A FURTHER APPLICATION OF
THE deLEEUW-GLICKSTEIN THEOREM

W.J. ANDERSON AND N. WARD-ANDERSON

ABSTRACT. The theorem of Ionescu Tulcea and Marinescu, a generalization of the ergodic theorem of Doeblin and Fortet, is derived as a result of the deLeeuw-Glicksberg decomposition. This complements recent work of Sine which derives asymptotic periodicity results for constricted Markov operators from the deLeeuw-Glickstein decomposition. A weak form of the Ionescu Tulcea and Marinescu theorem is obtained from the deLeeuw-Glickstein decomposition.

1. Introduction. A discrete time parameter Markov chain with stationary transition probabilities and state space E is specified by a transition function $P(x, A)$ on $E \times \mathcal{E}$, where $\mathcal{E}$ is a σ-algebra of subsets of E; or equivalently, the Markov operator P associated with $P(x, A)$ on some suitable function space X. A large portion of the theory of Markov chains deals with the asymptotic behavior of the iterates P^n of P as $n \to \infty$. Over the years, the accumulated evidence has been that, under certain not unrealistic general conditions, the iterates $P^n f$ approach a finite dimensional space which is either independent of f or, if not, dependent in an obvious way; we will refer to this as *asymptotic periodicity*. Asymptotic periodicity has been demonstrated in one form or another for distance-diminishing operators [3], quasicompact operators [2 and others since], uniform mean stable operators [16, 8, 17], and lately constricted operators [12, 9, 10].

A technique which is emerging as a useful tool is the deLeeuw-Glicksberg decomposition [1], which will be described now for future reference. Let X be a real or complex Banach space, let X' be the dual space, the continuous linear functionals on X, and let $[X]$ denote the set of all bounded linear operators $T : X \to X$. In addition to the strong, i.e., norm, topology of X, we shall also be interested in the weak topology in X. When X is considered with its strong topology, we shall consider $[X]$ with the strong operator topology; when X is considered with its weak topology, we shall take $[X]$ to be equipped with the weak

operator topology. With this convention, it will be unnecessary in this section to specify whether we are dealing with the weak or strong case. Let $\mathcal{S}$ be a semigroup in $[X]$. $\mathcal{S}$ is said to be *almost periodic* if, for every $x \in X$, the *orbit* $\mathcal{S}x = \{Tx \mid T \in \mathcal{S}\}$ of x is relatively compact in X. It can then be shown that $\overline{\mathcal{S}}$ is a compact semigroup in $[X]$ and so contains a closed nonempty ideal K. K is a group, and if U denotes the unit of K, then $K = U\overline{\mathcal{S}}$. The deLeeuw-Glickstein decomposition is then derived from the projection U.

Theorem 1.1 (deLeeuw-Glicksberg). *Let $\mathcal{S}$ be an Abelian almost periodic semigroup in $[X]$, where X is a Banach space. Then X can be written as the direct sum*

$$X = X_{\mathrm{fl}} \oplus X_{\mathrm{rev}},$$

where
(1.1)
$$X_{\mathrm{fl}} = \{x \in X \mid 0 \in \overline{\mathcal{S}x}\} \quad and \quad X_{\mathrm{rev}} = \{x \in X \mid y \in \overline{\mathcal{S}x} \Rightarrow x \in \overline{\mathcal{S}y}\}.$$

Members of X_{fl} are called *flight vectors*, and those of X_{rev} *reversible*. We shall also require the following properties in the rest of the paper: if $T \in \overline{\mathcal{S}}$, then

(1) $TX_{\mathrm{fl}} \subset X_{\mathrm{fl}}$ and $TX_{\mathrm{rev}} \subset X_{\mathrm{rev}}$.

(2) The map $T : X_{\mathrm{rev}} \to X_{\mathrm{rev}}$ is surjective.

(3) There is a $V \in K$ such that $T = V$ on X_{rev}. Hence the restriction of $\overline{\mathcal{S}}$ to X_{rev} is a group.

Our interest from now on will focus on the case where $\mathcal{S}$ is the cyclic semigroup $\mathcal{S} = \{I, T, T^2, \dots\}$ generated by a member T of $[X]$. In that case, we say T is *almost periodic* if $\mathcal{S}$ is. The following remarks will be useful in the next section.

(4) Suppose that $\sup_{n \geq 0} \|T^n\| = J < \infty$. Then every $S \in \overline{\mathcal{S}}$, including U, has $\|S\| \leq J$.

(5) If T is almost periodic in the strong sense, then $\|T^n x\| \to 0$ as $n \to \infty$ for every $x \in X_{\mathrm{fl}}$.

