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ON LIMITING BEHAVIOR OF PROBABILITY MEASURES ON LOCALLY COMPACT SEMIGROUPS

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ABSTRACT. In this paper we study the problem of summability in the weak and the vague topology of the sequence of convolution iterates (μ^n, $n \in \mathbf{N}$) where μ is a probability measure on locally compact semigroup S. Instead of the usual Cesàro, $(c, 1)$ summability of a sequence (μ^n, $n \in \mathbf{N}$), we consider the problem of the summability of this sequence by stochastic strongly regular matrices. Our Theorem 1 generalizes Theorem 3.1 in [**6**, p. 131]. Theorem 1 also holds if we make some changes on the topological assumptions on S, i.e., if we suppose that S is a Polish space.

1. Introduction and preliminaries. Throughout this paper S will denote a locally compact, Hausdorff, second countable semigroup. By a measure on S, we mean a finite regular nonnegative measure on the class $\mathcal{B}_S$ of all Borel sets in S. By $\mathcal{P}(S)$ we denote the set of all regular probability measures defined on S. We put $Q(S) = \{\mu : \mu \text{ is a measure on } S \text{ and } \mu(S) \leq 1\}$.

Let $\mathcal{C}(S) \supset \mathcal{K}(S)$ be the spaces of (real-valued) bounded continuous functions, and continuous functions with compact supports, respectively.

A net (μ_α) of measures converges *weakly* to a measure μ if $\lim_\alpha \int_S f d\mu_\alpha = \int_S f\, d\mu$ for each $f \in \mathcal{C}(S)$. Then we write $\mu = (w) \lim_\alpha \mu_\alpha$ or $\mu_\alpha \xrightarrow{w} \mu$. (μ_α) converges *vaguely* to μ if $\lim_\alpha \int_S f\, d\mu_\alpha = \int_S f\, d\mu$ for each $f \in \mathcal{K}(S)$. Then we write $\mu = (v) \lim_\alpha \mu_\alpha$ or $\mu_\alpha \xrightarrow{v} \mu$.

If $(\mu_\alpha) \subset \mathcal{P}(S)$ and $\mu_\alpha \xrightarrow{w} \mu$, then $\mu \in \mathcal{P}(S)$. By the Banach-Alaoglu's theorem, the set $Q(S)$ is compact in the vague topology.

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The *convolution* $\mu * \nu$ of two measures μ, ν is defined by

$$\int_S f \, d\mu * \nu = \int_S \int_S f(xy) \, d\mu(x) \, d\nu(y), \quad f \in \mathcal{K}(S). \tag{1.1}$$

For $A \subset S$ and $x \in S$ we write $Ax^{-1} = \{y : yx \in A\}$, $x^{-1}A = \{y : xy \in A\}$. Then we have

$$\mu * \nu(E) = \int_S \mu(Ex^{-1}) \, d\nu(x) = \int_S \nu(x^{-1}E) \, d\mu(x), \quad E \in \mathcal{B}_S. \tag{1.2}$$

If $\mu, \nu \in \mathcal{P}(S)$, then $\mu * \nu \in \mathcal{P}(S)$. Moreover, $\mathcal{P}(S)$ is a topological semigroup with respect to the convolution, i.e., the mapping $* : \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S)$ is jointly continuous in the vague topology, [**4**, p. 280]. $Q(S)$ although compact is not a topological semigroup since the convolution in $Q(S)$ is not vaguely jointly continuous [**4**, p. 281]. It can be shown that, in the general case, the convolution in $Q(S)$ is not even vaguely separately continuous, see [**9**].

When S is compact, $Q(S)$ as well as $\mathcal{P}(S)$ is a compact topological semigroup with respect to convolution and the vague (weak) topology (these two topologies obviously coincide in that case). $\mathcal{P}(S)$, despite being a topological semigroup, is compact if and only if S is compact. Otherwise, the closure of $\mathcal{P}(S)$ in the vague topology is equal to $Q(S)$ [**4**, p. 280].

In some respects convolutions on semigroups behave like convolutions on groups if the multiplication on S has the property that $S \times K \to S$ is *proper* for each compact $K \subset S$, i.e., if

$$\{(x,y) : xy \in K', y \in K\} \text{ is compact for all compact } K, K' \subset S. \tag{1.3}$$

If the multiplication in S satisfies (1.3) or, in particular, if S is a locally compact group, or if S is a compact semigroup, then the convolution is vaguely separately continuous on $Q(S)$ [**9**, pp. 367, 370 and 368]. Separate continuity in the weak topology takes place without any condition on the multiplication [**9**, p. 370].

$\mu * \mu * \cdots * \mu$ (with n terms) we denote by μ^n.

