ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 1, Spring 1999

ON HANKEL CONVOLUTION EQUATIONS IN DISTRIBUTION SPACES

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ABSTRACT. In this paper we study Hankel convolution equations in distribution spaces. Solvability conditions for Hankel convolution equations are obtained. Also we investigated hypoelliptic Hankel convolution equations.

1. Introduction. The Hankel integral transformation is usually defined by

$$h_\mu(\phi)(x) = \int_0^\infty (xt)^{1/2} J_\mu(xt)\phi(t)\,dt, \quad x \in I = (0,\infty),$$

where J_μ denotes the Bessel function of the first kind and order μ. Throughout this paper μ always will be greater than $-1/2$, and we will denote by I the real interval $(0,\infty)$.

Zemanian [**25, 26** and **27**] investigated the h_μ transformation on generalized function spaces. He introduced in [**25**] the space $\mathcal{H}_\mu$ constituted by all those complex valued and smooth functions ϕ defined on I such that, for every $m, k \in \mathbf{N}$,

$$\gamma^\mu_{m,k}(\phi) = \sup_{x\in(0,\infty)} \left| x^m \left(\frac{1}{x}D\right)^k [x^{-\mu-1/2}\phi(x)]\right| < \infty.$$

The space $\mathcal{H}_\mu$ is Fréchet when it is endowed with the topology generated by the family $\{\gamma^\mu_{m,k}\}_{m,k\in\mathbf{N}}$ of seminorms. It was established, [**25**, Lemma 8] that h_μ is an automorphism of $\mathcal{H}_\mu$. The Hankel transformation is defined on $\mathcal{H}'_\mu$, the dual space of $\mathcal{H}_\mu$, as the adjoint of the h_μ-transformation of $\mathcal{H}_\mu$, and it is denoted by h'_μ. More recently,

Received by the editors on March 9, 1996, and in revised form on March 25, 1997.
AMS *Mathematics Subject Classification.* 46F12.
Key words and phrases. Hankel, convolution equations, distribution, Bessel.
Partially supported by Consejería de Educación, Gobierno Autónomo de Canarias, Proyecto 967/15-9-95 and by DGICYT Grant PB 94-0591 (Spain).

Betancor and Rodríguez-Mesa [**8**] have studied h_μ on new spaces of functions and distributions. We defined the spaces $\mathcal{X}_\mu$ and $\mathcal{Q}_\mu$ as follows. A complex valued and smooth function ϕ defined on I is in $\mathcal{X}_\mu$ if and only if, for every $m, k \in \mathbf{N}$,

$$\eta^\mu_{m,k}(\phi) = \sup_{x\in I} \left| e^{mx} \left(\frac{1}{x}D\right)^k (x^{-\mu-1/2}\phi(x)) \right| < \infty.$$

$\mathcal{X}_\mu$ is equipped with the topology associated to the system $\{\eta^\mu_{m,k}\}_{m,k\in\mathbf{N}}$ of seminorms. Thus $\mathcal{X}_\mu$ is a Fréchet space.

The space $\mathcal{Q}_\mu$ is constituted by all those complex valued functions Φ satisfying the following two conditions

(i) $s^{-\mu-1/2}\Phi(s)$ is an even entire function, and

(ii) for every $m, k \in \mathbf{N}$

$$w^\mu_{m,k}(\Phi) = \sup_{|\operatorname{Im} s|\le k} (1+|s|^2)^m |s^{-\mu-1/2}\Phi(s)| < \infty.$$

$\mathcal{Q}_\mu$ is a Fréchet space when we consider the topology generated by the family of seminorms $\{w^\mu_{m,k}\}_{m,k\in\mathbf{N}}$ on $\mathcal{Q}_\mu$.

We established [**8**, Theorem 2.1] that h_μ is a homeomorphism from $\mathcal{X}_\mu$ onto $\mathcal{Q}_\mu$. Moreover, h_μ coincides with its inverse. The Hankel transform is defined on the dual spaces $\mathcal{X}'_\mu$ and $\mathcal{Q}'_\mu$ as the adjoint of the h_μ transformation and it is also denoted by h'_μ.

The convolution for a Hankel type transformation closely connected with h_μ was investigated by Hirschman [**13**], Haimo [**12**] and Cholewinski [**9**]. A straightforward manipulation in the convolution considered by the above authors allows us to obtain the convolution for h_μ that will be denoted by # and that is defined as follows: for every measurable function ϕ and ψ on I such that $x^{\mu+1/2}\phi$ and $x^{\mu+1/2}\psi$ are absolutely integrable on I, the convolution $\phi\#\psi$ of ϕ and ψ is given by

$$(\phi\#\psi)(x) = \int_0^\infty \phi(y)(\tau_x\psi)(y)\,dy, \quad x \in I,$$

where $(\tau_x\psi)(y) = \int_0^\infty D_\mu(x,y,z)\psi(z)\,dz$, $x, y \in I$ and

$$D_\mu(x,y,z) = \int_0^\infty t^{-\mu-1/2}(xt)^{1/2}J_\mu(xt)(yt)^{1/2}J_\mu(yt)(zt)^{1/2}J_\mu(zt)\,dt,$$

$$x, y, z \in I.$$

The study of the # convolution in distribution spaces was started by de Sousa-Pinto [**23**]. In a series of papers, Betancor and Marrero [**3, 4, 5, 6, 7**] and [**16**] have investigated the Hankel convolution on the Zemanian spaces. Also Betancor and González [**1**] studied the generalized Hankel convolution. Recently, Betancor and Rodríguez-Mesa [**8**] defined the # convolution on distributions of exponential growth.

In this paper we analyze Hankel convolution equations. Solvability conditions for the # convolution equations in $\mathcal{H}'_\mu$ and $\mathcal{X}'_\mu$ are investigated in Section 2. Also in Section 3 we study hypoelliptic Hankel convolution equations in $\mathcal{H}'_\mu$ and $\mathcal{X}'_\mu$.

Throughout this paper M will always denote a suitable positive constant not necessarily the same in each occurrence.

2. Solvability of Hankel convolution equations of distribution. In this section, inspired by the papers of Sznajder and Zielezny [**21, 22**] and Pahk and Sohn [**19**], we obtain necessary and sufficient conditions to solve Hankel convolution equations in $\mathcal{H}'_\mu$ and $\mathcal{X}'_\mu$.

Marrero and Betancor studied in [**16**] the Hankel convolution operators on $\mathcal{H}'_\mu$. They introduced, for every $m \in \mathbf{Z}$, the space $O_{\mu,m,\#}$ constituted by all those complex valued and smooth functions ϕ defined on I such that, for every $k \in \mathbf{N}$,

$$\delta_k^{\mu,m}(\phi) = \sup_{x\in I} |(1+x^2)^m x^{-\mu-1/2} S_\mu^k \phi(x)| < \infty,$$

where S_μ denotes the Bessel operator $x^{-\mu-1/2}Dx^{2\mu+1}Dx^{-\mu-1/2}$. We define $\mathcal{O}_{\mu,m,\#}$ as the closure of $\mathcal{H}_\mu$ in $O_{\mu,m,\#}$.