When X is a complex Banach space, one last important remark is in order. $x \in X$ is said to be a *unimodular* eigenvector of T if $x \neq 0$ and if there is a $\lambda \in \mathbf{C}$ with $|\lambda| = 1$ such that $Tx = \lambda x$. Then

(6) X_{rev} in the deLeeuw-Glicksberg decomposition is the closure of the subspace spanned by all unimodular eigenvectors of T.

The proof of this last remark, as well as an elegant development of deLeeuw-Glickstein theory, can be found in [**11**, pp. 103–112].

Rosenblatt [**14, 15**], followed by Jamison and Sine [**7**] showed how the deLeeuw-Glicksberg decomposition could be used to obtain asymptotic periodicity for an almost periodic Markov operator. Sine [**18**], after remarking that "This deLeeuw-Glicksberg decomposition is one of the most useful results of algebraic analysis," showed how the asymptotic periodicity results for strongly constricted operators [**12**] and weakly constricted operators [**9**] can be derived from the deLeeuw-Glicksberg decomposition. The purpose of this paper is to show in Section 2 that the theorem of Ionescu Tulcea and Marinescu [**5**], which is a Banach space generalization of the Doeblin-Fortet theorem, can also be derived from the strong form of the deLeeuw-Glicksberg decomposition. This places the deLeeuw-Glickstein decomposition in a central role in the asymptotic periodicity of Markov chains, very much like the role played by the renewal theorem in classical Markov chain theory. Finally, in Section 3, we show how the weak form of the deLeeuw-Glicksberg decomposition can be used to derive a weak form of the theorem of Ionescu Tulcea and Marinescu.

2. The theorem of Ionescu Tulcea and Marinescu. Let X and Y be complex Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, and assume that $X \subset Y$. Note that we can also consider X as a normed linear space with the norm $\| \cdot \|_Y$ inherited from Y. We shall assume the following compatibility condition is satisfied:

Compatibility condition. If $\{x_n, n \geq 1\} \subset X$, $\sup_{n \geq 1} \|x_n\|_X = c < \infty$, and $\lim_{n \to \infty} \|x_n - x\|_Y = 0$ for some $x \in Y$, then $x \in X$ and $\|x\|_X \leq c$.

Let $L_k(X, Y)$ denote the set of all linear operators $T : X \to X$ such that

(1) T is bounded with respect to both the norms $\| \cdot \|_X$ and $\| \cdot \|_Y$.

(2) $\sup_{n \geq 0} \|T^n\|_Y = H < \infty$.

(3) There are constants $r < 1$ and $R < \infty$ such that $\|T^k x\|_X \leq$

$r\|x\|_X + R\|x\|_Y$ for all $x \in X$.

Proposition 2.1. *Suppose that $T \in L_k(X, Y)$. Then*

(1) $\|T^{km}x\|_X \leq r^m\|x\|_X + R'\|x\|_Y$, $m \geq 1$, $x \in X$, *where $R' = RH/(1-r)$.*

(2) $\sup_{n \geq 0} \|T^n\|_X = J < \infty$.

Proof. (1) It is an easy matter to show inductively that

$$\|T^{km}x\|_X \leq r^m\|x\|_X + R \sum_{i=0}^{m-1} r^i\|T^{(m-1-i)k}x\|_Y.$$

The required inequality then follows from the fact that

$$\sum_{i=0}^{m-1} r^i\|T^{(m-1-i)k}x\|_Y \leq \sum_{i=0}^{\infty} r^i H\|x\|_Y = H\|x\|_Y/(1-r).$$

(2) Part (1), together with the principle of uniform boundedness, implies that $d \overset{\text{def}}{=} \sup_{m \geq 0} \|T^{km}\|_X < \infty$. Writing $n = km + j$, we have $\|T^n\|_X = \|T^{km}T^j\|_X \leq d\|T\|_X^j$, and so $\sup_{n \geq 0} \|T^n\|_X \leq d \sup_{0 \leq j \leq k-1}\{\|T\|_X^j\}$. $\quad\square$

Lemma 2.2. *Suppose that $k = 1$ and that TA has compact closure in $(Y, \|\cdot\|_Y)$ whenever A is a bounded subset of $(X, \|\cdot\|_X)$. Let $x \in X$, $y \in Y$, and suppose that $\{n_i, i \geq 1\}$ is such that $\|T^{n_i}x - y\|_Y \to 0$ as $i \to \infty$. Then $y \in X$ and $\|T^{n_i}x - y\|_X \to 0$ as $i \to \infty$.*

