ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 1, Spring 1999

THE ERGODIC HILBERT TRANSFORM ON THE WEIGHTED SPACES $\mathfrak{L}_p(G, w)$

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ABSTRACT. We extend a theorem of Hunt, Muckenhoupt and Wheeden on weighted norm inequalities for the Hilbert transform. Our generalization to locally compact abelian groups is formulated in terms of the ergodic Hilbert transform and the ergodic A_p-condition.

1. Introduction. We consider the question of the continuity of the ergodic Hilbert transform from a weighted $\mathfrak{L}_p$-space $\mathfrak{L}_p(G, w)$ into itself, where G is a locally compact abelian group. The classical result for $G = \mathbf{R}$ or $\mathbf{T}$ is that the A_p-condition for a weight w characterizes the continuity of the Hilbert transform. This result was given by Hunt, Muckenhoupt and Wheeden [**6**], which we state in the following theorem.

Theorem 1.1. *Let $G = \mathbf{R}$ or $\mathbf{T}$, let $T = H$ or MH, the Hilbert transform or maximal Hilbert transform, and let w be a nonnegative function in $\mathfrak{L}^1_{\text{loc}}(G)$. If $1 < p < \infty$, then the weighted norm inequality*

$$\int_G |Tf(t)|^p w(t)\,dt \le K_p \int_G |f(t)|^p w(t)\,dt \tag{1.1}$$

holds for every $f \in \mathfrak{L}_p(G, w)$ if and only if the weight w satisfies the A_p-condition

$$\sup_I \frac{1}{|I|}\int_I w(t)\,dt \left(\frac{1}{|I|}\int_I w^{-1/(p-1)}(t)\,dt\right)^{p-1} \le A_p. \tag{A_p}$$

When G is any locally compact abelian group, the ergodic A_p-condition, defined in terms of a continuous homomorphism from $\mathbf{R}$ into

Received by the editors on May 21, 1996.
AMS *Mathematics Subject Classification.* 43A15, 43A22, 44A15.

G, is analogous to the A_p-condition. Using the transference methods of Coifman and Weiss, we show that if a weight w satisfies the ergodic A_p-condition on a locally compact abelian group G, then the ergodic Hilbert transform is bounded from $\mathfrak{L}_p(G, w)$ into itself. This is an extension of a theorem of Lancien [**7**]. However, transference methods are not applicable in proving the converse.

In this paper we develop new methods to attack the converse. We define the *uniform* A_p*-condition*, uniform with respect to continuous homomorphisms from $\mathbf{R}$ into G, and construct examples of weights satisfying this condition. In the case $G = \mathbf{R}$ or $\mathbf{T}$, the uniform A_p-condition reduces to the A_p-condition. In our main theorem, Theorem 1.2, we consider a continuous weight w on a locally compact abelian group G. We show that the ergodic Hilbert transform is bounded uniformly with respect to continuous homomorphisms from $\mathbf{R}$ into G exactly when w satisfies the uniform A_p-condition. This reduces to the classical result for the cases of $G = \mathbf{R}$ or $\mathbf{T}$.

Before stating our main result, we need to define terms used throughout the paper. We let G denote a locally compact abelian group with character group $\hat{G}$ and Haar measure μ, and suppose that there is a continuous homomorphism $\varphi : \mathbf{R} \to G$. For a function $f \in \mathfrak{L}_1(G)$, the *nth truncated ergodic Hilbert transform*, with respect to the homomorphism φ, is given by

$$H_n^\varphi f(x) = \frac{1}{\pi} \int_{1/n \le |t| \le n} f(x - \varphi(t)) \frac{1}{t}\, dt, \quad x \in G.$$

The *ergodic Hilbert transform* is defined for almost every x by $H^\varphi f(x) = \lim_{n\to\infty} H_n^\varphi f(x)$, and the *maximal ergodic Hilbert transform* is defined for almost every x by $MH^\varphi f(x) = \sup_n H_n^\varphi f(x)$.

Now suppose $1 \le p < \infty$ and $w : G \to \mathbf{R}^+$ is a nonnegative locally integrable function. The weight w satisfies the ergodic A_p-condition, with respect to the homomorphism φ, if, for almost every $x \in G$,

(1.2)
$$\sup_I \frac{1}{|I|} \int_I w(x - \varphi(t))\, dt \left(\frac{1}{|I|} \int_I w^{-1/(p-1)}(x - \varphi(t))\, dt \right)^{p-1} \le A_p^\varphi,$$

where the constant A_p^φ is independent of x. In this case $w \in A_p^\varphi(G)$ and A_p^φ is the least constant such that (1.2) holds. (In other words,

$w \in A_p^\varphi(G)$ if and only if, for almost every $x \in G$, the functions $w_x : \mathbf{R} \to \mathbf{R}^+$, defined by $w_x(t) = w(x - \varphi(t))$, satisfy the A_p-condition on $\mathbf{R}$, with a constant A_p^φ independent of x.) Now, if $w \in A_p^\varphi(G)$ for all continuous homomorphisms $\varphi : \mathbf{R} \to G$ and $\mathfrak{A}_p = \sup_\varphi A_p^\varphi$ is finite, then we say that w satisfies the *uniform A_p-condition* or $w \in \mathfrak{A}_p(G)$.

If f is a ν-measurable function defined on G and $1 \leq p < \infty$, the Lorentz $\mathfrak{L}_{p,\infty}$ quasi-norm is defined by $\|f\|^*_{\mathfrak{L}_{p,\infty}(\nu)} = \sup_\tau \tau(\nu(\{x \in G : |f(x)| > \tau\}))^{1/p}$. See [**11**, Chapter 5, Section 3]; note that $\|\cdot\|^*_{\mathfrak{L}_{p,\infty}(\nu)}$ actually defines a norm when $1 < p < \infty$.

We now state our main result.

Theorem 1.2. *Let $1 \leq p < \infty$. Let G be a locally compact abelian group. Let $T^\varphi = H^\varphi$ or MH^φ and suppose that $w \in \mathfrak{C}^+(G)$. Then the weighted norm inequality*

$$\|T^\varphi f\|^*_{\mathfrak{L}_{p,\infty}(G,w)} \leq K_p \|f\|_{\mathfrak{L}_p(G,w)} \tag{1.3}$$

holds for all $f \in \mathfrak{L}_p(G,w) \cap \mathfrak{L}_1(G)$ and all continuous homomorphisms $\varphi : \mathbf{R} \to G$ if and only if the weight $w \in \mathfrak{A}_p(G)$.