Let $A = [a_{nj}]$, $n, j \in \mathbf{N}$, be a real infinite matrix. We say that A is *regular* or a Toeplitz matrix if it transforms convergent sequences into other convergent sequences and if it preserves limits. In other words, A is a regular matrix if

$$A - \lim(x) = A - \lim_{n\to\infty} x_n = \lim_{n\to\infty} \sum_{j=1}^{\infty} a_{nj} x_j = \lim_{n\to\infty} x_n,$$

for every convergent sequence $x = (x_n, n \in \mathbf{N})$ of real numbers. By the well-known Silvermann-Toeplitz theorem, we have that a matrix A is regular if and only if

(1.4)

$$\sup_{n\in\mathbf{N}} \sum_{j=1}^{\infty} |a_{nj}| < \infty; \quad \lim_{n\to\infty} a_{nj} = 0, \quad j \in \mathbf{N}; \quad \lim_{n\to\infty} \sum_{j=1}^{\infty} a_{nj} = 1.$$

A matrix A is called *strongly regular* if it is regular and if, in addition, it satisfies the following condition

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} |a_{nj} - a_{n,j+1}| = 0. \tag{1.5}$$

As an example of a stochastic strongly regular matrix, we have Cesàro $(c, 1)$ summability matrix $C = [a_{nj}]$ defined by $a_{nj} = 1/n$ for $j \le n$ and $a_{nj} = 0$ for $j > n$.

Put $a_{nj} = (e^{\varepsilon_n} - 1)e^{-j\varepsilon_n}$, $n, j \in \mathbf{N}$, where $(\varepsilon_n, n \in \mathbf{N})$ is a sequence of positive real numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$. Then $A = [a_{nj}]$, $n, j \in \mathbf{N}$, is a stochastic strongly regular matrix [**8**, p. 364].

Let $A = [a_{nj}]$, $n, j \in \mathbf{N}$, be a Toeplitz matrix. Then (1.5) is a necessary and sufficient condition such that for every bounded sequence $(z_j, j \in \mathbf{N}) \subset \mathbf{R}$ we have

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} a_{nj}(z_j - z_{j+1}) = 0, \tag{1.6}$$

[**3**, p. 119].

Let $A = [a_{nj}]$, $n, j \in \mathbf{N}$, be a stochastic matrix. We say that a sequence $(\mu_n,\ n \in \mathbf{N}) \subset \mathcal{P}(S)$ is *weakly A-convergent* to $\mu_0 \in \mathcal{P}(S)$, and we write $\mu_0 = (w)A - \lim_{n\to\infty} \mu_n$ if the sequence of real numbers $(\int_S f\, d\mu_n, n \in \mathbf{N})$ is A-convergent to $\int_S f\, d\mu_0$ for every $f \in \mathcal{C}(S)$, i.e.,

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} a_{nj} \int_S f\, d\mu_j = \int_S f\, d\mu_0, \quad f \in \mathcal{C}(S). \tag{1.7}$$

Let $(\mu_n, n \in \mathbf{N}) \subset \mathcal{P}(S)$ be such that $(w)\lim_{n\to\infty} \mu_n$ exists. Then it is obvious that, for any stochastic Toeplitz matrix A, we have

$$(w)A - \lim_{n\to\infty} \mu_n = (w) \lim_{n\to\infty} \mu_n.$$

We say that a sequence $(\mu_n, n \in \mathbf{N}) \subset Q(S)$ is *vaguely A-convergent* to $\mu_0 \in Q(S)$ and we write $\mu_0 = (v)A - \lim_{n\to\infty} \mu_n$ if we have

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} a_{nj} \int_S f\, d\mu_j = \int_S f\, d\mu_0, \quad f \in \mathcal{K}(S). \tag{1.8}$$

2. Results. First we have the following definition.

Definition. A family $\mathcal{M} \subset \mathcal{P}(\mathcal{S})$ is said to be *tight* if for each $\varepsilon > 0$ there exists a compact set K_ε such that

$$\mu(K_\varepsilon) > 1 - \varepsilon \quad \text{for every} \quad \mu \in \mathcal{M}.$$

Theorem 2.1. *Let S be a locally compact, Hausdorff, second countable semigroup. Let $\mu \in \mathcal{P}(S)$ be such that the sequence, $(\mu^n, n \in \mathbf{N})$, is tight, and let $A = [a_{nj}]$, $n, j \in \mathbf{N}$, be a stochastic strongly regular matrix. Then the sequence, $(\mu^n, n \in \mathbf{N})$, is weakly A-convergent. If we put $(w)A - \lim_{n\to\infty} \mu^n = \mu_0 \in \mathcal{P}(S)$, then μ_0 doesn't depend on matrix A, and we have*

$$\mu * \mu_0 = \mu_0 * \mu = \mu_0 = \mu_0^2. \tag{2.1}$$

Proof. Put $s_n(\mu) = \sum_{j=1}^{\infty} a_{nj} \mu^j$, $n \in \mathbf{N}$. Then $(s_n(\mu), n \in \mathbf{N}) \subset \mathcal{P}(S)$ and tightness of $(\mu^n, n \in \mathbf{N})$, implies the tightness of $(s_n(\mu),$