Proof. The compatibility condition implies that $y \in X$. Suppose that $T^{n_i}x$ does not converge to y; then there is a subsequence $\{n_{i_j}, j \geq 1\}$ of $\{n_i, i \geq 1\}$ and an $\varepsilon > 0$ such that $\|T^{n_{i_j}}x - y\|_X \geq \varepsilon$ for all $j \geq 1$. Let $m \geq 1$ be such that $2r^m\|x\|_X J < \varepsilon/2$, and let j^* be such that $n_{i_j} > m$ for all $j \geq j^*$. Because of part (2) of Proposition 2.1, the set $A = \{T^{n_{i_j}-m-1}x \mid j \geq j^*\}$ is bounded in $\|\cdot\|_X$ and so TA is relatively compact in $(Y, \|\cdot\|_Y)$. Hence there is a subsequence $\{n_{i_{j_k}}, k \geq 1\}$ and a $w \in Y$ such that $\|T^{n_{i_{j_k}}-m}x - w\|_Y \to 0$ as $k \to \infty$. By the compatibility

condition, we have $w \in X$. Also, we have $T^{n_{i_{j_k}}} x \xrightarrow{s} T^m w$ in $(Y, \|\cdot\|_Y)$ as $k \to \infty$, and so $T^m w = y$. By part (1) of Proposition 2.1,

$$
\begin{aligned}
\|T^{n_{i_{j_k}}} x - y\|_X &= \|T^m (T^{n_{i_{j_k}} - m} x - w)\|_X \\
&\leq r^m \|T^{n_{i_{j_k}} - m} x - w\|_X + R' \|T^{n_{i_{j_k}} - m} x - w\|_Y \\
&\leq 2r^m \|x\|_X J + R' \|T^{n_{i_{j_k}} - m} x - w\|_Y \\
&\leq \varepsilon/2 + R' \|T^{n_{i_{j_k}} - m} x - w\|_Y,
\end{aligned}
$$

and letting $k \to \infty$ gives

$$
\limsup_{k \to \infty} \|T^{n_{i_{j_k}}} x - y\|_X \leq \varepsilon/2,
$$

a contradiction.   □

Theorem 2.3 (Ionescu Tulcea and Marinescu [5]).   *Suppose there is an integer $k \geq 1$ such that $T \in L_k(X, Y)$ and $T^k A$ has compact closure in $(Y, \|\cdot\|_Y)$ whenever A is a bounded subset of $(X, \|\cdot\|_X)$. Then*

(1) the set G of eigenvalues of T of modulus 1 is finite, and if $\lambda \in G$, the eigenspace $D(\lambda) = \{x \in X \mid Tx = \lambda x\}$ is finite dimensional.

(2) for each $n \geq 1$, we have the representation

$$
(2.1) \qquad\qquad T^n = \sum_{\lambda \in G} \lambda^n T_\lambda + V^n,
$$

where $T_\lambda : X \to X$, $\lambda \in G$ are compact operators such that

(a) $\|T_\lambda\|_X < \infty$ for each $\lambda \in G$,

(b) $T_\lambda^2 = T_\lambda$ for all $\lambda \in G$ and $T_\lambda T_{\lambda'} = 0$ if $\lambda, \lambda' \in G$ with $\lambda \neq \lambda'$, and where the operator V satisfies $\|V\|_X < \infty$ and $V T_\lambda = T_\lambda V = 0$ for all $\lambda \in G$.

(3) $\|T^n - \sum_{\lambda \in G} \lambda^n T_\lambda\|_X \to 0$ as $n \to \infty$.

(4) $T : (X, \|\cdot\|_X) \to (X, \|\cdot\|_X)$ is quasi-compact.

Note.   An operator $T : X \to X$ on a Banach space X is *quasi-compact* if there is a sequence $\{T_n, n \geq 1\}$ of compact operators on X

such that $\|T^n - T_n\| \to 0$ as $n \to \infty$. It can be shown [**13**, p. 180] that T is quasi-compact if and only if there is an $n_0 \geq 1$ and a compact operator T' such that $\|T^{n_0} - T'\| < 1$.

Proof. First we will show that T is almost periodic as an operator on X, so that the deLeeuw-Glickstein decomposition applies. Let us assume first that $k = 1$. Let $x \in X$, and consider the sequence $A = \{T^n x, n \geq 0\}$. A is bounded in $(X, \|\cdot\|_X)$, and so TA is relatively compact in $(Y, \|\cdot\|_Y)$. Hence there is an $\{n_i, i \geq 1\}$ and a $y \in Y$ such that $\|T^{n_i} x - y\|_Y \to 0$ as $i \to \infty$. By the lemma, we have $y \in X$ and $\|T^{n_i} x - y\|_X \to 0$ as $i \to \infty$. Hence $\{T^n x, n \geq 0\}$ is relatively compact, so T is almost periodic. If $k > 1$, the above proof shows that T^k is almost periodic, which obviously implies that T is almost periodic.

We therefore have a deLeeuw-Glickstein decomposition $X = X_{\mathrm{fl}} \oplus X_{\mathrm{rev}}$ of X. Let U denote the deLeeuw-Glickstein projection. Note that $\|U\|_X \leq J$ by Remark 4 of Section 1.