The plan of the paper is as follows. In Section 2 we prove the necessity part of Theorem 1.2. In Section 3 we prove a result that shows that if the ergodic Hilbert transform is bounded from $\mathfrak{L}_p(G,w)$ into itself, then it is bounded on certain subspaces. In Section 4 we use this result, along with some other arguments, to prove our main theorem, Theorem 1.2. In Section 5 we construct weights that satisfy the uniform A_p-condition.

2. Weighted norm inequalities on locally compact abelian groups. In this section we show that, if G is a locally compact abelian group and w satisfies the ergodic A_p-condition, then the ergodic Hilbert transform is bounded from $\mathfrak{L}_p(G,w) \cap \mathfrak{L}_1(G)$ into $\mathfrak{L}_p(G,w)$. In the case that G is compact, Lancien [**7**] shows that the necessity part of our main theorem, Theorem 1.2, holds. We show that this is also true when G is a locally compact abelian group. Our proof is similar to the argument in [**7**], using the transference methods of Coifman and Weiss [**3**]; we include the proof here for completeness.

Theorem 2.1. *Let $1 \leq p < \infty$. Let G be a locally compact abelian group with Haar measure μ, let $\varphi : \mathbf{R} \to G$ be a continuous homomorphism, and let $T^\varphi = H^\varphi$ or MH^φ. If $w : G \to \mathbf{R}^+$ is in $A_p^\varphi(G)$, then the weighted norm inequality*

$$\|T^\varphi f\|^*_{\mathfrak{L}_{p,\infty}(G,w)} \leq A_p^\varphi \|f\|_{\mathfrak{L}_p(G,w)}$$

holds for all $f \in \mathfrak{L}_p(G,w) \cap \mathfrak{L}_1(G)$.

Proof. We show the theorem holds for the case $1 < p < \infty$. The case $p = 1$ follows by a similar argument. We assume that $w \in A_p^\varphi(G)$ with bound A_p^φ and show that the inequality

$$\|T^\varphi f\|_{\mathfrak{L}_p(G,w)} \leq A_p^\varphi \|f\|_{\mathfrak{L}_p(G,w)} \tag{2.1}$$

is valid for all $f \in \mathfrak{C}_{00}(G)$. (By a straightforward argument using [**5**, Theorem 33.11], it follows that (2.1) holds for all $f \in \mathfrak{L}_p(G,w) \cap \mathfrak{L}_1(G)$.) Let $K_n = \{t : 1/n \leq |t| \leq n\}$ and $k_n(t) = (1/(\pi t))\mathbf{1}_{K_n}(t)$. To see that (2.1) holds, it is enough to show that, for $N > 1$, the inequality

$$\int_G \Big(\max_{1 \leq n \leq N} |H_n^\varphi f(x)| \Big)^p w(x)\, d\mu(x) \leq (A_p^\varphi)^p \int_G |f(x)|^p w(x)\, d\mu(x) \tag{2.2}$$

is valid for all $f \in \mathfrak{C}_{00}(G)$. Then (2.1) follows from Fatou's lemma.

To see that (2.2) holds, let $N > 1$ and fix $f \in \mathfrak{C}_{00}(G)$. Since $\mathbf{R}$ is amenable, given $\varepsilon > 0$, we can choose a compact set K such that $|K - K_N|/|K| < 1 + \varepsilon$, see [**3**, p. 8]. Then, by the translation invariance of Haar measure μ and Fubini's theorem, we have the following:

$$\begin{aligned}
&\int_G \Big(\max_{1 \leq n \leq N} |H_n^\varphi f(x)| \Big)^p w(x)\, d\mu(x) \\
&= \frac{1}{|K|} \int_K \int_G \Big(\max_{1 \leq n \leq N} |H_n^\varphi f(x - \varphi(t))| \Big)^p w(x - \varphi(t))\, d\mu(x)\, dt \\
&= \frac{1}{|K|} \int_G \int_K \bigg(\max_{1 \leq n \leq N} \bigg| \int_{\mathbf{R}} f(x - \varphi(t-s)) k_n(s)\, ds \bigg| \bigg)^p \\
&\qquad\qquad \cdot w(x - \varphi(t))\, dt\, d\mu(x) \\
&= \frac{1}{|K|} \int_G \int_K \bigg(\max_{1 \leq n \leq N} \bigg| \int_{\mathbf{R}} f(x - \varphi(t-s)) \mathbf{1}_{K-K_N}(t-s) k_n(s)\, ds \bigg| \bigg)^p \\
&\qquad\qquad \cdot w(x - \varphi(t))\, dt\, d\mu(x).
\end{aligned}$$

Let $g_x(t) = f(x - \varphi(t))\mathbf{1}_{K-K_N}(t)$ and $w_x(t) = w(x - \varphi(t))$. We have assumed that $w \in A_p^\varphi(G)$, which means that, for μ-almost every $x \in G$, w_x satisfies the A_p-condition on $\mathbf{R}$ with bound A_p^φ. Then, by the above equalities and Theorem 1.1, we have the following

$$\begin{aligned}\int_G \Big(\max_{1\le n\le N} |H_n^\varphi f(x)|\Big)^p w(x)\,d\mu(x) \\ &= \frac{1}{|K|}\int_G\int_K \bigg(\max_{1\le n\le N}\bigg|\int_{\mathbf{R}} g_x(t-s)k_n(s)\,ds\bigg|\bigg)^p \\ &\qquad\qquad \cdot w_x(t)\,dt\,d\mu(x) \\ &\le \frac{(A_p^\varphi)^p}{|K|}\int_G\int_{\mathbf{R}} |g_x(t)|^p w_x(t)\,dt\,d\mu(x) \\ &= \frac{(A_p^\varphi)^p}{|K|}\int_{K-K_N}\int_G |f(x-\varphi(t))|^p \\ &\qquad\qquad \cdot w(x-\varphi(t))\,d\mu(x)\,dt \\ &= (A_p^\varphi)^p\frac{|K-K_N|}{|K|}\int_G |f(x)|^p w(x)\,d\mu(x).\end{aligned}$$

Since $(|K-K_N|/|K|) < 1+\varepsilon$, and ε is arbitrary, we have shown that (2.2) holds, completing the proof of the theorem. □

3. Subspaces of $\mathfrak{L}_p(G,w)$. The next theorem is a crucial result needed to prove our main theorem, Theorem 1.2. It is necessary for our analysis to define the function space $A\mathfrak{L}_1(G)$, where G is a locally compact abelian group: $A\mathfrak{L}_1(G) = \{f \in \mathfrak{L}_1(G) : \hat{f} \in \mathfrak{L}_1(\hat{G})\}$.