Note that $\mathcal{O}_{\mu,m,\#} \supset \mathcal{O}_{\mu,m+1,\#}$ for each $m \in \mathbf{Z}$. The space $\cup_{m\in\mathbf{Z}}\mathcal{O}_{\mu,m,\#}$ is denoted by $\mathcal{O}_{\mu,\#}$. The Hankel convolution operators of $\mathcal{H}'_\mu$ are the elements of $\mathcal{O}'_{\mu,\#}$, the dual space of $\mathcal{O}_{\mu,\#}$ [**4**]. Characterizations of $\mathcal{O}'_{\mu,\#}$ were obtained in Proposition 4.2 [**16**]. The next result was established in [**5**].

Proposition 2.1 [**5**, Theorem 3.1]. *For $S \in \mathcal{O}'_{\mu,\#}$, the following conditions are equivalent*

(i) *To every $k \in \mathbf{N}$ there correspond $m, n \in \mathbf{N}$ and a positive* *constant M such that*

$$\max_{0\le l\le m}\sup\left\{\left|\left(\frac{1}{t}D\right)^{l}[t^{-\mu-1/2}(h'_\mu S)(t)]\right| : t\in I, |x-t|\le (1+x^2)^{-k}\right\}$$
$$\ge (1+x^2)^{-n},$$

whenever $x\in I$, $x\ge M$.

(ii) *If $T\in\mathcal{O}'_{\mu,\#}$ and $S\#T\in\mathcal{H}_\mu$, then $T\in\mathcal{H}_\mu$.*

If $S\in\mathcal{O}'_{\mu,\#}$, the existence of solution for the convolution equation

$$u\#S=v, \tag{1}$$

for every $v\in\mathcal{H}'_\mu$, implies conditions (i) and (ii) in Proposition 2.1.

Proposition 2.2. *Let $S\in\mathcal{O}'_{\mu,\#}$. If $\mathcal{H}'_\mu\#S=\mathcal{H}'_\mu$, then conditions* (i) *and* (ii) *in Proposition* 2.1 *hold.*

Proof. It is sufficient to see that (ii) holds when $\mathcal{H}'_\mu\#S=\mathcal{H}'_\mu$. Note firstly that the mapping

$$\begin{aligned}F:\mathcal{H}'_\mu &\longrightarrow \mathcal{H}'_\mu=\mathcal{H}'_\mu\#S\\ u&\longrightarrow u\#S\end{aligned}$$

is the transpose of the mapping

$$\begin{aligned}G:\mathcal{H}_\mu &\longrightarrow \mathcal{H}_\mu\subset S\#\mathcal{H}_\mu\\ \phi&\longrightarrow S\#\phi.\end{aligned}$$

Then, by invoking [**10**, Corollary, p. 92] the mapping G is an isomorphism. In particular, the mapping $G^{-1}:S\#\mathcal{H}_\mu\to\mathcal{H}_\mu$ is continuous.

Assume now that $T\in\mathcal{O}'_{\mu,\#}$ is such that $T\#S\in\mathcal{H}_\mu$. Let $(\varphi_k)_{k=1}^\infty$ be a sequence of smooth functions such that the following three conditions are satisfied

(i) $c_\mu^{-1}\int_0^\infty x^{\mu+1/2}\varphi_k(x)\,dx=1$, where $c_\mu=2^\mu\Gamma(\mu+1)$,

(ii) $0\le\varphi_k(x)$, $x\in I$,

(iii) $\varphi_k(x) = 0$, $x \notin (1/(k+1), 1/k)$,

for every $k \in \mathbf{N}$.

According to [**4**, p. 1148], for each $\phi \in \mathcal{H}_\mu$,

$$\varphi_k \# \phi \longrightarrow \phi, \quad \text{as } k \longrightarrow \infty, \quad \text{in } \mathcal{H}_\mu. \tag{2}$$

Moreover, by invoking [**16**, Proposition 4.7], we can write

$$S\#(T\#\varphi_k) = (S\#T)\#\varphi_k = (T\#S)\#\varphi_k, \quad \text{for every } k \in \mathbf{N}. \tag{3}$$

Since $T\#\varphi_k \in \mathcal{H}_\mu$, $k \in \mathbf{N}$, by taking into account that G^{-1} is continuous and by (2) and (3), we conclude that $(T\#\varphi_k)_{k=1}^\infty$ converges in $\mathcal{H}_\mu$. Also by (2) again $T\#\varphi_k \to T$, as $k \to \infty$, in $\mathcal{H}'_\mu$ when we consider in $\mathcal{H}'_\mu$ the weak $*$ (or the strong) topology. Hence $T \in \mathcal{H}_\mu$. Thus the proof is finished. $\square$

The authors in [**8**] have defined the Hankel convolution of distributions of exponential growth. We introduce [**8**, Section 3] a subspace $\mathcal{X}'_{\mu,\#}$ of $\mathcal{X}'_\mu$ consisting of $S \in \mathcal{X}'_\mu$ such that $S\#\phi \in \mathcal{X}_\mu$ for every $\phi \in \mathcal{X}_\mu$.

In the following we establish a condition that $S \in \mathcal{X}'_{\mu,\#}$ satisfies when the equation (1) admits a solution for every $v \in \mathcal{X}'_\mu$.

Proposition 2.3. *Let $S \in \mathcal{X}'_{\mu,\#}$. If $\mathcal{X}'_\mu\#S = \mathcal{X}'_\mu$, then S verifies the following property: $T \in \mathcal{X}_\mu$ provided that $T \in \mathcal{X}'_{\mu,\#}$ and $T\#S \in \mathcal{X}_\mu$.*

Proof. This result can be proved in a similar way to Proposition 2.2. It is sufficient to see that if $(\varphi_k)_{k=1}^\infty$ is a sequence of smooth functions verifying the three conditions listed in the proof of Proposition 2.2 then, for every $\phi \in \mathcal{X}_\mu$,

$$\phi\#\varphi_k \longrightarrow \phi, \quad \text{as } k \longrightarrow \infty, \quad \text{in } \mathcal{X}_\mu. \tag{4}$$

By virtue of [**8**, Theorem 2.1] and by the interchange formula [**13**, Theorem 2d] to show (4) it is equivalent to see that, for every $\Psi \in \mathcal{Q}_\mu$,

$$s^{-\mu-1/2} h_\mu(\varphi_k)\Psi \longrightarrow \Psi, \quad \text{as } k \longrightarrow \infty, \quad \text{in } \mathcal{Q}_\mu. \tag{5}$$