$n \in \mathbf{N})$. It follows by Prokhorov's theorem that the set $(s_n(\mu),\, n \in \mathbf{N})$, is relatively compact in $\mathcal{P}(S)$ in the weak topology. Hence, there exists $\mu_0 \in \mathcal{P}(S)$ and a subsequence

$$\left(s_{n_i}(\mu) = \sum_{j=1}^{\infty} a_{n_i j}\mu^j, \quad i \in \mathbf{N}\right),$$

such that

$$\mu_0 = (w) \lim_{i\to\infty} s_{n_i}(\mu). \tag{2.2}$$

Since the sequence $(\int_S f\,d\mu^j,\ j \in \mathbf{N}) \subset \mathbf{R}$ is bounded for every $f \in \mathcal{C}(S)$ and since A is a strongly regular matrix, we conclude from (1.6) that

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{n_i j}\left(\int_S f\,d\mu^j - \int_S f\,d\mu^{j+1}\right) = 0, \tag{2.3}$$

for every $f \in \mathcal{C}(S)$.

It follows from (2.2) and (2.3) that $\mu_0 = (w)\lim_{i\to\infty}\sum_{j=1}^{\infty} a_{n_i j}\mu^{j+1}$. Now, by separate continuity of the convolution in the weak topology, we get

$$\begin{aligned}\mu * \mu_0 &= \mu * \left((w) \lim_{i\to\infty} \sum_{j=1}^{\infty} a_{n_i j}\mu^j\right)\\ &= (w) \lim_{i\to\infty} \sum_{j=1}^{\infty} a_{n_i j}\mu^{j+1} = \mu_0.\end{aligned}$$

Analogously, we prove $\mu_0 * \mu = \mu_0$ and therefore $\mu^j * \mu_0 = \mu_0 * \mu^j = \mu_0$, $j \in \mathbf{N}$. Further, we have

$$\begin{aligned}\mu_0 * \mu_0 &= (w) \lim_{i\to\infty} \left(\mu_0 * \sum_{j=1}^{\infty} a_{n_i j}\mu^j\right)\\ &= \left(\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{n_i, j}\right)\mu_0 = \mu_0,\end{aligned}$$

so μ_0 is an idempotent in $\mathcal{P}(S)$. Considering now another subsequence $(\sum_{j=1}^{\infty} a_{m_i j}\mu^j, j \in \mathbf{N})$ converging weakly to $\mu_1 \in \mathcal{P}(S)$, we conclude that μ_1 is again an idempotent such that $\mu * \mu_1 = \mu_1 * \mu = \mu_1$ and therefore $\mu^j * \mu_1 = \mu_1 * \mu^j = \mu_1$ for each j, so we have

$$s_{n_i}(\mu) * \mu_1 = \bigg(\sum_{j=1}^{\infty} a_{n_i j}\bigg)\mu_1 = \mu_1,$$

for each i.

Analogously we get $\mu_0 * s_{m_i}(\mu) = \mu_0$, $i \in \mathbf{N}$.

It follows that

$$\begin{aligned}\mu_1 &= (w)\lim_{i\to\infty}(s_{n_i}(\mu) * \mu_1) = \mu_0 * \mu_1\\ &= (w)(\lim_{i\to\infty}(\mu_0 * s_{m_i}(\mu)) = \mu_0.\end{aligned}$$

Thus, every weakly convergent subsequence of the sequence $(s_n(\mu), n \in \mathbf{N})$ converges to the same element in $\mathcal{P}(S)$, denote it by μ_0. Since $(s_n(\mu), n \in \mathbf{N})$ is relatively compact in the weak topology, it follows easily that $\mu_0 = (w)\lim_{n\to\infty} s_n(\mu)$. We have also proved that (2.1) holds true.

Let $B = [b_{nj}]$, $n, j \in \mathbf{N}$, be another stochastic strongly regular matrix, let $\tilde{s}_n(\mu) = \sum_{j=1}^{\infty} b_{nj}\mu^j$, $n \in \mathbf{N}$, and let $\mu_1 = (w)B - \lim_{n\to\infty}\mu^n = (w)\lim_{n\to\infty}\tilde{s}_n(\mu)$. For arbitrary $j \in \mathbf{N}$, we have again $\mu^j * \mu_1 = \mu_1 = \mu_1 * \mu^j$, and therefore $s_n(\mu) * \mu_1 = \mu_1 * s_n(\mu) = (\sum_{j=1}^{\infty} a_{nj})\mu_1 = \mu_1$, $n \in \mathbf{N}$. Analogously, we get $\mu_0 * \tilde{s}_n(\mu) = \mu_0$, $n \in \mathbf{N}$.