Theorem 3.1. *Let $1 \le p < \infty$. Let G be a locally compact abelian group, and let H be a closed subgroup of G. Let $\beta : \mathbf{R} \to H$ be a continuous homomorphism from $\mathbf{R}$ into H. Let $T^\beta = H^\beta$ or MH^β and $w \in \mathfrak{C}^+(G)$. Suppose the weighted norm inequality*

$$\|T^\beta f\|^*_{\mathfrak{L}_{p,\infty}(G,w)} \le K_p\|f\|_{\mathfrak{L}_p(G,w)} \tag{3.1}$$

holds for all $f \in \mathfrak{L}_p(G,w)\cap\mathfrak{L}_1(G)$. Then, for every $y \in G$, the weighted norm inequality

$$\|T^\beta f\|^*_{\mathfrak{L}_{p,\infty}(H,w_y)} \le K_p\|f\|_{\mathfrak{L}_p(H,w_y)} \tag{3.2}$$

holds for all $f \in \mathfrak{L}_p(H, w_y) \cap \mathfrak{L}_1(H)$*, where* $w_y : H \to \mathbf{R}^+$ *is the translate of* w *defined by* $w_y(h) = w(y+h)$*.*

Proof. It is enough to show that, for any $y \in G$, we have for all $F \in \mathfrak{C}_{00}(H) \cap A\mathfrak{L}_1(H)$

(3.3)
$$\int_H \mathbf{1}_{\{h:\max_{1\le n\le N} |H_n^\beta F(h)|>1\}}(h) w_y(h)\, d\mu_H(h) \le K_p^p \|F\|^p_{\mathfrak{L}_p(H,w_y)},$$

where K_p is independent of F, y and N. (By a straightforward argument using [**5**, Theorem 33.11], it follows that (3.3) holds for all $F \in \mathfrak{L}_p(H, w_y) \cap \mathfrak{L}_1(H)$. Then the theorem follows by Fatou's lemma.)

Fix $N \in \mathbf{N}$, and fix $F \in \mathfrak{C}_{00}(H) \cap A\mathfrak{L}_1(H)$. Let $\Lambda = A(\hat{G}, H)$ be the annihilator of H in $\hat{G}$. By [**9**, p. 35], we have $\hat{G}/\Lambda \cong \hat{H}$ and Λ is closed. Since $F \in A\mathfrak{L}_1(H)$, we have $\hat{F} \in \mathfrak{L}_1(\hat{H}) \cong \mathfrak{L}_1(\hat{G}/\Lambda)$. By the Weil formula [**5**, Theorem 28.54] it is easy to see that there is a function $u \in \mathfrak{L}_1(\hat{G})$ such that

$$\hat{u}(-h) = F(h), \quad h \in H. \tag{3.4}$$

Now if $-K_0 = \operatorname{supp} F$, by [**5**, Theorem 31.37], there is a function $g \in \mathfrak{L}_1(\hat{G})$ such that $\hat{g} \in \mathfrak{C}_{00}(G)$ and $\hat{g} \equiv 1$ on $-K_0$. Define $f \in \mathfrak{C}_{00}(G) \cap A\mathfrak{L}_1(G)$ by

$$f(x) = \hat{u}(-x)\hat{g}(-x), \quad x \in G.$$

Clearly we have $f(h) = F(h)$ for all $h \in H$.

Let K be a compact neighborhood of the identity in G/H. Let U be an open neighborhood containing K. By [**1**, Proposition 2.13], there is a measure $v \in M(\hat{G})$ such that

$$\operatorname{supp} \hat{v} \subset \eta_H^{-1}(U), \quad 0 \le \hat{v} \le 1 \quad \text{and} \quad \hat{v} \equiv 1 \text{ on } \eta_H^{-1}(K),$$

where $\eta_H : G \to G/H$ is the natural map from G onto G/H. Then, clearly, for all $x \in \eta_H^{-1}(K)$ and $h \in H$,

$$\begin{aligned}(3.5)\quad \max_{1\le n\le N} |H_n^\beta(\hat{v}f)(x+h)| &= \max_{1\le n\le N}\left|\frac{1}{\pi}\int_{1\le|s|\le n} \hat{v}f(x+h-\beta(s))\frac{1}{s}\,ds\right| \\ &= \max_{1\le n\le N}\left|\frac{1}{\pi}\int_{1\le|s|\le n} f(x+h-\beta(s))\frac{1}{s}\,ds\right| \\ &= \max_{1\le n\le N}|H_n^\beta(f)(x+h)|.\end{aligned}$$

Fix $y \in G$. By the hypothesis and the fact that $0 \le \hat{v} \le 1$,

$$\begin{aligned}(3.6)\quad \int_G \mathbf{1}_{\{x\in G:\max_{1\le n\le N}|H_n^\beta(\hat{v}f)(x)|>1\}}(x)w_y(x)\,d\mu(x) &= \int_G \mathbf{1}_{\{x\in G:\max_{1\le n\le N}|H_n^\beta(\hat{v}f)_{-y}(x)|>1\}}(x)w(x)\,d\mu(x) \\ &\le K_p^p\int_G |(\hat{v}f)_{-y}(x)|^p w(x)\,d\mu(x) \\ &= K_p^p\int_G |\hat{v}f(x)|^p w_y(x)\,d\mu(x) \\ &\le K_p^p\int_G |f(x)|^p w_y(x)\,d\mu(x) \\ &< \infty.\end{aligned}$$

Let $A = \{x \in G : \max_{1\le n\le N}|H_n^\beta(\hat{v}f)(x)| > 1\}$. By (3.6), we have $\mathbf{1}_A w_y \in \mathfrak{L}_1(G)$, so we can apply the Weil formula [**5**, Theorem 28.54], to get

$$\begin{aligned}\int_{G/H}\int_H \mathbf{1}_A(x+h)w_y(x+h)\,d\mu_H(h)\,d\mu_{G/H}(x+H) &= \int_G \mathbf{1}_A(x)w_y(x)\,d\mu(x) \\ &\le K_p^p\int_G |\hat{v}f(x)|^p w_y(x)\,d\mu(x) \\ &= K_p^p\int_{G/H}\int_H |\hat{v}(x+h)f(x+h)|^p \\ &\qquad \cdot w_y(x+h)\,d\mu_H(h)\,d\mu_{G/H}(x+H).\end{aligned}$$