(1) Because of Remark 6, it will suffice to show that X_{rev} is finite dimensional, and for this, that the unit ball in X_{rev} is compact. By Remarks 3 and 4 of Section 1, the restriction of $\overline{\mathcal{S}}$ to X_{rev} is a group whose members are bounded in norm by J. It follows that

$$(2.2) \qquad \|T^{-n} x\|_X \leq J \|x\|_X, \quad n \geq 0, \quad x \in X_{\mathrm{rev}},$$

and from the inequality in part (1) of Proposition 2.1, we get

$$\|x\|_X = \|T^{-km} T^{km} x\|_X \leq J \|T^{km} x\|_X \leq J r^m \|x\|_X + J R' \|x\|_Y,$$
$$m \geq 1, \quad x \in X_{\mathrm{rev}}.$$

Letting $m \to \infty$ shows that

$$(2.3) \qquad \|x\|_X \leq J R' \|x\|_Y, \quad x \in X_{\mathrm{rev}}.$$

Combining (2.2) and (2.3), we have

$$(2.4) \quad \|x\|_X = \|T^{-k} T^k x\|_X \leq J \|T^k x\|_X \leq J^2 R' \|T^k x\|_Y, \quad x \in X_{\mathrm{rev}}.$$

Now suppose that $\{x_n, n \geq 1\} \subset B(X_{\mathrm{rev}})_X$ (the closed unit ball in X_{rev} in the $\|\cdot\|_X$-norm). By the compactness condition, there is a subsequence $\{n_i, i \geq 1\}$ such that $\{T^k x_{n_i}, i \geq 1\}$ converges in $\|\cdot\|_Y$.

$\{T^k x_{n_i}, i \geq 1\}$ is therefore a Cauchy sequence in $\|\cdot\|_Y$. From (2.4), we have $\|x_{n_i} - x_{n_j}\|_X \leq J^2 R' \|T^k x_{n_i} - T^k x_{n_j}\|_Y \to 0$ as $i, j \to \infty$. Hence $\{x_{n_i}, i \geq 1\}$ is a Cauchy sequence in the norm $\|\cdot\|_X$ and, since $(X, \|\cdot\|_X)$ is complete and X_{rev} is closed in it, converges to a point in X_{rev}.

(2) From part (1), X_{rev} is finite-dimensional. Since T maps X_{rev} into itself, the restriction T_{rev} of T to X_{rev} is well defined, and it is easy to verify that $\sigma(T_{\mathrm{rev}})$, the set of eigenvalues of T_{rev}, coincides with G. Hence, we can use Proposition 4.1 of the Appendix to obtain the spectral representation

$$T^n_{\mathrm{rev}} = \sum_{\lambda \in G} \lambda^n E(\lambda), \quad n \geq 0,$$

of T_{rev}, where the projection operators $E(\lambda)$, $\lambda \in G$, have the properties specified there. By multiplying both sides from the right by U, using the fact that $T_{\mathrm{rev}} U = T U$ (and therefore $T^n_{\mathrm{rev}} U = T^n U$), and setting $T_\lambda = E(\lambda) U$ for $\lambda \in G$, we get

$$T^n U = \sum_{\lambda \in G} \lambda^n T_\lambda, \quad n \geq 0.$$

Each T_λ is a compact operator since it has a finite dimensional range. Moreover, we have

(a) $\|T_\lambda\|_X \leq \|E(\lambda)\|_X \|U\|_X < \infty$,

(b) $T_\lambda^2 = E(\lambda) U E(\lambda) U = E(\lambda)^2 U = E(\lambda) U = T_\lambda$ and $T_\lambda T_{\lambda'} = E(\lambda) U E(\lambda') U = E(\lambda) E(\lambda') U = 0$ if $\lambda \neq \lambda'$ (since $U E(\lambda) = E(\lambda)$ on X_{rev}),

as required. Define

$$V = T - TU = T - \sum_{\lambda \in G} \lambda T_\lambda,$$

so that (2.1) holds for $n = 1$. Then

$$VT_\lambda = TT_\lambda - \sum_{\lambda' \in G} \lambda' T_{\lambda'} T_\lambda = TT_\lambda - \lambda T_\lambda^2$$
$$= TT_\lambda - \lambda T_\lambda = (T - \lambda I) T_\lambda$$
$$= (T - \lambda I) E(\lambda) U = 0,$$

and similarly $T_\lambda V = 0$. Next, since U is a projection, then $(I - U)^n = I - U$ for any $n \geq 0$; since U and T commute, then $V^n = [T(I-U)]^n = T^n(I-U)^n = T^n(I-U) = T^n - T^n U$ for all $n \geq 0$, thus verifying (2.1) for all $n \geq 1$.

