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SYMMETRY INCREASING BIFURCATIONS VIA COLLISIONS OF ATTRACTORS

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ABSTRACT. In 1988, Chossat and Golubitsky observed numerically, in discrete dynamical systems equivariant under the action of a finite group, a phenomenon for which they coined the name *symmetry increasing bifurcation*. They observed that, while varying a parameter, conjugate attractors of such a system may collide yielding an attractor with larger symmetry group than before.

One of the questions arising in this context is the following: Given a group Γ, which subgroups Σ of Γ are admissible in the sense that Σ-symmetric attractors of a Γ-equivariant mapping may undergo a symmetry increasing bifurcation?

In this paper we extend the approach to solve this problem made by Dellnitz and Heinrich. We construct collisions of attractors at arbitrary reflection hyperplanes, as well as collisions which take place at points of trivial isotropy. Combining these results we are able to give necessary and sufficient criterions for admissibility of these collisions.

1. Introduction. Discrete dynamical systems on $\mathbf{R}^n$ equivariant under the action of a finite group Γ typically possess attractors displaying symmetry. More precisely, these attractors as a set are invariant under the action of a subgroup of Γ. If a parameter is introduced into the system, preserving the symmetry, then one can often observe that these attractors collide with conjugate attractors yielding an attractor with larger symmetry than before. This phenomenon was observed by Grebogi, Ott, Romeiras and Yorke [**11**] and Chossat and Golubitsky [**6**], who named these transitions *symmetry increasing bifurcations*.

Since then, other mechanisms by which symmetry can be increased have been observed. In particular, three mechanisms have been described in [**7**]. Apart from collisions, also "explosions" of attractors may take place, see also King and Stewart [**13**], and, for continuous groups, an attractor may start to "drift" along its group orbit yielding larger symmetry than before, see Dellnitz, Golubitsky and Melbourne

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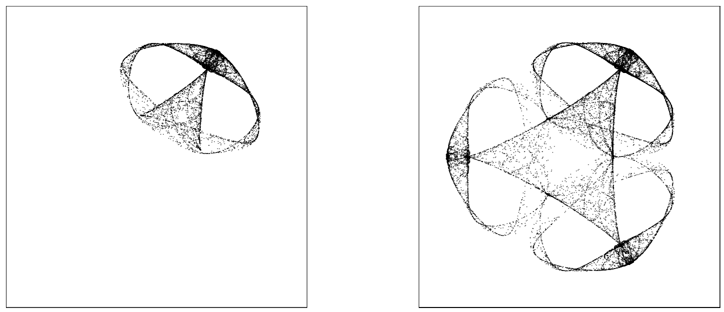


FIGURE 1. A symmetry increasing bifurcation. An attractor collides with two conjugate counterparts yielding an attractor with increased symmetry.

[**7**] and Aston [**4**]. Contrary to symmetry increasing bifurcations caused by a collision of attractors, one may also observe, for so called *Milnor attractors*, that these attractors touch their conjugate counterpart but do not merge with it to form an attractor with increased symmetry, see Ashwin [**2**].

In this paper we consider the special type of symmetry increasing bifurcation caused by a collision of attractors. The group Γ is always assumed to be a finite subgroup of $\mathbf{O}(n)$ with the standard action on $\mathbf{R}^n$. We investigate the following question. Given a group Γ, a subgroup Σ of Γ and an element $\kappa \in \Gamma \backslash \Sigma$, when does there exist a Γ-equivariant, parameter dependent, continuous system on $\mathbf{R}^n$ in which we may observe a Σ-symmetric attractor A colliding with its conjugate attractor κA and merging to form an attractor with symmetry $\langle \Sigma \cup \{\kappa\} \rangle$?

To answer this question, another problem has to be considered first, namely, which subgroups Σ of Γ are *admissible* in the sense that there exists a Σ-symmetric attractor of a continuous Γ-equivariant system. Melbourne, Dellnitz and Golubitsky [**14**] stated necessary conditions for admissible subgroups and Ashwin and Melbourne [**3**] proved that these conditions were sufficient as well. They showed that admissibility depends crucially on the reflection hyperplanes of elements in $\Gamma \backslash \Sigma$. The Σ-symmetric attractor may not intersect these hyperplanes, and hence it must be contained in certain connected components bounded by these reflection hyperplanes. For diffeomorphisms, the question of admissible subgroups was treated by Field, Melbourne and Nicol [**9**].

To prove their results, Ashwin and Melbourne use embedded Σ-symmetric graphs, on which they define the dynamics. We apply their methods to construct parameter dependent embeddings. This approach was first made by Dellnitz and Heinrich [**8**]. The authors considered collisions taking place at precisely one reflection hyperplane and came to the conclusion that in this case a symmetry increasing bifurcation could take place if and only if κ commuted with all elements in Σ.

In this paper we extend this approach. We not only admit collisions at arbitrary reflection hyperplanes, but we also look at collisions where κ is not a reflection, thus allowing κ to be any element in $\Gamma\backslash\Sigma$. Moreover, we extend the construction in [**8**] beyond the critical point of collision, that is, we construct attractors with symmetry group $\langle \Sigma \cup \{\kappa\}\rangle$ after the collision. Our main results are two theorems, Theorems 4.10 and 4.21, which treat the two separate cases depending on whether κ is a reflection or not. Each of these theorems gives a necessary and sufficient condition for admissibility of symmetry increasing bifurcations.