Note that $\hat{v}(x+h) = 0$ if $x + H \notin U$ and $h \in H$. So we have

$$\int_K \int_H \mathbf{1}_A(x+h) w_y(x+h)\, d\mu_H(h)\, d\mu_{G/H}(x+H) \\ \le K_p^p \int_U \int_H |\hat{v}(x+h) f(x+h)|^p \\ \cdot w_y(x+h)\, d\mu_H(h)\, d\mu_{G/H}(x+H).$$

Let $B = \{x \in G : \max_{1\le n\le N} |H_n^\beta f(x)| > 1\}$. From (3.5), the above, and $0 \le \hat{v} \le 1$,

$$\begin{aligned}\int_K \int_H & \mathbf{1}_B(x+h) w_y(x+h)\, d\mu_H(h)\, d\mu_{G/H}(x+H) \\ &= \int_K \int_H \mathbf{1}_A(x+h) w_y(x+h)\, d\mu_H(h)\, d\mu_{G/H}(x+H) \\ &\le K_p^p \int_U \int_H |\hat{v}(x+h) f(x+h)|^p w_y(x+h)\, d\mu_H(h)\, d\mu_{G/H}(x+H) \\ &\le K_p^p \int_U \int_H |f(x+h)|^p w_y(x+h)\, d\mu_H(h)\, d\mu_{G/H}(x+H).\end{aligned}$$

Since U is an arbitrary open neighborhood containing K, we can replace U by K to get

$$\int_K \int_H \mathbf{1}_B(x+h) w_y(x+h)\, d\mu_H(h)\, d\mu_{G/H}(x+H) \\ \le K_p^p \int_K \int_H |f(x+h)|^p w_y(x+h)\, d\mu_H(h)\, d\mu_{G/H}(x+H). \tag{3.7}$$

It is easy to see that the function $M(x) = \max_{1\le n\le N} |H_n^\beta f(x)|$ is a continuous function on G and has compact support. Also, since $\mathbf{1}_B$ is the characteristic function of an open σ-compact subset of G, there is a sequence $(u_j)_{j=1}^\infty \subset \mathfrak{C}_{00}(G)$ such that $0 \le u_j \le u_{j+1}$ and $u_j(x) \to \mathbf{1}_B(x)$ for all $x \in G$.

By (3.7) we have

$$\int_K \int_H u_j(x+h) w_y(x+h)\, d\mu_H(h)\, d\mu_{G/H}(x+H) \\ \le K_p^p \int_K \int_H |f(x+h)|^p w_y(x+h)\, d\mu_H(h)\, d\mu_{G/H}(x+H).$$

Now, for any function $v \in \mathfrak{C}_{00}(G)$, the function $x \mapsto \int_H v(x+h)\,d\mu_H(h)$ is continuous on G and constant on cosets of H. Hence, the function $x+H \mapsto \int_H v(x+h)\,d\mu_H(h)$ is well-defined and continuous on G/H. So, since w is continuous on G and K is an arbitrary compact neighborhood of the identity in G/H, we have

$$\int_H u_j(h)w_y(h)\,d\mu_H(h) \le K_p^p \int_H |f(h)|^p w_y(h)\,d\mu_H(h).$$

But $f(h) = F(h)$ for all $h \in H$, and so, by the monotone convergence theorem,

$$\begin{aligned}\int_H \mathbf{1}_{\{h\in H:\max_{1\le n\le N}|H_n^\beta F(h)|>1\}}(h)w_y(h)\,d\mu_H(h)& \\ &= \int_H \mathbf{1}_B(h)w_y(h)\,d\mu_H(h)\\ &\le K_p^p\int_H |F(h)|^p w_y(h)\,d\mu_H(h),\end{aligned}$$

where K_p is independent of F, y and N. Thus we have shown that (3.3) holds, completing the proof of the theorem. □

Corollary 3.2. *Let $1 \le p < \infty$. Let G be a locally compact abelian group, and let $\varphi : \mathbf{R} \to G$ be a topological isomorphism. Let $T^\varphi = H^\varphi$ or MH^φ, and let $w \in \mathfrak{C}^+(G)$. Suppose the weighted norm inequality*

$$\|T^\varphi f\|^*_{\mathfrak{L}_{p,\infty}(G,w)} \le K_p\|f\|_{\mathfrak{L}_p(G,w)}$$

holds for all $f \in \mathfrak{L}_p(G,w)\cap\mathfrak{L}_1(G)$. Then $w \in A_p^\varphi(G)$, where $A_p^\varphi \le K_p$ and the inequality in (1.2) holds for all $x \in G$.

Proof. This follows immediately from Theorem 3.1 and Theorem 1.1. □

4. Proof of main theorem. To prove our main theorem, Theorem 1.2, we first prove the case of $G = \mathbf{T}^n$ and then use this to prove the theorem for certain compact groups, which we then use to prove the theorem for any locally compact abelian group.

The following notation will be used. If $\varphi_b : \mathbf{R} \to \mathbf{T}^n$ is defined by $\varphi_b(t) = (e^{i\alpha t}, e^{ic_1\alpha t}, \dots , e^{ic_{n-1}\alpha t})$, we say that φ_b is a one-dimensional homomorphism. In the next theorem we prove the case of $G = \mathbf{T}^n$ by approximating continuous homomorphisms by one-dimensional homomorphisms.

Theorem 4.1. *Let* $1 \le p < \infty$. *Let* $w \in \mathfrak{C}^+(\mathbf{T}^n)$, *and let* $T^\varphi = H^\varphi$ *or* MH^φ. *Suppose that, for all continuous homomorphisms* $\varphi : \mathbf{R} \to \mathbf{T}^n$, *the weighted norm inequality*

$$\|T^\varphi f\|^*_{\mathfrak{L}_{p,\infty}(\mathbf{T}^n,w)} \le K_p \|f\|_{\mathfrak{L}_p(\mathbf{T}^n,w)},$$

holds for all $f \in \mathfrak{L}_p(\mathbf{T}^n, w) \cap \mathfrak{L}_1(\mathbf{T}^n)$, *where* K_p *is independent of* f *and* φ. *Then* $w \in \mathfrak{A}_p(\mathbf{T}^n)$ *and* $\mathfrak{A}_p \le K_p^2(4\pi)^{2p}$.