We now prove (5). Let $(\varphi_k)_{k=1}^{\infty}$ be a sequence in the proof of Proposition 2.2, and let $\Psi \in \mathcal{Q}_\mu$. Since $\int_0^\infty t^{\mu+1/2}\varphi_k(t)\,dt = c_\mu$, for every $k \in \mathbf{N}$, where $c_\mu = 2^\mu\Gamma(\mu+1)$, we can write

$$\begin{aligned} s^{-\mu-1/2}h_\mu(\varphi_k)(s) - 1 &= \int_0^\infty (st)^{-\mu}J_\mu(st)t^{\mu+1/2}\varphi_k(t)\,dt - 1 \\ &= \int_{1/(k+1)}^{1/k}\left[(st)^{-\mu}J_\mu(st) - \frac{1}{c_\mu}\right]t^{\mu+1/2}\varphi_k(t)\,dt, \end{aligned}$$
$$\text{for every } k \in \mathbf{N} \text{ and } s \in \mathbf{C}.$$

Let K be a compact subset of $\mathbf{C}$, and let $\varepsilon > 0$. There exists $t_0 > 0$ such that $|(st)^{-\mu}J_\mu(st) - (1/c_\mu)| < \varepsilon$ for each $0 < t < t_0$ and $s \in K$. Hence we can find $k_0 \in \mathbf{N}$ such that, for every $k \geq k_0$ and $s \in K$,

$$\begin{aligned} |s^{-\mu-1/2}h_\mu(\varphi_k)(s) - 1| &\leq \int_{1/(k+1)}^{1/k}\left|(st)^{-\mu}J_\mu(st) - \frac{1}{c_\mu}\right|t^{\mu+1/2}\varphi_k(t)\,dt \\ &< \varepsilon c_\mu. \end{aligned}$$

Moreover, from [**15**, Lemma 4], we deduce

$$\begin{aligned} |s^{-\mu-1/2}h_\mu(\varphi_k)(s) - 1| &\leq \int_0^\infty (|(st)^{-\mu}J_\mu(st)| + 1)t^{\mu+1/2}\varphi_k(t)\,dt \\ &\leq Me^{|\operatorname{Im} s|}\int_0^\infty t^{\mu+1/2}\varphi_k(t)\,dt \\ &= Me^{|\operatorname{Im} s|}, \end{aligned}$$
$$\text{for every } k \in \mathbf{N} \text{ and } s \in \mathbf{C}.$$

Hence, for each $m \in \mathbf{N}$ there exists $\alpha > 0$ such that

$$\frac{1}{1+|s|^2}|s^{-\mu-1/2}h_\mu(\varphi_k)(s) - 1| < \varepsilon,$$
$$\text{for every } k \in N,\ |\operatorname{Re} s| > \alpha \text{ and } |\operatorname{Im} s| \leq m.$$

We can conclude that, for every $m \in \mathbf{N}$, there exists $k_0 \in \mathbf{N}$ such that

$$\frac{1}{1+|s|^2}|s^{-\mu-1/2}h_\mu(\varphi_k)(s) - 1| < \varepsilon,$$
$$\text{for every } k \geq k_0 \text{ and } |\operatorname{Im} s| \leq m.$$

Now let $m, n \in \mathbf{N}$. We have, for every $\Psi \in \mathcal{Q}_\mu$,

$$
\begin{aligned}
&w^{\mu}_{n,m}(\Psi(s)[s^{-\mu-1/2}h_\mu(\varphi_k)(s)-1])\\
&\quad\leq \sup_{|\operatorname{Im} s|\leq m}(1+|s|^2)^{n+1}|s^{-\mu-1/2}\Psi(s)|\\
&\qquad\cdot \sup_{|\operatorname{Im} s|\leq m}\frac{1}{1+|s|^2}|s^{-\mu-1/2}h_\mu(\varphi_k)(s)-1| \longrightarrow 0, \quad \text{as } k\longrightarrow\infty.
\end{aligned}
$$

Thus (5) is established. $\square$

We now give a condition for $S \in \mathcal{X}'_{\mu,\#}$ that implies the solvability of equation (1) for every $v \in \mathcal{X}'_\mu$.

Proposition 2.4. *Let $S \in \mathcal{X}'_{\mu,\#}$. If there exist N, r, C positive constants such that*

$$
(6) \qquad \sup_{s\in\mathbf{C},|s|\leq r}|(\xi+s)^{-\mu-1/2}h'_\mu(S)(\xi+s)| \geq \frac{C}{(1+|\xi|^2)^N}, \quad \xi\in\mathbf{R},
$$

then $\mathcal{X}'_\mu \# S = \mathcal{X}'_\mu$.

Proof. According to [**10**, Corollary, p. 92], we see $\mathcal{X}'_\mu = \mathcal{X}'_\mu \# S$, it is sufficient to prove that the linear mapping

$$
\begin{aligned}
G : \mathcal{X}_\mu &\longrightarrow S\#\mathcal{X}_\mu \subset \mathcal{X}_\mu\\
\phi &\longrightarrow S\#\phi
\end{aligned}
$$

is a homeomorphism.

Note firstly that G is a continuous mapping. In effect, by invoking [**8**, Proposition 3.4], we obtain

$$
(7) \qquad G(\phi) = S\#\phi = h_\mu(s^{-\mu-1/2}h'_\mu(S)h_\mu(\phi)), \quad \text{for every } \phi\in\mathcal{X}_\mu.
$$

Since $s^{-\mu-1/2}h'_\mu(S)$ is a continuous multiplier from $\mathcal{Q}_\mu$ into itself [**8**, Theorem 3.1], from [**8**, Theorem 2.1] it infers that G is continuous.

Moreover, from (7), we can deduce that G is one-to-one. In fact, if $\phi \in \mathcal{X}_\mu$ being $G(\phi) = 0$ then $s^{-\mu-1/2}h'_\mu(S)h_\mu(\phi) = 0$. Since $S \neq 0$, $h_\mu(\phi) = 0$ and hence $\phi = 0$.

To finish the proof, we have to prove that the mapping

$$\begin{aligned} G^{-1} : S\#\mathcal{X}_\mu &\longrightarrow \mathcal{X}_\mu \\ S\#\phi &\longrightarrow \phi \end{aligned}$$

is continuous, or equivalently, according to [**8**, Proposition 3.4 and Theorem 2.1], we have to see that the mapping

$$\begin{aligned} F : s^{-\mu-1/2}h'_\mu(S)\mathcal{Q}_\mu &\longrightarrow \mathcal{Q}_\mu \\ s^{-\mu-1/2}h'_\mu(S)\Phi &\longrightarrow \Phi \end{aligned}$$

is continuous. Let $\Phi \in \mathcal{Q}_\mu$ and define $\Psi = s^{-\mu-1/2}h'_\mu(S)\Phi$. Let $k \in \mathbf{N}$. By invoking a lemma of Hormander [**14**, Lemma 3.2], we obtain

$$\begin{aligned} |s^{-\mu-1/2}\Phi(s)| \leq &\sup_{|z-s|<4(k+r)} |z^{-\mu-1/2}h'_\mu(S)(z)z^{-\mu-1/2}\Phi(z)| \\ &\cdot \frac{\sup_{|z-s|<4(k+r)} |z^{-\mu-1/2}h'_\mu(S)(z)|}{[\sup_{|z-s|<k+r} |z^{-\mu-1/2}h'_\mu(S)(z)|]^2}, \quad s \in \mathbf{C}. \end{aligned} \tag{8}$$