Now we have

$$\mu_1 = (w)\lim_{n\to\infty}(s_n(\mu) * \mu_1) = \mu_0 * \mu_1 = (w)\lim_{n\to\infty}(\mu_0 * \tilde{s}_n(\mu)) = \mu_0.$$

□

Remark 2.1. Theorem 3.1 in [**6**, p. 131] follows from Theorem 1 if we take for A the Cèsaro $(c, 1)$ summability matrix.

Remark 2.2. Let us make some changes on the topological assumptions on S, i.e., suppose that S is a Polish space. Prokhorov's theorem holds on all Polish spaces, and joint continuity of $(\mu,\nu)\mapsto\mu*\nu$ holds in that case by [**7**, Theorem 1]. Therefore, Theorem 1 holds on all Polish spaces in the same proof.

Note also that on Polish spaces all finite (even signed) measures are automatically regular.

In the case of the vague topology we have the following theorem.

Theorem 2.2. *Let S be a locally compact, Hausdorff, second countable semigroup with the multiplication satisfying* (1.3), *and let $A=[a_{nj}]$, $n,j\in\mathbf{N}$, be a stochastic strongly regular matrix. Then the sequence $(\mu^n, n\in\mathbf{N})$ is vaguely A-convergent for each $\mu\in\mathcal{P}(S)$. If we put $\mu_0=(v)A-\lim_{n\to\infty}\mu^n$, then $\mu_0\in Q(S)$ is an idempotent not depending on the matrix A and μ_0 is either the null-measure or $\mu_0\in\mathcal{P}(S)$.*

Proof. We follow the ideas in [**2**]. We have $(s_n(\mu)=\sum_{j=1}^{\infty}a_{nj}\mu^j, n\in\mathbf{N})\subset Q(S)$ and, since $Q(S)$ is compact in the vague topology, the set $(s_n(\mu), n\in\mathbf{N})$ is relatively compact in the vague topology. Condition (1.3) implies that the convolution is vaguely separately continuous on $Q(S)$. Thus, in the same way as in Theorem 1, we prove that the sequence $(\mu^n, n\in\mathbf{N})$ is vaguely A-convergent to $\mu_0\in Q(S)$ satisfying (2.1) and not dependent on A.

By the Riesz representation theorem there is a one-to-one correspondence between the set $\mathcal{M}^1(S)$ of all regular, real-valued, signed measures on $\mathcal{B}_S$ and the dual of $\mathcal{K}(S)$. Elements from $\mathcal{P}(S)$ correspond to positive functionals with norm one. By [**1**, p. 142], $\mathcal{M}^1(S)$ is a Banach algebra with convolution as multiplication and norm defined by

$$\|\mu\|=|\mu|(S),\quad \mu\in\mathcal{M}^1(S), \tag{2.4}$$

where $|\mu|$ is the total variation of μ. Now it follows from (2.1) that $\|\mu_0\|=\|\mu_0^2\|\le\|\mu_0\|^2$, and therefore $\|\mu_0\|=0\Leftrightarrow\mu_0=0$ or $\|\mu_0\|\ge 1$. But $\mu_0\in Q(S)$ so we have $\|\mu_0\|=\mu_0(S)\le 1$. Thus we have $\mu_0=0$ or $\mu_0\in\mathcal{P}(S)$. □

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REFERENCES

1. N. Bourbaki, *Integration*, Hermann, Paris, 1963.

2. D. Butković, *On the summability of convolution sequences of measures*, Glas. Mat. Ser. III **13** (1978), 69–74.

3. R.G. Cooke, *Infinite matrices and sequence spaces*, Macmillan, London, 1950.

4. I. Csiszár, *On infinite products of random elements and infinite convolutions of probability distributions on locally compact groups*, Z. Wahrscheinlichkeitstheorie Verw. Gebiete **5** (1966), 279–295.

5. U. Grenander, *Probabilities on algebraic structures*, John Wiley & Sons, New York, 1963.

6. Lin Jin-e and Xu Kan, *A weak limit behaviour of a probability measure on locally compact topological semigroups*, Proceedings of the International Symposium on the Semigroup Theory and its Related Fields (Kyoto, 1990), 129–136, Shimane Univ., Matsue, 1990.

7. P. Ressel, *Some continuity and measurability results on spaces of measures*, Math. Scand. **40** (1977), 69–78.

8. N. Sarapa, *On the* (c,α) *summability of probability distributions defined on compact semigroups*, Glas. Mat. Ser. III **12** (1977), 359–372.

9. J. Yuan, *On the continuity of convolution*, Semigroup Forum **10** (1975), 367–372.

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