Proof. Let $\varphi : \mathbf{R} \to \mathbf{T}^n$ be a continuous homomorphism, where $\varphi(t) = (e^{i\alpha_1 t}, \dots , e^{i\alpha_n t})$. To show that $w \in A_p^\varphi(\mathbf{T}^n)$, fix an interval $I = [c, d]$. Since w is continuous on the compact metric space $\mathbf{T}^n$, it is easy to see that there is a sequence of continuous one-dimensional homomorphisms $\varphi_b^k : \mathbf{R} \to \mathbf{T}^n$ such that, for all $x \in \mathbf{T}^n$ and all $t \in I$, $w(x\varphi_b^k(-t)) \to w(x\varphi(-t))$ and $w^{-1/(p-1)}(x\varphi_b^k(-t)) \to w^{-1/(p-1)}(x\varphi(-t))$. Fixing k, by the hypothesis and Theorem 3.1, we have for each $x \in \mathbf{T}^n$, the weighted norm inequality

$$\|T^{\varphi_b^k} f\|^*_{\mathfrak{L}_{p,\infty}(\overline{\varphi_b^k(\mathbf{R})},w_x)} \le K_p \|f\|_{\mathfrak{L}_p(\overline{\varphi_b^k(\mathbf{R})},w_x)}$$

holds for all $f \in \mathfrak{L}_p(\overline{\varphi_b^k(\mathbf{R})}, w_x) \cap \mathfrak{L}_1(\overline{\varphi_b^k(\mathbf{R})})$, where K_p is independent of f and x. But $\overline{\varphi_b^k(\mathbf{R})} = \varphi_b^k(\mathbf{R}) \cong \mathbf{T}$, so by Theorem 1.1 we have, for all $x \in \mathbf{T}^n$, $w_x \in A_p(\varphi_b^k(\mathbf{R}))$. (The bound A_p is less than or equal to $K_p^2(4\pi)^{2p}$; see the proof of Theorem 1.1 in [**6**].) This is true for all $k \in \mathbf{N}$ so, by Fatou's lemma, we have for all $x \in \mathbf{T}^n$,

$$\begin{aligned}\frac{1}{|I|}\int_I w(x\varphi(-t))\,dt\biggl(\frac{1}{|I|}\int_I w^{-1/(p-1)}(x\varphi(-t))\,dt\biggr)^{p-1}\\ \le \frac{1}{|I|}\liminf_k \int_I w(x\varphi_b^k(-t))\,dt\\ \cdot\biggl(\liminf_k \frac{1}{|I|}\int_I w^{-1/(p-1)}(x\varphi_b^k(-t))\,dt\biggr)^{p-1}\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{|I|}\limsup_k \int_I w(x\varphi_b^k(-t))\,dt \\
&\quad \cdot \limsup_k \left(\frac{1}{|I|}\int_I w^{-1/(p-1)}(x\varphi_b^k(-t))\,dt\right)^{p-1} \\
&\le K_p^2(4\pi)^{2p}.
\end{aligned}$$

We can do this for all intervals I, hence $w \in A_p^\varphi(\mathbf{T}^n)$ and the inequality in (1.2) holds for all $x \in \mathbf{T}^n$. Since this is true for all continuous homomorphisms $\varphi : \mathbf{R} \to \mathbf{T}^n$, we have $w \in \mathfrak{A}_p(\mathbf{T}^n)$ and $\mathfrak{A}_p \le K_p^2(4\pi)^{2p}$. $\square$

A group G is *solenoidal* if it contains a dense homomorphic image of $\mathbf{R}$. In the next theorem we prove the sufficiency part of our main theorem, Theorem 1.2, in the case when G is a compact solenoidal group. We need the following lemma.

Lemma 4.2. *Let $1 \le p < \infty$. Let G be a compact solenoidal abelian group and let $T^\varphi = H^\varphi$ or MH^φ. Let $\hat{S} = \langle\{\chi_j\}_{j=1}^k\rangle$ be a finitely generated subgroup of $\hat{G}$, let $G_0 = A(G, \hat{S})$ be the annihilator of $\hat{S}$ in G, and let μ_0 be Haar measure on G_0. Suppose $w \in \mathfrak{C}^+(G)$ and for all continuous homomorphisms $\varphi : \mathbf{R} \to G$, the weighted norm inequality*

$$\|T^\varphi f\|^*_{\mathfrak{L}_{p,\infty}(G,w)} \le K_p \|f\|_{\mathfrak{L}_p(G,w)}$$

holds for all $f \in \mathfrak{L}_p(G,w) \cap \mathfrak{L}_1(G)$. Then $w_0 \in \mathfrak{A}_p(G/G_0)$, where $\mathfrak{A}_p \le K_p^2(4\pi)^{2p}$ and $w_0 : G/G_0 \to \mathbf{R}^+$ is defined by $w_0(x + G_0) = \int_{G_0} w(x+y)\,d\mu_0(y)$.

Proof. It suffices to show that, for all continuous homomorphisms $\beta : \mathbf{R} \to G/G_0$, the weighted norm inequality

$$\|T^\beta f\|^*_{\mathfrak{L}_{p,\infty}(G/G_0,w_0)} \le K_p \|f\|_{\mathfrak{L}_p(G/G_0,w_0)} \tag{4.1}$$

holds for all $f \in \mathfrak{L}_p(G/G_0, w_0) \cap \mathfrak{L}_1(G/G_0)$, where K_p is independent of f and β. This is because $\hat{S}$ is finitely generated and $\hat{G}$ is torsion free, see [**4**, Theorem 25.18], so that $\hat{S}$ is isomorphic to $\mathbf{Z}^k$ and $G/G_0 \cong S \cong \mathbf{T}^k$; then we can use Theorem 4.1 to get $w_0 \in \mathfrak{A}_p(G/G_0)$ with $\mathfrak{A}_p \le K_p^2(4\pi)^{2p}$.

To show that (4.1) holds, fix a continuous homomorphism $\beta : \mathbf{R} \to G/G_0$ with adjoint $\tilde{\beta} : \hat{S} \to \mathbf{R}$. By [**4**, p. 441], there is a continuous homomorphism $\psi : \hat{G} \to \mathbf{R}$ that extends $\tilde{\beta}$. So then we can realize $\tilde{\beta}$ by $\tilde{\beta} : \hat{S} \xrightarrow{\iota} \hat{G} \xrightarrow{\psi} \mathbf{R}$, where ι is the identity map from $\hat{S}$ into $\hat{G}$. Then we have $\beta : \mathbf{R} \xrightarrow{\varphi} G \xrightarrow{\eta} G/G_0$, where η is the natural map from G onto G/G_0 and $\varphi : \mathbf{R} \to G$ is the adjoint of ψ. Now fix a trigonometric polynomial $f \in \mathfrak{T}(G)$, so then $f * \mu_0 \in \mathfrak{L}_p(G, w) \cap \mathfrak{L}_1(G)$. Letting $B = \{x \in G : |T^\varphi(f * \mu_0)(x)| > 1\}$ and $B_y = \{x \in G : |T^\varphi(f_y * \mu_0)(x)| > 1\}$, we have by the translation invariance of μ, Fubini's theorem, and the hypothesis,