Also, according to (6), one has

$$\begin{aligned} \sup_{|z-s|<k+r} |z^{-\mu-1/2}h'_\mu(S)(z)| &= \sup_{|z|<k+r} |(s+z)^{-\mu-1/2}h'_\mu(S)(s+z)| \\ &\geq \sup_{|z|<r} |(\operatorname{Re} s+z)^{-\mu-1/2}h'_\mu(S)(\operatorname{Re}\ s+z)| \\ &\geq \frac{C}{(1+|\operatorname{Re} s|^2)^N} \qquad (9) \\ &\geq \frac{C}{(1+|s|^2)^N}, \quad |\operatorname{Im} s| \leq k. \end{aligned}$$

Moreover, according to [**8**, Theorem 3.1], there exists $n \in \mathbf{N}$ such that

$$\sup_{|\operatorname{Im} z|\leq 5k+4r} (1+|z|^2)^{-n}|z^{-\mu-1/2}h'_\mu(S)(z)| < \infty.$$

Then

$$\begin{aligned} \sup_{|z-s|<4(k+r)} |z^{-\mu-1/2}h'_\mu(S)(z)| &= \sup_{|z|<4(k+r)} |(s+z)^{-\mu-1/2}h'_\mu(S)(s+z)| \\ &\leq M \sup_{|z|<4(k+r)} (1+|s+z|^2)^n \\ &\leq M(1+|s|^2)^n, \quad |\operatorname{Im} s| \leq k. \qquad (10) \end{aligned}$$

Hence, from (8), (9) and (10), we conclude that

$$
\begin{aligned}
&|s^{-\mu-1/2}\Phi(s)| \leq M(1+|s|^2)^{n+2N} \\
(11)\qquad &\qquad\cdot \sup_{|z|<4(k+r)} |(z+s)^{-\mu-1/2}\Psi(z+s)|, \quad |\mathrm{Im}\ s| \leq k.
\end{aligned}
$$

Now let $m \in \mathbf{N}$. By (11) one has

$$
\begin{aligned}
&\sup_{|\mathrm{Im}\, s|\leq k} (1+|s|^2)^m |s^{-\mu-1/2}\Phi(s)| \\
&\qquad \leq M \sup_{|\mathrm{Im}\, s|\leq k} (1+|s|^2)^{n+2N+m} \\
&\qquad\quad \cdot \sup_{|z|<4(k+r)} |(z+s)^{-\mu-1/2}\Psi(z+s)| \leq M \sup_{|\mathrm{Im}\, s|\leq k} \\
&\qquad\quad \cdot \sup_{|z|<4(k+r)} (1+|z+s|^2)^{n+2N+m} |(z+s)^{-\mu-1/2}\Psi(z+s)| \\
&\qquad \leq M \sup_{|\mathrm{Im}\, s|\leq 5k+4r} (1+|s|^2)^{n+2N+m} |s^{-\mu-1/2}\Psi(s)|.
\end{aligned}
$$

Thus we prove that F is continuous, and we conclude that G is a homeomorphism. $\square$

Remark 1. It is an open problem whether the conditions presented in Propositions 2.3 and 2.4 are equivalent. We conjecture that the answer is affirmative.

3. Hypoelliptic Hankel convolution equations. Sampson and Zielezny [**20**], Zielezny [**28**] and [**29**] and Pahk [**17**] and [**18**], amongst others, have investigated hypoelliptic (usual) convolution equations in certain spaces of generalized functions.

In this section we investigate hypoelliptic conditions for the Hankel convolution equations in $\mathcal{H}'_\mu$ and $\mathcal{X}'_\mu$.

Let $S \in \mathcal{O}'_{\mu,\#}$. We say that S (or the Hankel convolution equation $u\#S = v$) is hypoelliptic in $\mathcal{H}'_\mu$ if all solutions $u \in \mathcal{H}'_\mu$ of $u\#S = v$ are in $\mathcal{O}_{\mu,\#}$ whenever $v \in \mathcal{O}_{\mu,\#}$.

Note that, conversely, $v \in \mathcal{O}_{\mu,\#}$ provided that the equation $u\#S = v$ admits a solution $u \in \mathcal{O}_{\mu,\#}$.

Proposition 3.1. *If $f \in \mathcal{O}_{\mu,\#}$ and $S \in \mathcal{O}'_{\mu,\#}$, then $f \# S \in \mathcal{O}_{\mu,\#}$.*

Proof. A straightforward modification in the proof of [**16**, Proposition 4.2] allows us to see that, for every $m \in \mathbf{N}$, there exist $k = k(m)$ and continuous functions f_p on I, $0 \le p \le k$, such that

$$S = \sum_{p=0}^{k} S_\mu^p f_p$$

and

$$(1+x^2)^m x^{-\mu-1/2} f_p \quad \text{is bounded on} \quad I, \quad 0 \le p \le k.$$

Claim 1. *Let $l \in \mathbf{Z}$, and let f be in $\mathcal{O}_{\mu,l,\#}$. If $S \in \mathcal{O}'_{\mu,\#}$, then*

$$f \# S = \sum_{p=0}^{k} S_\mu^p (f \# f_p)$$

where $(f_p)_{p=0}^k$ is a family of continuous functions on $(0,\infty)$ such that

$$S = \sum_{p=0}^{k} S_\mu^p f_p \tag{12}$$

and $(1+x^2)^m x^{-\mu-1/2} f_p$ is bounded on I, for every $p = 0, 1, \dots, k$, and being $m > |l| + \mu + 1$.

Proof. Let $\phi \in \mathcal{H}_\mu$. By (12) we can write

$$\begin{aligned}\langle f \# S, \phi \rangle = \langle f, S \# \phi \rangle &= \int_0^\infty f(x) \sum_{p=0}^{k} \int_0^\infty f_p(y)(\tau_x S_\mu^p \phi)(y)\, dy\, dx \\ &= \sum_{p=0}^{k} \int_0^\infty (S_\mu^p \phi)(x) \int_0^\infty f_p(y)(\tau_x f)(y)\, dy\, dx.\end{aligned}$$

Since $m > |l| + \mu + 1$, it follows for every $p = 0, 1, \dots, k$,

$$\begin{aligned}
&\int_0^\infty \int_0^\infty |f_p(y)||f(x)| \int_0^\infty D_\mu(x,y,z)|S^p_{\mu,z}\phi(z)|\,dz\,dx\,dy \\
&\quad = \int_0^\infty \int_0^\infty |f_p(y)||S^p_{\mu,z}\phi(z)| \\
&\quad\quad \cdot \int_{|z-y|}^{z+y} D_\mu(x,y,z)x^{\mu+1/2}x^{-\mu-1/2}|f(x)|\,dx\,dy\,dz \\
&\quad \le M \int_0^\infty \int_0^\infty |f_p(y)||S^p_{\mu,z}\phi(z)|(1+(z+y)^2)^{|l|}(zy)^{\mu+1/2}\,dz\,dy \\
&\quad \le M \int_0^\infty y^{\mu+1/2}(1+y^2)^{|l|}|f_p(y)|\,dy \\
&\quad\quad \cdot \int_0^\infty z^{\mu+1/2}(1+z^2)^{|l|}|S^p_{\mu,z}\phi(z)|\,dz < \infty,
\end{aligned}$$

and the interchange in the order of integrations is justified.