(3) First note that, since $V^k = T^k(I - U)$, then V satisfies the same compactness condition that T does. Also, from Remark (5), we have $\|V^n x\|_X \to 0$ as $n \to \infty$ for all $x \in X$. We will need these facts in the argument to follow.

We are to prove that $\|V^n\|_X \to 0$. It suffices to show that there is an $n \geq 1$ such that $\|V^{kn}\|_X < 1$. Suppose there is no such n. Then for every n, we can find $x_n \in X$ such that $\|x_n\|_X = 1$ and $\|V^{kn} x_n\|_X \geq \varepsilon > 0$. By the compactness condition, there is an $x \in Y$ and a subsequence $\{n_i, i \geq 1\}$ such that $\|V^k x_{n_i} - x\|_Y \to 0$ as $i \to \infty$. By the compatibility condition, we have $x \in X$ also. Then

$$
\begin{aligned}
\|V^{km} x_{n_i} - V^{k(m-1)} x\|_X &= \|V^{k(m-1)}(V^k x_{n_i} - x)\|_X \\
&= \|T^{k(m-1)}[I - U](V^k x_{n_i} - x)\|_X \\
&\leq \|I - U\|_X \|T^{k(m-1)}(V^k x_{n_i} - x)\|_X \\
&\leq \|I - U\|_X [r^{m-1}\|V^k x_{n_i} - x\|_X \\
&\quad + R'\|V^k x_{n_i} - x\|_Y] \longrightarrow 0
\end{aligned}
$$

as $m, i \to \infty$. In particular, we have $\|V^{kn_i} x_{n_i} - V^{k(n_i-1)} x\|_X \to 0$ as $i \to \infty$, and then

$$
\|V^{kn_i} x_{n_i}\|_X \leq \|V^{kn_i} x_{n_i} - V^{k(n_i-1)} x\|_X + \|V^{k(n_i-1)} x\|_X \longrightarrow 0
$$

as $i \to \infty$, contradicting the choice of the x_n's.

(4) Since each T_λ is a compact operator, then so is $\sum_{\lambda \in G} \lambda^n T_\lambda$ for each n. There is an n large enough that $\|T^n - \sum_{\lambda \in G} \lambda^n T_\lambda\|_X = \|V^n\|_X < 1$ so that T is quasi-compact by the note preceding this proof. $\square$

For an appreciation of the theorem of Ionescu Tulcea and Marinescu, see Iosifescu [6]. Therein, it is pointed out that applications of the ITM theorem go beyond the Doeblin-Fortet theorem, which will now be stated for completeness.

Application. The Doeblin-Fortet theorem. Let (E, d) be a compact metric space. Let $C(E)$ be the set of all continuous $f : E \to \mathbf{C}$, and define $\|f\| = \sup_{y \in E} |f(y)|$ for $f \in C(E)$. If $f \in C(E)$, define

$$s(f) = \sup_{\substack{y_1, y_2 \in E \\ y_1 \neq y_2}} \frac{f(y_1) - f(y_2)}{d(y_1, y_2)},$$

and let $L(E) = \{f \in C(E) \mid s(f) < \infty\}$ be the set of all Lipschitz continuous functions on E. If $f \in L(E)$, we define $\|f\|_L = \|f\| + s(f)$. We are going to apply the Ionescu Tulcea and Marinescu theorem with $X = (L(E), \|\cdot\|_L)$ and $Y = (C(E), \|\cdot\|)$. It is easily shown that the compatibility condition holds.

Theorem 2.4 (Doeblin-Fortet). *Let $T : L(E) \to L(E)$ be a linear operator satisfying*

(1) $\|Tf\| \leq \|f\|$ *for all $f \in L(E)$,*

(2) *there exist $r, R > 0$ with $r < 1$ such that $\|Tf\|_L \leq r\|f\|_L + R\|f\|$, $f \in L(E)$. Then*

(a) *the set G of eigenvalues of T of modulus 1 is finite, and if $\lambda \in G$, the eigenspace $D(\lambda) = \{x \in X \mid Tx = \lambda x\}$ is finite dimensional.*

(b) *For each $n \geq 1$, we have the representation*

$$T^n = \sum_{\lambda \in G} \lambda^n T_\lambda + V^n,$$

where $T_\lambda : L(E) \to L(E)$, $\lambda \in G$ are compact operators such that

(i) $\|T_\lambda\|_L < \infty$ *for each $\lambda \in G$,*

(ii) $T_\lambda^2 = T_\lambda$ *for all $\lambda \in G$ and $T_\lambda T_{\lambda'} = 0$ if $\lambda, \lambda' \in G$ with $\lambda \neq \lambda'$, and where the operator V satisfies $\|V\|_L < \infty$ and $VT_\lambda = T_\lambda V = 0$ for all $\lambda \in G$.*