$$\begin{aligned}
\int_G \mathbf{1}_B(x) w * \mu_0(x)\, d\mu(x) &= \int_G \int_{G_0} \mathbf{1}_B(x) w(x-y)\, d\mu_0(y)\, d\mu(x) \\
&= \int_{G_0} \int_G \mathbf{1}_{B_y}(x) w(x)\, d\mu(x)\, d\mu_0(y) \\
&\le K_p^p \int_{G_0} \int_G |f_y * \mu_0(x)|^p w(x)\, d\mu(x)\, d\mu_0(y) \\
&= K_p^p \int_G |f * \mu_0(x)|^p w * \mu_0(x)\, d\mu(x).
\end{aligned}$$

Using the Weil formula [**5**, Theorem 28.54] and the fact that $\beta = \eta \circ \varphi$, it is easy to see that this is enough to show that (4.1) holds, completing the proof of the lemma. □

Theorem 4.3. *Let $1 \le p < \infty$. Let G be a compact solenoidal abelian group equipped with Haar measure μ and dual $\hat{G}$. Let $T^\varphi = H^\varphi$ or MH^φ. Suppose $w \in \mathfrak{C}^+(G)$ and that whenever $\varphi : \mathbf{R} \to G$ is a continuous homomorphism, we have for all $f \in \mathfrak{L}_p(G, w) \cap \mathfrak{L}_1(G)$,*

$$\|T^\varphi f\|_{\mathfrak{L}_{p,\infty}(G,w)} \le K_p \|f\|_{\mathfrak{L}_p(G,w)},$$

where K_p is independent of f and φ. Then $w \in \mathfrak{A}_p(G)$, where the bound $\mathfrak{A}_p \le K_p^2 (4\pi)^{2p}$.

Proof. By the Stone-Weierstrass theorem, the set of trigonometric polynomials is dense in $\mathfrak{C}(G)$, see [**9**, p. 24]. So, since $w \in \mathfrak{C}^+(G)$, there is a trigonometric polynomial h_n such that $\|h_n - w\|_u < 1/n$. Let $w_n = |h_n| + 1/n$, and we have

$$\|w_n - w\|_u = \||h_n| + 1/n - w\|_u \le 1/n + \||h_n| - w\|_u \le 2/n. \tag{4.2}$$

We claim that, for any continuous homomorphism $\varphi : \mathbf{R} \to G$, we have for all $f \in \mathfrak{L}_p(G, w_n) \cap \mathfrak{L}_1(G)$,

$$\|T^\varphi f\|^*_{\mathfrak{L}_{p,\infty}(G,w_n)} \le \overline{C}_p \|f\|_{\mathfrak{L}_p(G,w_n)}, \tag{4.3}$$

where $\overline{C}_p$ is a constant independent of f, n and φ. To prove the claim, let $\varphi : \mathbf{R} \to G$ be a continuous homomorphism, and let $f \in \mathfrak{L}_p(G, w_n) \cap \mathfrak{L}_1(G)$. By [**2**, Theorem 6.5], we have $\|T^\varphi f\|^*_{\mathfrak{L}_{p,\infty}(G,\mu)} \le \overline{M}_p \|f\|_{\mathfrak{L}_p(G,\mu)}$, where $\overline{M}_p$ is a constant independent of f and φ. By (4.2), the hypothesis and [**2**, Theorem 6.5],

$$\begin{aligned}
(\|T^\varphi f\|^*_{\mathfrak{L}_{p,\infty}(G,w_n)})^p
&= \sup_\tau \tau^p \int_G \mathbf{1}_{\{x \in G : |T^\varphi f| > \tau\}}(x) w_n(x)\, d\mu(x) \\
&\le \sup_\tau \tau^p \int_G \mathbf{1}_{\{x \in G : |T^\varphi f| > \tau\}}(x) |w_n(x) - w(x)|\, d\mu(x) \\
&\quad + \sup_\tau \tau^p \int_G \mathbf{1}_{\{x \in G : |T^\varphi f| > \tau\}}(x) w(x)\, d\mu(x) \\
&\le 2\overline{M}_p^p \frac{1}{n} \int_G |f(x)|^p\, d\mu(x) + K_p^p \int_G |f(x)|^p w(x)\, d\mu(x) \\
&\le 2\overline{M}_p^p \int_G |f(x)|^p w_n(x)\, d\mu(x) \\
&\quad + K_p^p \int_G |f(x)|^p |w(x) - w_n(x)|\, d\mu(x) \\
&\quad + K_p^p \int_G |f(x)|^p w_n(x)\, d\mu(x) \\
&\le 2\overline{M}_p^p \|f\|^p_{\mathfrak{L}_p(G,w_n)} + 2K_p^p \|f\|^p_{\mathfrak{L}_p(G,w_n)} + K_p^p \|f\|^p_{\mathfrak{L}_p(G,w_n)} \\
&= \overline{C}_p^p \|f\|^p_{\mathfrak{L}_p(G,w_n)},
\end{aligned}$$

where $\overline{C}_p$ is independent of f, n and φ. Thus we have proved that (4.3) holds for any continuous homomorphism $\varphi : \mathbf{R} \to G$.

Fix $n \in \mathbf{N}$. We have $w_n(x) = |h_n(x)| + 1/n = |\sum_{j=1}^{l(n)} a_j \chi_j(x)| + 1/n$, where $a_j \in \mathbf{C}$ and $\chi_j \in \hat{G}$. Define $\hat{S}_n \subset \hat{G}$ to be the subgroup generated by $\{\chi_j\}_{j=1}^{l(n)}$. If we let $G_0^n = A(G, \hat{S}_n)$, the annihilator of $\hat{S}_n$ in G, then by (4.3) and Lemma 4.2, we have $w_0^n \in \mathfrak{A}_p(G/G_0^n)$, where $w_0^n(x+G_0^n) = \int_{G_0^n} w_n(x+y)\,d\mu_0^n(y)$ and $\mathfrak{A}_p \le (\overline{C}_p)^2(4\pi)^{2p}$. Now, by the definition of G_0^n and w_n, we have $w_n(x+y) = w_n(x)$ for all $y \in G_0^n$, and hence $w_0^n(x+G_0^n) = w_n(x)$ for all $x \in G$. Then clearly it follows that $w_n \in \mathfrak{A}_p(G)$, where the constant $\mathfrak{A}_p$ is independent of n. By an argument similar to that used in Theorem 4.1, it follows that $w \in \mathfrak{A}_p(G)$. □

Now if G is any locally compact abelian group and $\varphi : \mathbf{R} \to G$ is a continuous homomorphism, then φ is either a topological isomorphism or $\overline{\varphi(\mathbf{R})}$ is compact [**4**, Theorem 9.1]. We use this fact in the proof of our main theorem, Theorem 1.2.

