx \right|^2 dt \\ &\quad + \int_0^1 q(t) \left| \int_0^t w(x)\,dx \right| |f(t,0,0)|\,dt \\ &\le \frac{a_0}{\lambda_1} \|w\|_{L^2}^2 + \|w\|_{L^2} \left(\frac{1}{\lambda_1} \int_0^1 q(t)\,dt \right)^{1/2} (\max_{[0,1]} |f(t,0,0)|).\end{aligned}$$

Thus, since $a_0 < \lambda_1$, there exists a constant M_0, independent of λ, with $\|w\|_{L^2} < M_0$, for any solution w to $(4.10)_\lambda$. Let

$$U = \{u \in L_0^2([0,1], H) : \|u\|_{L^2} < M_0\}.$$

Now all the conditions of Theorem 3.4 are satisfied. Finally notice that condition (A2) cannot occur since, if there exist $\lambda_0 \in (0,1)$ and $u \in \partial U$ with $u = \lambda_0 F_1(u)$, then $\|u\|_{L^2} = M_0$ and u is a solution of $(4.10)_{\lambda_0}$, which is a contradiction. Thus, F_1 has a fixed point in $\overline{U}$. □

Theorem 4.3. *Suppose that $f : [0,1] \times H \times H \to H$ is continuous. Also assume the following conditions are satisfied*:

$$\text{(4.12)} \qquad \begin{cases} \textit{for each } r > 0 \textit{ there exists } h_r \in L^1[0,1] \textit{ with} \\ |f(t,u,v)| \le h_r(t) \textit{ for all } t \in [0,1], |u| \le r \\ \textit{and } |v| \le r, \end{cases}$$

and

$$\text{(4.13)} \qquad \begin{cases} \alpha(f(I \times A \times B)) \le k \max\{\alpha(A), \alpha(B)\} \\ \textit{for all bounded subsets } A, B \textit{ of } H; \\ \textit{here } I = [0,1] \textit{ and } 0 \le k < 1. \end{cases}$$

In addition, suppose there is a constant M_0, independent of λ, with

$$|w|_1 = \max\{\sup_{[0,1]} |w(t)|, \sup_{[0,1]} |w'(t)|\} \neq M_0$$

for any solution w to $(4.10)_\lambda$. *Then* (4.2) *has a solution.*

Proof. Let $E = C = C^1([0,1], H)$ and

$$U = \{u \in C^1([0,1], H) : |u|_1 < M_0\}.$$

Define $F_2 : \overline{U} \to E$ by

$$F_2 w(t) = \int_0^1 \int_0^z f\left(s, \int_0^s w(x)\,dx, w(s)\right) ds\,dz \\ - \int_0^t f\left(s, \int_0^s w(x)\,dx, w(s)\right) ds.$$

Let $F_1 = 0$. We claim that $F = F_1 + F_2 = F_2$ has a fixed point $w \in \overline{U}$. If the claim is true, then w is a solution of (4.2) and we are finished.

To show $F = F_2$ has a fixed point in $\overline{U}$ we will apply Theorem 3.2. Notice first $(I - \lambda F_1)^{-1}(\lambda F_2(\cdot)) = \lambda F_2(\cdot) : \overline{U} \to C = E$ for any $\lambda \in [0,1]$. We next show that $F_2 : \overline{U} \to C^1([0,1], H)$ is a condensing map. Let $G : \overline{U} \to E$ and $H : \overline{U} \to E$ be defined by

$$Gw(t) = \int_0^t f\left(s, \int_0^s w(x)\,dx, w(s)\right) ds$$

and

$$Hw(t) = \int_0^1 \int_0^z \left(s, \int_0^s w(x)\,dx, w(s)\right) ds\,dz$$

so $F_2 = H - G$. Now let $\Omega \subseteq C^1([0,1], H)$ be bounded. Then $G(\Omega)$ is clearly bounded and equicontinuous because of assumption (4.12). Notice as well that, for $w \in \Omega$, we have

$$\frac{1}{s} \int_0^s w(x)\,dx \subseteq \overline{\mathrm{co}}(\mathrm{range}\, w(x)) \subseteq \overline{\mathrm{co}}(\Omega(I))$$

so

$$\int_0^s w(x)\,dx \subseteq s\overline{\text{co}}(\Omega(I)) \subseteq \overline{\text{co}}(\overline{\text{co}}(\Omega(I)) \cup \{0\}).$$

Remark. Here $I = [0,1]$ and $\Omega(I) = \cup_{t\in I}\{\phi(t) : \phi \in \Omega\}$.

Also, for $w \in \Omega$ and $t \in [0,1]$, we have

$$\begin{aligned}\frac{1}{t}\int_0^t f\bigg(s, \int_0^s w(x)\,dx, w(s)\bigg)\,ds \\ &\subseteq \overline{\text{co}}\bigg(\text{range}\, f\bigg(s, \int_0^s w(x)\,dx, w(s)\bigg)\bigg) \\ &\subseteq \overline{\text{co}}(f(I \times \overline{\text{co}}(\overline{\text{co}}(\Omega(I)) \cup \{0\}) \times \Omega(I)))\end{aligned}$$

and so

$$\begin{aligned}\int_0^t f\bigg(s, \int_0^s w(x)\,dx, w(s)\bigg)\,ds \\ &\subseteq \overline{\text{co}}(\overline{\text{co}}(f(I \times \overline{\text{co}}(\overline{\text{co}}(\Omega(I)) \cup \{0\}) \times \Omega(I))) \cup \{0\}).\end{aligned}$$