(iii) $\|T^n - \sum_{\lambda \in G} \lambda^n T_\lambda\|_L \to 0$ *as $n \to \infty$.*

Proof. There are just three small details to take care of. First, since $\|f\| \leq \|f\|_L$ for every $f \in L(E)$, then

$$\|Tf\|_L \leq r\|f\|_L + R\|f\| \leq (r + R)\|f\|_L, \quad f \in L(E),$$

and therefore $\|T\|_L < \infty$. Secondly, the H in condition (2) of the definition of $L_k(X,Y)$ in this case is 1. Lastly, if A is a bounded subset of $(L(E), \|\cdot\|_L)$, then so is TA. But a bounded subset of $(L(E), \|\cdot\|_L)$ is bounded (since $\|f\| \le \|f\|_L$) and equicontinuous in $(C(E), \|\cdot\|)$, and therefore (by the Ascoli-Arzelà theorem) relatively compact in $(C(E), \|\cdot\|)$. $\square$

3. A weak form of the theorem of Ionescu Tulcea and Marinescu. Assume the same conditions as in Section 2, except that (3) in the definition of $L_k(X,Y)$ is replaced by (3′); there is an $R < \infty$ and for each $x' \in X'$ an $r_{x'} < 1$ such that

$$(3.1) \qquad |x'(T^k x)| \le r_{x'}|x'(x)| + R\|x\|_Y, \quad x \in X.$$

Then, just as in part (1) of Proposition 2.1, we find that there is an $R' < \infty$ such that

$$(3.2) \quad |x'(T^{km}x)| \le r_{x'}^m |x'(x)| + R'\|x\|_Y, \quad x' \in X', x \in X, m \ge 1.$$

By taking the supremum over all $x' \in X'$ with $\|x'\|_{X'} \le 1$, we find that $\|T^{km}x\|_X \le \|x\|_X + R'\|x\|_Y$ for all $x \in X$ and $m \ge 1$, and hence that part (2) of Proposition 2.1 also holds as is.

Lemma 3.1. *Suppose that $k = 1$, that $x \in X$, $y \in Y$ and that $T^{n_i}x \xrightarrow{s} y$ in $(Y, \|\cdot\|_Y)$ as $i \to \infty$. Then $y \in X$ and $T^{n_i}x \xrightarrow{w} y$ in $(X, \|\cdot\|_X)$ as $i \to \infty$.*

Proof. The compatibility condition implies that $y \in X$. Suppose that $T^{n_i}x$ does not converge weakly to y; then there is an $x' \in X'$, a subsequence $\{n_{i_j}, j \ge 1\}$, and an $\varepsilon > 0$ such that $|x'(T^{n_{i_j}}x - y)| \ge \varepsilon$ for all $j \ge 1$. Let $m \ge 1$ be such that $2r_{x'}^m\|x'\|_{X'}J < \varepsilon/2$, and let j^* be such that $n_{i_j} > m$ for all $j \ge j^*$. The set $A = \{T^{n_{i_j}-m-1}x | j \ge j^*\}$ is bounded in $\|\cdot\|_X$ and so TA is relatively compact in $(Y, \|\cdot\|_Y)$. Hence there is a subsequence $\{n_{i_{j_k}}, k \ge 1\}$ and a $w \in Y$ such that $\|T^{n_{i_{j_k}}-m}x - w\|_Y \to 0$ as $k \to \infty$. By the compatibility condition, we have $w \in X$. Also, we have $T^{n_{i_{j_k}}}x \xrightarrow{s} T^m w$ in $(Y, \|\cdot\|_Y)$ as $k \to \infty$,

and so $T^m w = y$. By (3.1),

$$
\begin{aligned}
|x'(T^{n_{i_{j_k}}} x - y)| &= |x'[T^m(T^{n_{i_{j_k}} - m} x - w)]| \\
&\leq r_{x'}^m |x'(T^{n_{i_{j_k}} - m} x - w)| \\
&\quad + R' \|T^{n_{i_{j_k}} - m} x - w\|_Y \\
&\leq 2 r_{x'}^m \|x'\|_{X'} J + R' \|T^{n_{i_{j_k}} - m} x - w\|_Y \\
&\leq \varepsilon/2 + R' \|T^{n_{i_{j_k}} - m} x - w\|_Y,
\end{aligned}
$$

and letting $k \to \infty$ gives

$$
\limsup_{k \to \infty} |x'(T^{n_{i_{j_k}}} x - y)| \leq \varepsilon/2,
$$

a contradiction.   □

In the following theorem, we obtain weak versions of some of the parts of Theorem 2.3.