Proof of Theorem 1.2. The necessity part of the theorem follows from Theorem 2.1. We show that the sufficiency part of the theorem holds. Let $\varphi : \mathbf{R} \to G$ be a continuous homomorphism. If φ is a topological isomorphism, then $w \in A_p^{\varphi}(G)$ by Corollary 3.2, so suppose $\overline{\varphi(\mathbf{R})}$ is compact. Consider a continuous homomorphism $\beta : \mathbf{R} \to \overline{\varphi(\mathbf{R})}$. Since we also have $\beta : \mathbf{R} \to G$, given any $f \in \mathfrak{L}_p(G,w) \cap \mathfrak{L}_1(G)$, we have by the hypothesis

$$\|T^{\beta} f\|^*_{\mathfrak{L}_{p,\infty}(G,w)} \le K_p \|f\|_{\mathfrak{L}_p(G,w)},$$

where K_p is independent of f and β. Then, by Theorem 3.1, for all $y \in G$, the weighted norm inequality

$$\|T^{\beta} f\|^*_{\mathfrak{L}_{p,\infty}(\overline{\varphi(\mathbf{R})},w_y)} \le K_p \|f\|_{\mathfrak{L}_p(\overline{\varphi(\mathbf{R})},w_y)}$$

holds for all $f \in \mathfrak{L}_p(\overline{\varphi(\mathbf{R})}, w_y) \cap \mathfrak{L}_1(\overline{\varphi(\mathbf{R})})$, where K_p is independent of f, y and β. This is true for all continuous homomorphisms $\beta : \mathbf{R} \to \overline{\varphi(\mathbf{R})}$ so, by Theorem 4.3, $w_y \in \mathfrak{A}_p(\overline{\varphi(\mathbf{R})})$ for all $y \in G$, where the constant $\mathfrak{A}_p \le (\overline{C}_p)^2(4\pi)^{2p}$. In particular, $w_y \in A_p^{\varphi}(\overline{\varphi(\mathbf{R})})$ for all $y \in G$. Then, given $y \in G$, we have, for some $x = \varphi(s) \in \overline{\varphi(\mathbf{R})}$,

$$(\overline{C}_p)^2(4\pi)^{2p} \ge \sup_I \frac{1}{|I|}\int_I w_y(\varphi(s-t))\,dt \cdot \left(\frac{1}{|I|}\int_I w_y^{-1/(p-1)}(\varphi(s-t))\,dt\right)^{p-1}$$

$$= \sup_I \frac{1}{|I|} \int_{s-I} w_y(\varphi(t))\,dt \cdot \left(\frac{1}{|I|} \int_{s-I} w_y^{-1/(p-1)}(\varphi(t))\,dt \right)^{p-1}$$
$$= \sup_I \frac{1}{|I|} \int_I w(y - \varphi(t))\,dt \cdot \left(\frac{1}{|I|} \int_I w^{-1/(p-1)}(y - \varphi(t))\,dt \right)^{p-1}.$$

(Refer to the proofs of Theorem 4.1 and Theorem 4.3 to see that the above inequality actually holds for all $x \in \overline{\varphi(\mathbf{R})}$.) This is true for any $y \in G$, so $w \in A_p^\varphi(G)$ and A_p^φ is independent of φ. Thus, for all continuous homomorphisms $\varphi : \mathbf{R} \to G$, we have $w \in A_p^\varphi(G)$ and $\sup_\varphi A_p^\varphi$ is finite. Hence, $w \in \mathfrak{A}_p(G)$, completing the proof of the theorem. □

5. Examples of weights satisfying the uniform A_p-condition. The following question is still open: If we suppose that (1.3) holds for a weight w and a fixed homomorphism $\varphi : \mathbf{R} \to G$, does this necessarily imply that w satisfies the ergodic A_p-condition? The proof of Theorem 1.2 shows the importance of the case of $\mathbf{T}^n$ in considering this question. In this section we construct examples of weights on $\mathbf{T}^n$ that provide insight into the above question.

Stein [**10**] and Muckenhoupt [**8**] show that, if $-1 < \alpha < p-1$ and $W_\alpha(x) = |x|^\alpha$, then the weight W_α satisfies the A_p-condition on $\mathbf{R}$. In the next lemma we show that the functions defined by $w_\alpha(t) = |\sin t|^\alpha$ satisfy the A_p-condition on $\mathbf{T}$ if and only if $-1 < \alpha < p-1$. Then, defining the weights $w : \mathbf{T}^n \to \mathbf{R}^+$ by $w(t_1, \dots, t_n) = |\sin t_1|^{\alpha_1} + \cdots + |\sin t_n|^{\alpha_n}$, we have $w \in \mathfrak{A}_p(\mathbf{T}^n)$ if and only if $-1 < \alpha_j < p-1$.

Lemma 5.1. *Define the weight $w_\alpha(t) = |\sin t|^\alpha$. If $1 < p < \infty$, then $w_\alpha \in A_p(\mathbf{T})$ if and only if $-1 < \alpha < p-1$. If $p = 1$, we have $w_\alpha \in A_p(\mathbf{T})$ if and only if $-1 < \alpha \le 0$.*

Proof. We show that $w_\alpha \in A_p(\mathbf{T})$ for $0 < \alpha < p-1$. The case of $-1 < \alpha < 0$ follows by a similar argument. Let $I = (a,b) \subset (-\pi/2, \pi/2)$. Recall that the weight $W_\alpha(x) = |x|^\alpha$ is in $A_p(\mathbf{R})$, and let $A_p(W_\alpha)$

denote the uniform bound. Note that $|t| > |\sin t| > \sqrt{2}|t|/4$ for all $-\pi/2 < t < \pi/2$. Then we have

$$\begin{aligned}
(5.1)\quad & \frac{1}{b-a}\int_a^b |\sin t|^\alpha\,dt\left(\frac{1}{b-a}\int_a^b |\sin t|^{-\alpha/(p-1)}\,dt\right)^{p-1} \\
&\le \frac{1}{b-a}\int_a^b |t|^\alpha\,dt\left(\frac{1}{b-a}\int_a^b \left(\frac{\sqrt{2}}{4}|t|\right)^{-\alpha/(p-1)} dt\right)^{p-1} \\
&\le \left(\frac{\sqrt{2}}{4}\right)^{-\alpha} A_p(W_\alpha).
\end{aligned}$$