Thus the claim is established. □

Claim 2. *Let $l \in \mathbf{Z}$. If g is a continuous function on I such that $(1+x^2)^\alpha x^{-\mu-1/2}g(x)$ is bounded on I, for some $\alpha > |l| + \mu + 1$, and $f \in \mathcal{O}_{\mu,l,\#}$ then $f\#g \in \mathcal{O}_{\mu,\#}$.*

Proof. Let $\beta \in \mathbf{N}$. Since, by proceeding as in the proof of [**1**, Lemma 3.1], we can see that the operators τ_x and S_μ commute on $\mathcal{O}_{\mu,\#}$, for each $x \in I$, we can write

$$S^\beta_\mu(f\#g)(x) = \int_0^\infty g(y)\tau_x(S^\beta_\mu f)(y)\,dy, \quad x \in I.$$

For every $x, y \in I$ one has

$$\begin{aligned}
|\tau_x(S^\beta_\mu f)(y)| &\le \int_{|x-y|}^{x+y} D_\mu(x,y,z)|(S^\beta_\mu f)(z)|\,dz \\
&\le M(1+(x+y)^2)^{|l|}(xy)^{\mu+1/2} \\
&\le M(xy)^{\mu+1/2}(1+x^2)^{|l|}(1+y^2)^{|l|}.
\end{aligned}$$

Hence we obtain that

$$
\begin{aligned}
|S_\mu^\beta(f\#g)(x)| &\le \int_0^\infty |g(y)||(\tau_x S_\mu^\beta f)(y)|\,dy \\
&\le M x^{\mu+1/2}(1+x^2)^{|l|} \\
&\quad \cdot \sup_{z\in I}(1+z^2)^l z^{-\mu-1/2}|S_\mu^\beta f(z)| \\
&\quad \cdot \int_0^\infty y^{\mu+1/2}(1+y^2)^{|l|}|g(y)|\,dy, \quad x\in I.
\end{aligned}
\tag{13}
$$

Then $f\#g \in O_{\mu,-|l|,\#}$.

Moreover, $f\#g \in \mathcal{O}_{\mu,-|l|,\#}$. In effect, if $(\phi_n)_{n=0}^\infty \subset \mathcal{H}_\mu$ and $\phi_n \to f$, as $n\to\infty$, in $O_{\mu,l,\#}$, then from (13) we can infer that $\phi_n\#g \to f\#g$, as $n\to\infty$, in $O_{\mu,-|l|,\#}$. Also, according to [**6**, Proposition 2], there exists an $s\in\mathbf{Z}$ such that $\phi_n\#g\in\mathcal{O}_{\mu,s,\#}$ for every $n\in\mathbf{N}$. Hence, since $\mathcal{O}_{\mu,\#}$ is complete, [**6**, Corollary 3], $f\#g\in\mathcal{O}_{\mu,\#}$.

Now, by taking into account that, for every $\phi\in\mathcal{H}_\mu$ and $f\in\mathcal{O}_{\mu,\#}$,

$$\int_0^\infty f(x)S_\mu\phi(x)\,dx = \int_0^\infty S_\mu f(x)\phi(x)\,dx,$$

Claims 1 and 2 allow us to conclude that $f\#S\in\mathcal{O}_{\mu,\#}$. □

We say that $S\in\mathcal{O}'_{\mu,\#}$ has the property (HE) if and only if there exist $B,C>0$ such that $|h'_\mu(S)(y)|\ge y^{-B}$ for every $y\ge C$.

We now prove that the property (HE) is a necessary and sufficient condition in order that $S\in\mathcal{O}'_{\mu,\#}$ is hypoelliptic in $\mathcal{H}'_\mu$.

The following result will allow us to prove the necessity of the condition (HE).

Proposition 3.2. *Assume that $\xi_1>1$, $\xi_j-\xi_{j-1}>1$ for every $j=2,3,\dots$, and $(a_j)_{j=1}^\infty\subset\mathbf{C}$ such that $|a_j|=O(\xi_j^\gamma)$, as $j\to\infty$, for some $\gamma>0$. Denote by δ_μ the element of $\mathcal{H}'_\mu$ defined by*

$$\langle\delta_\mu,\phi\rangle = c_\mu\lim_{x\to0^+}x^{-\mu-1/2}\phi(x),\quad \phi\in\mathcal{H}_\mu,$$

being $c_\mu=2^\mu\Gamma(\mu+1)$. Then $\sum_{j=1}^\infty a_j\tau_{\xi_j}\delta_\mu\in\mathcal{H}'_\mu$. Moreover, if $T=h'_\mu(\sum_{j=1}^\infty a_j\tau_{\xi_j}\delta_\mu)$, then $T\in\mathcal{O}_{\mu,\#}$ if and only if $|a_j|=O(\xi_j^{-\nu})$ as $j\to\infty$, for each $\nu\in\mathbf{N}$.

Proof. The series $\sum_{j=1}^{\infty} a_j \tau_{\xi_j} \delta_\mu$ converges in $\mathcal{H}'_\mu$, when we consider in $\mathcal{H}'_\mu$ the weak $*$ topology. In effect, for every $\phi \in \mathcal{H}_\mu$ and $\xi \in I$ according to [**5**, (2.1)], one has

$$\begin{aligned}\langle \tau_\xi \delta_\mu, \phi\rangle &= c_\mu \lim_{x\to 0^+} x^{-\mu-1/2}(\tau_\xi \phi)(x)\\ &= c_\mu \lim_{x\to 0^+} h_\mu[(xt)^{-\mu} J_\mu(xt) h_\mu(\phi)(t)](\xi)\\ &= \phi(\xi).\end{aligned} \tag{14}$$

Hence for each $n \in \mathbf{N}$,

$$\Big\langle \sum_{j=1}^{n} a_j \tau_{\xi_j} \delta_\mu, \phi \Big\rangle = \sum_{j=1}^{n} a_j \phi(\xi_j), \quad \phi \in \mathcal{H}_\mu,$$

and since $|a_j| = O(\xi_j^\gamma)$ as $j \to \infty$, for some $\gamma > 0$, the last sequence converges as $n \to \infty$, for every $\phi \in \mathcal{H}_\mu$. Therefore $\sum_{j=1}^{\infty} a_j \tau_{\xi_j} \delta_\mu \in \mathcal{H}'_\mu$.