Theorem 3.2. *There exists a weak deLeeuw-Glickstein decomposition $X = X_{\mathrm{fl}} \oplus X_{\mathrm{rev}}$, where X_{rev} is weakly compact in $(X, \|\cdot\|)$, and therefore reflexive. If, moreover, $\|T\|_Y \leq 1$, then $T^n x \xrightarrow{w} 0$ in $(X, \|\cdot\|_X)$ for every $x \in X_{\mathrm{fl}}$.*

Proof. First we will show that T is weakly almost periodic as an operator on X, so that the deLeeuw-Glickstein decomposition applies. Let us assume first that $k = 1$. Let $x \in X$, and consider the sequence $A = \{T^n x, n \geq 0\}$. A is bounded in $(X, \|\cdot\|_X)$ because of part (2) of Proposition 2.1, and so TA is relatively compact in $(Y, \|\cdot\|_Y)$. Hence there is an $\{n_i, i \geq 1\}$ and a $y \in Y$ such that $\|T^{n_i} x - y\|_Y \to 0$ as $i \to \infty$. By the lemma, we have $y \in X$ and $T^{n_i} x \xrightarrow{w} y$. Hence, $\{T^n x, n \geq 0\}$ is weakly relatively compact, and so T is weakly almost periodic. If $k > 1$, the above proof shows that T^k is weakly almost periodic, which obviously implies that T is weakly almost periodic. We therefore have a deLeeuw-Glickstein decomposition $X = X_{\mathrm{fl}} \oplus X_{\mathrm{rev}}$ of X.

Next we will show that the unit ball $(X_{\mathrm{rev}})_1^X$ in X_{rev} is weakly compact, which will imply that X_{rev} is reflexive. Recall that the

restriction of $\overline{S}$ to X_{rev} is a group whose members are bounded in norm by J. Let $\{x_n, n \geq 0\} \subset (X_{\text{rev}})_1^X$. Since $\{T^{-k}x_n, n \geq 1\}$ is bounded in $(X, \|\cdot\|_X)$, there is a sequence $\{n_i^0, i \geq 1\} \subset \mathbf{Z}^+$ such that $\{x_{n_i^0}, i \geq 1\}$ converges in $(Y, \|\cdot\|_Y)$. By the same token, since $\{T^{2k}x_{n_i^0}, i \geq 1\}$ is bounded in $(X, \|\cdot\|_X)$, there is a subsequence $\{n_i^1, i \geq 1\}$ of $\{n_i^0, i \geq 1\}$ such that $\{T^{-k}x_{n_i^1}, i \geq 1\}$ converges in $(Y, \|\cdot\|_Y)$. Repeating this procedure, and using the diagonalization argument, one can find a sequence $\{n_i^d, i \geq 1\}$ such that $\{T^{-km}x_{n_i^d}, i \geq 1\}$ converges in $(Y, \|\cdot\|_Y)$, say to y_m, for every $m \geq 0$. By the compatibility condition, $y_m \in X$ for every m. It is easy to see that $T^{km}y_m$ is independent of m; let $z = T^{km}y_m \in X$. Now let $x' \in X'$, and let $\varepsilon > 0$. Let m be such that $2r_{x'}^m \|x'\|_{X'}J < \varepsilon$; then

$$
\begin{aligned}
|x'(x_{n_i^d}) - x'(z)| &= |x'[T^{km}(T^{-km}x_{n_i^d} - y_m)] \\
&\leq r_{x'}^m |x'(T^{-km}x_{n_i^d} - y_m)| \\
&\quad + R'|(T^{-km}x_{n_i^d} - y_m)\|_Y \\
&\leq \varepsilon + R'|(T^{-km}x_{n_i^d} - y_m)\|_Y,
\end{aligned}
$$

from which we find that $|x'(x_{n_i^d}) - x'(z)| \to 0$ as $i \to \infty$. Hence $x_{n_i^d} \xrightarrow{w} z$ as $i \to \infty$, so X_{rev} is weakly compact.

Finally, assume that $\|T\|_Y \leq 1$, and let $x \in X_{\text{fl}}$. Then there is an $\{n_i, i \geq 1\}$ such that $T^{n_i}x \xrightarrow{w} 0$ in $(X, \|\cdot\|_X)$. The sequence $\{T^{n_i}x, i \geq 1\}$ is bounded in $(X, \|\cdot\|_X)$, so there is a subsequence $\{n_{i_j}, j \geq 1\}$ and $y \in Y$ such that $T^{n_{i_j}}x \xrightarrow{s} y$ in $(Y, \|\cdot\|_Y)$ as $j \to \infty$. By the lemma we must have $y = 0$, so $\|T^{n_{i_j}}x\|_Y \to 0$ as $j \to \infty$. Finally, since $\|T\|_Y \leq 1$, we obtain $\|T^n x\|_Y \to 0$, so again by the lemma, we get $T^n x \xrightarrow{w} 0$ in $(X, \|\cdot\|_X)$. $\square$

APPENDIX

A proof of the following can be found in Dunford and Schwartz [**4**, pp. 556–559].