This together with the properties of α and (4.13) implies for $w \in \Omega$ and $t \in [0,1]$ that

$$\begin{aligned}\alpha(G(w(t)) &\leq \alpha(\overline{\text{co}}(f(I \times \overline{\text{co}}(\overline{\text{co}}(\Omega(I)) \cup \{0\}) \times \Omega(I)))) \\ &= \alpha(f(I \times \overline{\text{co}}(\overline{\text{co}}(\Omega(I)) \cup \{0\}) \times \Omega(I))) \\ &\leq k \max\{\alpha(\overline{\text{co}}(\Omega(I)) \cup \{0\}), \alpha(\Omega(I))\} \\ &= k\alpha(\Omega(I)).\end{aligned}$$

Thus for $w \in \Omega$, we have

$$\alpha(G(w(t)) \leq k\alpha(\Omega(I)). \tag{4.14}$$

Also since Ω is bounded in $C^1([0,1], H)$ then Theorem 1.5 implies that $\alpha(\Omega) \geq \alpha(\Omega(I))$ and this, together with (4.14), implies for $w \in \Omega$ and $t \in [0,1]$ that

$$\alpha(G(w(t)) \leq k\alpha(\Omega).$$

Consequently,

$$\sup_{t\in[0,1]} \alpha(G\Omega(t)) \le k\alpha(\Omega). \tag{4.15}$$

Remark. Recall $G\Omega(t) = \{\phi(t) : \phi \in G\Omega\}$.

Also notice if $w \in \Omega$ and $t \in [0,1]$, them

$$(Gw)'(t) = f\left(t, \int_0^t w(x)\,dx, w(t)\right) \tag{4.16}$$

so this, together with (4.12), implies that there exists $h \in L^1[0,1]$ with

$$|(Gw)'(t)| \le h(t) \quad \text{for a.e.} \quad t \in [0,1] \quad \text{and all} \quad w \in \Omega. \tag{4.17}$$

Consequently, since $f : [0,1] \times H \times H \to H$ is continuous, we have that $G\Omega$ is a bounded subset of $C^1([0,1], H)$ and, from (4.17), we have that $(G\Omega)' = \{\phi' : \phi \in G\Omega\}$ is an equicontinuous set. Now Theorem 1.4 implies that

$$\alpha(G(\Omega)) = \max\{\sup_{t\in I} \alpha(G\Omega(t)), \sup_{t\in I} \alpha((G\Omega)'(t))\}. \tag{4.18}$$

From above, for $w \in \Omega$ and $t \in [0,1]$, we have that

$$(Gw)'(t) \subseteq f(I \times \overline{\mathrm{co}}\,(\overline{\mathrm{co}}\,(\Omega(I)) \cup \{0\}) \times \Omega(I))$$

and so

$$\begin{aligned}\alpha((Gw)'(t)) &\le \alpha(f(I \times \overline{\mathrm{co}}\,(\overline{\mathrm{co}}\,(\Omega(I)) \cup \{0\}) \times \Omega(I)))\\ &\le k \max\{\alpha(\overline{\mathrm{co}}\,(\Omega(I)) \cup \{0\}), \alpha(\Omega(I))\}\\ &= k\alpha(\Omega(I)).\end{aligned}$$

This, together with Theorem 1.5, implies for $w \in \Omega$ and $t \in [0,1]$ that

$$\alpha((Gw)'(t)) \le k\alpha(\Omega)$$

and so

$$\sup_{t\in[0,1]} \alpha((G\Omega)'(t)) \le k\alpha(\Omega). \tag{4.19}$$

Now (4.15), (4.18) and (4.19) yields

$$\alpha(G\Omega) \leq k\alpha(\Omega). \tag{4.20}$$

Also $H\Omega$ is a bounded subset of $C^1([0,1], H)$ and $(H\Omega)'$ is an equicontinuous set. Also, for $w \in \Omega$ and $t \in [0,1]$, we have

$$Hw(t) = \int_0^1 \int_0^z f\bigg(s, \int_0^s w(x)\,dx, w(s)\bigg)\,ds\,dz$$

and so

$$\alpha(H(w(t)) = 0.$$

Consequently, $\sup_{t\in I} \alpha(H\Omega(t)) = 0$. Similarly $\sup_{t\in I} \alpha((H\Omega)'(t)) = 0$ and so

$$\alpha(H\Omega) = 0. \tag{4.21}$$

Now $F_2 = H - G$ so (4.20) and (4.21) imply

$$\alpha(F_2\Omega) = \alpha(G\Omega) \leq k\alpha(\Omega)$$

so F_2 is a condensing map.

Hence, all the conditions of Theorem 3.2 are satisfied. Finally, notice that condition (A2) cannot occur since if there exists a $\lambda_0 \in (0,1)$ and $u \in \partial U$ with $u = \lambda_0 F_2(u)$, then $|u|_1 = M_1$ and u is a solution of $(4.10)_{\lambda_0}$, which is a contradiction. Thus, $F = F_2$ has a fixed point in $\overline{U}$, i.e.,

$$\begin{aligned} u = &\int_0^1 \int_0^z f\bigg(s, \int_0^s u(x)\,dx, u(s)\bigg)\,ds\,dz \\ &- \int_0^t f\bigg(s, \int_0^s u(x)\,dx, u(s)\bigg)\,ds \end{aligned}$$

and so $\int_0^1 u(x)\,dx = 0$. □

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