Moreover, from (14) we deduce

$$\begin{aligned}\langle T, \phi\rangle &= \Big\langle \sum_{j=1}^{\infty} a_j \tau_{\xi_j} \delta_\mu, h_\mu \phi \Big\rangle\\ &= \sum_{j=1}^{\infty} a_j h_\mu(\phi)(\xi_j)\\ &= \Big\langle \sum_{j=1}^{\infty} a_j (x\xi_j)^{1/2} J_\mu(x\xi_j), \phi(x) \Big\rangle, \quad \phi \in \mathcal{H}_\mu.\end{aligned}$$

Thus it is established that $T = \sum_{j=1}^{\infty} a_j (x\xi_j)^{1/2} J_\mu(x\xi_j)$.

It is not hard to see that, if $|a_j| = O(|\xi_j|^{-\nu})$ as $j \to \infty$, for each $\nu \in \mathbf{N}$, then by invoking well-known properties of the Bessel function [**27**, Section 5.1, (6) and (7)] for every $\beta \in \mathbf{N}$ the series

$$S_\mu^\beta T(x) = \sum_{j=1}^{\infty} a_j (-\xi_j^2)^\beta (x\xi_j)^{1/2} J_\mu(x\xi_j),$$

converges uniformly in $x \in I$ and $x^{-\mu-1/2} S_\mu^\beta T$ is bounded on I. Hence, $T \in O_{\mu,\#} = \cup_{m\in\mathbf{Z}} O_{\mu,m,\#}$. Moreover, by proceeding as in the proof of [**1**, Lemma 2.1] we can conclude that $T \in \mathcal{O}_{\mu,\#}$.

Assume now that $T \in \mathcal{O}_{\mu,\#}$. Let $k \in \mathbf{N}$ and $\phi \in \mathcal{H}_\mu$. According to [**5**, (2.1)] and by (14) we can write

$$\begin{aligned}
\langle x^{-\mu-1/2}(xh)^{1/2} & J_\mu(xh) S_\mu^k T(x), \phi(x)\rangle \\
&= \langle S_\mu^k T(x), h_\mu(\tau_h h_\mu \phi)(x)\rangle \\
&= \langle (h'_\mu T)(x), (-x^2)^k \tau_h(h_\mu \phi)(x)\rangle \\
&= \sum_{j=1}^{\infty} a_j \langle \delta_\mu, \tau_{\xi_j}((-x^2)^k \tau_h(h_\mu\phi)(x))\rangle \\
&= \sum_{j=1}^{\infty} a_j (-\xi_j^2)^k \tau_{\xi_j}(h_\mu\phi)(h) \\
&= \int_0^\infty (xh)^{1/2} J_\mu(xh)(S_\mu^k T)(x) x^{-\mu-1/2}\phi(x)\, dx, \quad h \in I.
\end{aligned}$$

Since $x^{-\mu-1/2}\phi(x)(S_\mu^k T)(x)$ is absolutely integrable on I, the Riemann-Lebesgue lemma for the Hankel transform [**24**, Section 14.41], leads to

$$\sum_{j=1}^{\infty} a_j(-\xi_j^2)^k \tau_{\xi_j}(h_\mu \phi)(h) \longrightarrow 0, \quad \text{as } h \longrightarrow \infty. \tag{15}$$

We choose a function $\phi \in \mathcal{H}_\mu$ such that $\phi \not\equiv 0$, $h_\mu(\phi)(x) = 0$ for every $x \geq 1$, and $h_\mu(\phi) \geq 0$. It is not hard to see that such a function ϕ can be found.

Then, if $x, y \in I$ and $x - y > 1$, we have

$$\begin{aligned}
\tau_x(h_\mu\phi)(y) &= \int_{x-y}^{x+y} (h_\mu\phi)(z) D_\mu(x,y,z)\, dz \\
&= \int_1^\infty (h_\mu\phi)(z) D_\mu(x,y,z)\, dz = 0.
\end{aligned} \tag{16}$$

Moreover, if $x \geq 1/2$ from (3) [**24**, Section 13.45] it infers

$$\begin{aligned}\tau_x(h_\mu\phi)(x) &= \int_0^{2x} (h_\mu\phi)(z) D_\mu(x,x,z)\,dz \\ &= \frac{x^{1-2\mu}}{2^{3\mu-1}\Gamma(\mu+1/2)\sqrt{\pi}} \\ &\quad \cdot \int_0^{2x} z^{\mu-1/2}(4x^2-z^2)^{\mu-1/2}(h_\mu\phi)(z)\,dz \\ &= \frac{1}{2^{3\mu-1}\Gamma(\mu+1/2)\sqrt{\pi}} \\ &\quad \cdot \int_0^1 z^{\mu-1/2}\left(4-\left(\frac{z}{x}\right)^2\right)^{\mu-1/2}(h_\mu\phi)(z)\,dz.\end{aligned}$$

Hence,

$$\tau_x(h_\mu\phi)(x) \longrightarrow \frac{2^{-\mu}}{\Gamma(\mu+1/2)\sqrt{\pi}}\int_0^1 z^{\mu-1/2}(h_\mu\phi)(z)\,dz, \qquad \text{as } x \longrightarrow \infty. \tag{17}$$

Note that $\int_0^1 z^{\mu-1/2}(h_\mu\phi)(z)\,dz \in I$.

By virtue of (16), for every $l \in \mathbf{N}$,

$$\sum_{j=1}^{\infty} a_j(-1)^k \xi_j^{2k}\tau_{\xi_j}(h_\mu\phi)(\xi_l) = a_l(-1)^k\xi_l^{2k}\tau_{\xi_l}(h_\mu\phi)(\xi_l).$$

Therefore, (15) and (17) imply that $a_l\xi_l^{2k} \to 0$ as $l \to \infty$, and the proof is complete. □

In the following we establish that (HE) is necessary and sufficient in order that $S \in \mathcal{O}'_{\mu,\#}$ be hypoelliptic in $\mathcal{H}'_\mu$.

Proposition 3.3. *Let $S \in \mathcal{O}'_{\mu,\#}$. Then S is hypoelliptic in $\mathcal{H}'_\mu$ if and only if S satisfies* (HE).

Proof. Assume firstly that S does not verify (HE). Then, for every $j \in \mathbf{N}$ there exists $\xi_j \in I$ for which

$$\xi_j^{-\mu-1/2}|h'_\mu(S)(\xi_j)| \leq \xi_j^{-j}$$

and $\xi_j - \xi_{j-1} > 1$, $j = 2, 3, \dots$ and $\xi_1 > 1$.

We now consider $u \in \mathcal{H}'_\mu$ such that $h'_\mu(u) = \sum_{j=1}^\infty \tau_{\xi_j}\delta_\mu$. According to Proposition 3.2, $u \notin \mathcal{O}_{\mu,\#}$. Moreover, by invoking [**16**, Proposition 4.5]

$$h'_\mu(u\#S) = x^{-\mu-1/2}h'_\mu(u)h'_\mu(S) = \sum_{j=1}^\infty \xi_j^{-\mu-1/2}h'_\mu(S)(\xi_j)\tau_{\xi_j}\delta_\mu,$$

and Proposition 3.2 implies that $u\#S \in \mathcal{O}_{\mu,\#}$. Hence S is not hypoelliptic in $\mathcal{H}'_\mu$.