Proposition 4.1. *Let T be a linear operator on a finite-dimensional vector space X. Let $\sigma(T) = \{\lambda_1, \ldots, \lambda_r\}$ denote the spectrum of T, i.e., the set of eigenvalues of T. For any $i \geq 1$ and $\lambda \in \sigma(T)$, let*

$\mathcal{N}_\lambda^i = \ker[(T - \lambda I)^i]$. *Finally, let* $\nu(\lambda)$ *be the index of* λ; *that is, the smallest value of* i *such that* $\mathcal{N}_\lambda^i = \mathcal{N}_\lambda^{i+1}$. *Then* X *can be written*

$$X = \mathcal{N}_{\lambda_1}^{\nu(\lambda_1)} \oplus \cdots \oplus \mathcal{N}_{\lambda_r}^{\nu(\lambda_r)}.$$

For any $\lambda \in \sigma(T)$, *let* $E(\lambda)$ *denote the projection of* X *onto* $\mathcal{N}_\lambda^{\nu(\lambda)}$. *Then the operators* $E(\lambda)$, $\lambda \in \sigma(T)$ *on* X *satisfy*

(1) $E(\lambda)^2 = E(\lambda)$ *for all* $\lambda \in \sigma(T)$ *and* $E(\lambda)E(\lambda') = 0$ *for all* $\lambda, \lambda' \in \sigma(T)$ *such that* $\lambda \neq \lambda'$.

(2) $\sum_{\lambda \in \sigma(T)} E(\lambda) = I$, *the identity mapping.*

(3) $\sum_{\lambda \in \sigma(T)} \lambda^n E(\lambda) = T^n$, $n \geq 0$.

REFERENCES

1. K. deLeeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta Math. **105** (1961), 63–97.

2. W. Doeblin, *Elements d'une theorie generale des chaines simple constants de Markoff*, Ann. Sci. École Norm. Sup. (4) **57** (1940), 61–111.

3. W. Doeblin and R. Fortet, *Sur des chaines à liaisons complètes*, Bull. Soc. Math. France **65** (1937), 132–148.

4. N. Dunford and J.T. Schwartz, *Linear operators* Part I: *General theory*, Interscience Publishers, New York, 1967.

5. C.T. Ionescu Tulcea and G. Marinescu, *Théorie ergodique pour des classes d'opérations non complètement continues*, Ann. of Math. **52** (1951), 140–147.

6. M. Iosifescu, *A basic tool in mathematical chaos theory: Doeblin and Fortet's ergodic theorem and Ionescu Tulcea and Marinescu's generalization*, Contem. Math. **149** (1993), 111–124.

7. B. Jamison and R. Sine, *Irreducible almost periodic Markov operators*, J. Math. Mech. **18** (1969), 1043–1057.

8. ———, *Sample path convergence of stable Markov operators*, Z. Wahr. **28** (1974), 173–177.

9. J. Komornik, *Asymptotic periodicity of the iterates of weakly constrictive Markov operators*, Tohoku Math. J. **38** (1986), 15–27.

10. J. Komornik and A. Lasota, *Asymptotic decomposition of Markov operators*, Bull. Polish Acad. Sci. Math. **35** (1987), 321–327.

11. U.K. Krengel, *Ergodic theorems*, Walter de Gruyter & Co., Berlin, 1985.

12. A. Lasota, T.Y. Li and J.A. Yorke, *Asymptotic periodicity of the iterates of Markov operators*, Trans. Amer. Math. Soc. **286** (1984), 751–764.

13. J. Neveu, *Mathematical foundations of the calculus of probability*, Holden-Day, San Francisco, 1965.

14. M. Rosenblatt, *Almost periodic transition operators acting on the continuous functions on a compact space*, J. Math. Mech. **13** (1964), 837–847.

15. ——, *Equicontinuous Markov operators*, in Theory Probab. Appl. **9** (1964), 180–197.

16. R. Sine, *Geometric theory of a single Markov operator*, Pacific J. Math. **27** (1968), 155–166.

17. ——, *Sample path convergence of stable Markov operators* II, Indiana Univ. J. Math. **25** (1976), 23–43.

18. ——, *Constricted systems*, Rocky Mountain J. Math. **21** (1991), 1373–1383.

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, MONTREAL, QUEBEC, CANADA H3Z 2K5
E-mail address: BILL@MARKOV.MATH.MCGILL.CA

DEPARTMENT OF MATHEMATICS AND STATISTICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, CANADA