Next, consider an interval $I = (\pi/2 - a, \pi/2 + b)$, where $0 < a < b < \pi/2$. Let $I_1 = (\pi/2 - b, \pi/2 + b)$ and $J_1 = (\pi/2 - b, \pi/2)$. Then we have

$$\begin{aligned}
\frac{1}{|I|}\int_I |\sin t|^\alpha\,dt\left(\frac{1}{|I|}\int_I |\sin t|^{-\alpha/(p-1)}\,dt\right)^{p-1} & \\
&\le \frac{1}{|I|}\int_{I_1} |\sin t|^\alpha\,dt\left(\frac{1}{|I|}\int_{I_1} |\sin t|^{-\alpha/(p-1)}\,dt\right)^{p-1} \\
&= \frac{2}{|I|}\int_{J_1} |\sin t|^\alpha\,dt\left(\frac{2}{|I|}\int_{J_1} |\sin t|^{-\alpha/(p-1)}\,dt\right)^{p-1} \\
&\le \frac{2}{|I|^p}|J_1|^p\left(\frac{\sqrt{2}}{4}\right)^{-\alpha} A_p(W_\alpha) \\
&\le \frac{2}{|I|^p}|I|^p\left(\frac{\sqrt{2}}{4}\right)^{-\alpha} A_p(W_\alpha) \\
&= 2\left(\frac{\sqrt{2}}{4}\right)^{-\alpha} A_p(W_\alpha).
\end{aligned}$$

(The first equality follows from the fact that $\sin(\pi/2 - x) = \sin(\pi/2 + x)$; the second inequality follows from (5.1).)

Now if I is any interval in $\mathbf{R}$ with $|I| < \pi/2$, by the periodicity of $|\sin x|$ and the translation invariance of Lebesgue measure, we have $(1/|I|)\int_I |\sin t|^\alpha\,dt((1/|I|)\int_I |\sin t|^{-\alpha/(p-1)}\,dt)^{p-1}$ bounded by a constant independent of I. This is enough to show that $w_\alpha \in A_p(\mathbf{T})$ for $0 < \alpha < p-1$. (By a simple change of variable, the boundedness for intervals of length less than $\pi/2$ implies the boundedness for intervals of any length.)

To show the converse, we consider the case $\alpha < -1$. The other cases follow by a similar argument. Let (a_n) be a sequence of positive real numbers converging to 0, such that $0 < a_n < \pi/8$, and let $I_n = (a_n, \pi/4)$. Since $1+\alpha < 0$, we have $\sup_n (\sin a_n)^{1+\alpha} = \infty$. Also, $\alpha/(p-1) < 1$ so that $-C = (1/-(\alpha+1))(1/(1-\alpha/(p-1)))^{p-1} > 0$. Then

$$\begin{aligned}
\sup_I \frac{1}{|I|}\int_I |\sin t|^\alpha \, dt\biggl(\frac{1}{|I|}\int_I |\sin t|^{-\alpha/(p-1)}\, dt\biggr)^{p-1} \\
&\geq \sup_n \frac{1}{|I_n|}\int_{I_n} \cos t(\sin t)^\alpha \, dt \\
&\quad \cdot \biggl(\frac{1}{|I_n|}\int_{I_n} \cos t(\sin t)^{-\alpha/(p-1)}\, dt\biggr)^{p-1} \\
&= \sup_n \frac{1}{|I_n|^p}(-C)((\sin a_n)^{1+\alpha} - (\sin(\pi/4))^{1+\alpha}) \\
&\quad \cdot ((\sin(\pi/4))^{1-\alpha/(p-1)} - (\sin a_n)^{1-\alpha/(p-1)})^{p-1} \\
&\geq \sup_n \frac{1}{(\pi/4)^p}(-C)((\sin a_n)^{1+\alpha} - (\sin(\pi/4))^{1+\alpha}) \\
&\quad \cdot ((\sin(\pi/4)^{1-\alpha/(p-1)} - (\sin(\pi/8))^{1-\alpha/(p-1)})^{p-1} \\
&= \infty.
\end{aligned}$$

Thus $w_\alpha \notin A_p(\mathbf{T})$ if $\alpha < -1$. $\square$

Now $\mathfrak{A}_p(\mathbf{T}^n) \subsetneqq A_p^\varphi(\mathbf{T}^n)$. For example, define the function $w(e^{ix}, e^{iy}) = |\sin x|^{\alpha_1} + |\sin y|^{\alpha_2}$ on $\mathbf{T}^2$, where $-1 < \alpha_1 < p-1$ and $\alpha_2 \geq p-1$, and consider the continuous homomorphisms $\varphi(t) = (e^{it}, 1)$ and $\varphi_1(t) = (1, e^{it})$. By Lemma 5.1, it is easy to see that $w \in A_p^\varphi(\mathbf{T}^2)$ but $w \notin A_p^{\varphi_1}(\mathbf{T}^2)$ and hence $w \notin \mathfrak{A}_p(\mathbf{T}^2)$.

Also there are weights w such that H^φ is bounded (with respect to w) for some homomorphism $\varphi : \mathbf{R} \to \mathbf{T}^n$, but H^β is not bounded (with respect to w) for some one-to-one homomorphism $\beta : \mathbf{R} \to \mathbf{T}^n$. For example, define the weight $w(e^{ix}, e^{iy}) = |\sin x|^{\alpha_1} + |\sin y|^{\alpha_2}$ on $\mathbf{T}^2$, where $-1 < \alpha_1 < p-1$ and $\alpha_2 \geq p-1$. If we suppose that H^α is bounded (with respect to w) for all homomorphisms $\alpha : \mathbf{R} \to \mathbf{T}^2$, then by Theorem 1.2, we have $w \in \mathfrak{A}_p(\mathbf{T}^2)$, which is a contradiction to the above. Then it is easy to see that there is a continuous one-to-one homomorphism $\beta : \mathbf{R} \to \mathbf{T}^2$ such that H^β is not bounded (with respect to w).

Acknowledgments. The author would like to thank Professor Nakhlé Asmar for his advice in the preparation of this paper. The results in this paper represent a portion of the author's doctoral dissertation, prepared under the direction of Professor Nakhlé Asmar.

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