Consider now that S satisfies (HE), and let ϕ be a smooth function defined on I such that

$$\phi(x) = \begin{cases} x^{\mu+1/2} & \text{for } 0 < x < C, \\ 0 & \text{for } x \geq C+1, \end{cases}$$

where C is the positive constant that appears in property (HE). Note that $\phi \in \mathcal{H}_\mu$.

Also we define

$$P(x) = \begin{cases} 0 & \text{for } 0 < x \leq C, \\ (x^{\mu+1/2} - \phi(x))/(x^{-\mu-1/2}h'_\mu(S)(x)) & \text{for } x > C. \end{cases}$$

According to [**16**, Proposition 4.2], $x^{-\mu-1/2}h'_\mu(S)(x)$ is a multiplier of $\mathcal{H}_\mu$. Hence, since S satisfies (HE), P is smooth on I. Moreover, $x^{-\mu-1/2}P$ is a multiplier of $\mathcal{H}_\mu$. In effect, according to [**2**, Theorem 2.3], for every $k \in \mathbf{N}$ there exists an $n_k \in \mathbf{N}$ such that

$$(1+x^2)^{-n_k}\left(\frac{1}{x}D\right)^k [x^{-\mu-1/2}h'_\mu(S)(x)]$$

is bounded on I. Hence, since S verifies (HE) by virtue of Theorem 2.3 [**2**], $x^{-\mu-1/2}P$ is a multiplier of $\mathcal{H}_\mu$.

We have that

$$x^{-\mu-1/2}P(x)h'_\mu(S)(x) = x^{\mu+1/2} - \phi(x), \quad x \in I. \tag{18}$$

By applying the Hankel transformation to (18), it obtains

$$Q\#S = \delta_\mu - \psi$$

where $Q = h'_\mu(P) \in \mathcal{O}'_{\mu,\#}$, [**16**, Proposition 4.2], and $\psi = h_\mu(\phi) \in \mathcal{H}_\mu$, [**25**, Lemma 8].

Suppose now that $u\#S = v$ where $u \in \mathcal{H}'_\mu$ and $v \in \mathcal{O}_{\mu,\#}$. Then, according to [**16**, Proposition 4.7], we can write

$$u = u\#\delta_\mu = u\#(Q\#S) + u\#\psi = (u\#S)\#Q + u\#\psi = v\#Q + u\#\psi.$$

Proposition 3.1 implies that $v\#Q \in \mathcal{O}_{\mu,\#}$ and [**6**, Proposition 2] leads to $u\#\psi \in \mathcal{O}_{\mu,\#}$. Thus, the hypoellipticity of S is proved. □

Remark 2. Note that, by proceeding as in the proof of Proposition 3.3, we can also prove that if $S \in \mathcal{O}'_{\mu,\#}$ and there exist $Q \in \mathcal{O}'_{\mu,\#}$ and $R \in \mathcal{H}_\mu$ such that

$$Q\#S = \delta_\mu - R$$

then S is hypoelliptic in $\mathcal{H}'_\mu$.

In [**8**] we introduced for every $m \in \mathbf{Z}$ the space $X_{\mu,m,\#}$ that is formed by all those complex valued and smooth functions ϕ defined on I such that for every $k \in \mathbf{N}$,

$$\beta_k^{\mu,m}(\phi) = \sup_{x \in I} |e^{mx} x^{-\mu-1/2} S_\mu^k \phi(x)| < \infty.$$

It is clear that $X_{\mu,m+1,\#}$ is contained in $X_{\mu,m,\#}$. By $\mathcal{X}_{\mu,m,\#}$ we denote the closure of $\mathcal{X}_\mu$ into $X_{\mu,m,\#}$. The space $\mathcal{X}_{\mu,\#} = \cup_{m\in\mathbf{Z}} \mathcal{X}_{\mu,m,\#}$ is endowed with the inductive topology.

Let $S \in \mathcal{X}'_{\mu,\#}$. We say that S (or the Hankel convolution equation $u\#S = v$) is hypoelliptic in $\mathcal{X}'_\mu$ when $v \in \mathcal{X}_{\mu,\#}$ implies that any solution $u \in \mathcal{X}'_\mu$ of $u\#S = v$ belongs to $\mathcal{X}_{\mu,\#}$.

The following property is analogous to the one presented in Proposition 3.1.

Proposition 3.4. *If $f \in \mathcal{X}_{\mu,\#}$ and $S \in \mathcal{X}'_{\mu,\#}$, then $f\#S \in \mathcal{X}_{\mu,\#}$.*

Proof. This result can be proved in a similar way to Proposition 3.1. □

After establishing the following proposition (similar to Proposition 3.2) we will prove that (HE) is also a necessary condition for the hypoellipticity of S in $\mathcal{X}'_\mu$.

Proposition 3.5. *Let $\mu \geq 1/2$. Assume that $\xi_j > 2\xi_{j-1}$, $j = 2, 3, \dots$, and $\xi_1 > 1$. Let $(a_j)_{j=1}^\infty$ be a complex sequence such that $|a_j| = O(\xi_j^\gamma)$ as $j \to \infty$, for some $\gamma > 0$. Then $\sum_{j=1}^\infty a_j \tau_{\xi_j} \delta_\mu \in \mathcal{Q}'_\mu$. Moreover, if $T = h'_\mu(\sum_{j=1}^\infty a_j \tau_{\xi_j} \delta_\mu)$, then $T \in \mathcal{X}_{\mu,\#}$ if and only if $|a_j| = O(\xi_j^{-\nu})$ as $j \to \infty$, for every $\nu \in \mathbf{N}$.*

Proof. Since $\mathcal{Q}_\mu \subset \mathcal{H}_\mu$, [**8**, Corollary 2.1] from Proposition 3.2 it is inferred that the series $\sum_{j=1}^\infty a_j \tau_{\xi_j} \delta_\mu$ converges in $\mathcal{Q}'_\mu$ when we consider in $\mathcal{Q}'_\mu$ the weak $*$ topology. Then, by [**8**, Theorem 2.1]

$$T = \sum_{j=1}^\infty a_j (x\xi_j)^{1/2} J_\mu(x\xi_j) \in \mathcal{X}'_\mu.$$

Moreover, if $|a_j| = O(\xi_j^{-\nu})$, as $j \to \infty$, for each $\nu \in \mathbf{N}$, then it is easy to see that if $T \in \mathcal{X}_{\mu,\#}$. Suppose now that $T \in \mathcal{X}_{\mu,\#}$. Let $k \in \mathbf{N}$ and $\phi \in \mathcal{X}_\mu$. We have

$$\begin{aligned} &\sum_{j=1}^\infty a_j (-\xi_j^2)^k \tau_{\xi_j}(h_\mu \phi)(h) \\ &\qquad = \int_0^\infty (xh)^{1/2} J_\mu(xh)(S_\mu^k T)(x) x^{-\mu-1/2} \phi(x)\, dx \longrightarrow 0, \\ &\qquad\qquad \text{as } h \longrightarrow \infty. \end{aligned} \tag{19}$$

Define $\phi(x) = e^{-x^2} x^{\mu+1/2}$, $x \in I$. According to (10) [**11**, Section 8.6],

$$h_\mu(\phi)(y) = \frac{y^{\mu+1/2}}{2^{\mu+1}} e^{-y^2/4}, \quad y \in I.$$

Hence, since $h_\mu(\phi) \in \mathcal{X}_\mu$, $\phi \in \mathcal{Q}_\mu$, [**8**, Theorem 2.1]. Note that $h_\mu(\phi)(y) y^{-\mu-1/2} > 0$ for every $y \in I$.

Let $m \in \mathbf{N}$. We can write

$$\tau_x(h_\mu\phi)(y) = \int_{|x-y|}^{x+y} D_\mu(x,y,z)h_\mu(\phi)(z)\,dz \tag{20}$$
$$\leq M(xy)^{\mu+1/2}(1+|x-y|^2)^{-m}, \quad x,y \in I.$$

Moreover, for each $x \in I$,

$$\begin{aligned}\tau_x(h_\mu\phi)(x) &= \int_0^{2x} D_\mu(x,x,z)h_\mu(\phi)(z)\,dz \\ &= \frac{x^{1-2\mu}}{2^{3\mu-1}\Gamma(\mu+1/2)\sqrt{\pi}} \\ &\quad \cdot \int_0^{2x} z^{\mu-1/2}((2x)^2-z^2)^{\mu-1/2}h_\mu(\phi)(z)\,dz \\ &= \frac{2^{-\mu}}{\Gamma(\mu+1/2)\sqrt{\pi}} \\ &\quad \cdot \int_0^{2x} z^{\mu-1/2}\left(1-\left(\frac{z}{2x}\right)^2\right)^{\mu-1/2} h_\mu(\phi)(z)\,dz.\end{aligned}$$

Hence

$$\tau_x(h_\mu\phi)(x) \longrightarrow \frac{2^{-\mu}}{\Gamma(\mu+1/2)\sqrt{\pi}}\int_0^\infty z^{\mu-1/2}(h_\mu\phi)(z)\,dz, \tag{21}$$
$$\text{as } x \longrightarrow \infty.$$

Let l and $k \in \mathbf{N}$. From (20) we deduce

$$\begin{aligned}&\left|\sum_{j=1}^\infty a_j(-1)^k\xi_j^{2k}(\tau_{\xi_j}h_\mu\phi)(\xi_l)\right| \\ &\qquad \geq |a_l|\xi_l^{2k}(\tau_{\xi_l}h_\mu\phi)(\xi_l) - \sum_{\substack{j=1\\ j\neq l}}^\infty |a_j|\xi_j^{2k}(\tau_{\xi_j}h_\mu\phi)(\xi_l) \\ &\qquad \geq |a_l|\xi_l^{2k}(\tau_{\xi_l}h_\mu\phi)(\xi_l) \\ &\qquad\quad - M\xi_l^{\mu+1/2}\sum_{\substack{j=1\\ j\neq l}}^\infty |a_j|\xi_j^{2k+\mu+1/2}(1+|\xi_j-\xi_l|^2)^{-m}\end{aligned} \tag{22}$$

$$\geq |a_l|\xi_l^{2k}(\tau_{\xi_l}h_\mu\phi)(\xi_l) \\ - M\xi_l^{\mu+1/2}\sum_{\substack{j=1\\j\neq l}}^{\infty}|a_j|\xi_j^{2k+\mu+1/2}|\xi_j-\xi_l|^{-m}.$$

Since $|a_j| = O(\xi_j^\gamma)$, as $j \to \infty$, with $\gamma > 0$, one has

$$\sum_{\substack{j=1\\j\neq l}}^{\infty}|a_j|\xi_j^{2k+\mu+1/2}|\xi_j-\xi_l|^{-m} \leq M\sum_{\substack{j=1\\j\neq l}}^{\infty}\xi_j^{2k+\gamma+\mu+1/2}|\xi_j-\xi_l|^{-m}. \tag{23}$$

By taking into account that

$$\xi_j - \xi_{j-1} \geq 2\xi_{j-1} - \xi_{j-1} = \xi_{j-1} \geq 2^{j-1}, \quad j = 2, 3, \dots,$$

we can obtain

$$|\xi_j - \xi_l| \geq 2^{l-1}, \quad \text{for each } j \in \mathbf{N} - \{l\}.$$

Hence, by choosing $m \in \mathbf{N}$ such that $m \geq 2(2k+\gamma+\mu+3/2)$ it follows

$$\begin{aligned}\sum_{\substack{j=1\\j\neq l}}^{\infty}\xi_j^{2k+\gamma+\mu+1/2}|\xi_j-\xi_l|^{-m} &\leq \sum_{\substack{j=1\\j\neq l}}^{\infty}|\xi_j-\xi_l|^{-1} \\ &\quad \cdot \left|1-\frac{\xi_l}{\xi_j}\right|^{-(2k+\gamma+\mu+1/2)}|\xi_j-\xi_l|^{-(2k+\gamma+\mu+3/2)} \\ &\leq M2^{-l}.\end{aligned} \tag{24}$$

By combining (22), (23) and (24), we conclude

$$\begin{aligned}&\left|\sum_{\substack{j=1\\j\neq l}}^{\infty}a_j(-1)^k\xi_j^{2k}(\tau_{\xi_j}h_\mu\phi)(\xi_l)\right| \\ &\quad \geq \xi_l^{\mu+1/2}(|a_l|\xi_l^{2k-\mu-1/2}\tau_{\xi_l}(h_\mu\phi)(\xi_l) - M2^{-l}) \longrightarrow 0, \\ &\qquad\qquad l \longrightarrow \infty.\end{aligned} \tag{25}$$

Hence, from (19), (21) and (25), we deduce that $|a_l|\xi_l^{2k-\mu-1/2} \to 0$, as $l \to \infty$. Thus the desired result is established. $\square$

The following proposition can be proved as Proposition 3.3.

Proposition 3.6. *Let $\mu \geq 1/2$ and $S \in \mathcal{X}'_{\mu,\#}$. If S is hypoelliptic in $\mathcal{X}'_\mu$, then S satisfies the property* (HE).

Remark 3. Finally we want to remark that, as in $\mathcal{H}'_\mu$, if $S \in \mathcal{X}'_{\mu,\#}$ and there exist $Q \in \mathcal{X}'_{\mu,\#}$ and $R \in \mathcal{X}_\mu$ such that

$$Q\#S = \delta_\mu - R, \tag{26}$$

then S is hypoelliptic in $\mathcal{X}'_\mu$. However, we do not know how to define $Q \in \mathcal{X}'_{\mu,\#}$ and $R \in \mathcal{X}_\mu$ satisfying (26) when S verifies (HE). We think that the condition (HE) must be replaced by other analogous but stronger conditions than (HE) involving complex values. This is still an open problem.

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