An example of an admissible symmetry increasing bifurcation can be observed in Figure 1. Here, the underlying group Γ is $\mathbf{D}_3$, and the group Σ of the attractor before the collision is $\mathbf{D}_1$. From the picture, it appears that a transition from $\mathbf{D}_1$ to full $\mathbf{D}_3$-symmetry is admissible, and indeed it follows from Theorem 4.10 that this is the case. On the other hand, it will turn out that a transition from $\mathbf{Z}_3$ to $\mathbf{D}_3$-symmetry is not possible although both groups may be symmetry groups of attractors of a $\mathbf{D}_3$-equivariant system. Other examples can be found in Section 4.5, where all admissible transitions of the tetrahedral group have been classified.

This paper is an excerpt from my Ph.D. Thesis [**12**].

2. Symmetry increasing bifurcations. In this section we are going to define what we mean by a *symmetry increasing bifurcation*. Roughly we wish to apply this term to the phenomenon of a *collision of chaotic attractors* of discrete dynamical systems.

2.1. Preliminaries. Throughout the paper we consider discrete dynamical systems, given by a continuous parameter dependent mapping $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$. Now suppose that $\Gamma < \mathbf{O}(n)$ is a finite group with the standard action on $\mathbf{R}^n$ and that f is *equivariant* under this action, i.e.,

$$f(\gamma x, \lambda) = \gamma f(x, \lambda)$$

for all $x \in \mathbf{R}^n$, $\lambda \in \mathbf{R}$ and $\gamma \in \Gamma$. For a fixed parameter value, there may be *attractors* associated to the dynamical system. We define an attractor to be a stable ω-limit set, precisely

Definition 2.1. A set $A \subset \mathbf{R}^n$ is called *stable* (in the sense of Liapunov) if, for any neighborhood U of A, there exists a neighborhood V of A such that for any $j \in \mathbf{N}$ we have $f^j(V) \subset U$.

The ω-limit set $\omega(x)$ for any $x \in \mathbf{R}^n$ is the set of accumulation points of the *orbit* of x, that is, of the sequence $(f^j(x))_{j \in \mathbf{N}}$.

An *attractor* is a stable ω-limit set.

An attractor of a mapping f has the symmetry of a subgroup of the group Γ under which f is equivariant. By this we mean the following: We define the *symmetry group* of a set $B \subset \mathbf{R}^n$ to be the subgroup of Γ which fixes B as a set, that is,

$$\Sigma_B = \{\gamma \in \Gamma \mid \gamma B = B\}.$$

If $\Sigma = \Sigma_B$, we say that B is Σ-invariant. Moreover, we call a Σ-invariant ω-limit set B Σ-*symmetric*, if B contains points of trivial isotropy.

If an attractor A has a symmetric group Σ which is strictly smaller than Γ, it is easy to check that there must exist *conjugate* attractors γA for any $\gamma \in \Gamma \backslash \Sigma$. In this context the following result is most relevant, which was first proved by Chossat and Golubitsky [**6**] and was then reproved in [**14**] in the current setting.

Proposition 2.2. *Let A be an attractor of a Γ-equivariant mapping f. Then, for any $\gamma \in \Gamma$, we have either $A = \gamma A$ or $A \cap \gamma A = \varnothing$.*

This proposition helps us to understand why symmetry increasing bifurcations occur. Suppose that we observe a Σ-symmetric attractor A in a Γ-equivariant system, and let $\kappa \in \Gamma \backslash \Sigma$. Hence, there exists a conjugate attractor κA which, by the above proposition, does not intersect A. Now suppose that, while varying a parameter, these two conjugate attractors collide. Then, by Proposition 2.2, we would expect them to merge and form an attractor with a larger symmetry group, namely $\langle \Sigma \cup \{\kappa\} \rangle$.

2.2. Admissible subgroups of Γ. To understand the setting of a symmetry increasing bifurcation, it is necessary to look at the symmetry groups and specific properties of the attractors involved. A first observation is that a Σ-symmetric attractor A may not, by Proposition 2.2, intersect any reflection hyperplanes of reflections that are not contained in Σ. Recall that a *reflection* is defined as a group element $\tau \in \mathbf{O}(n)$ such that $\dim(\mathrm{Fix}(\tau)) = n - 1$, where we let

$$\mathrm{Fix}(\tau) \stackrel{\mathrm{def}}{=} \{x \in \mathbf{R}^n \mid \tau x = x\}.$$

If A is a Δ-symmetric attractor, the reflection hyperplanes of the set

$$K_\Delta \stackrel{\mathrm{def}}{=} \{\tau \in \Gamma \backslash \Delta \mid \tau \text{ is a reflection}\}$$

cannot intersect A. That is, $A \cap L_\Delta = \varnothing$, where

$$L_\Delta \stackrel{\mathrm{def}}{=} \bigcup_{\tau \in K_\Delta} \mathrm{Fix}(\tau). \tag{2.1}$$

The set L_Δ separates $\mathbf{R}^n$ into connected components. These components play a crucial role in determining which symmetry groups of attractors are *admissible* in the sense that there exists a continuous Γ-equivariant dynamical system with a Σ-symmetric attractor. Precisely we define, using the notation of [**3**],

Definition 2.3. A subgroup Σ of a finite group $\Gamma < \mathbf{O}(n)$ is called *admissible* if there exists a continuous Γ-equivariant mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ with a Σ-symmetric attractor A.

Moreover, we call such a subgroup Σ *strongly admissible*, if f and A can be chosen such that A is connected.

An admissible subgroup which is not strongly admissible will be denoted *weakly admissible*.

Now we are in a position to state the main result in [**3**], which gives a classification of the admissible groups.

Theorem 2.4. *Let Γ be a finite subgroup of $\mathbf{O}(n)$, $n \geq 3$.*

(a) *A subgroup $\Delta < \Gamma$ is strongly admissible if and only if Δ fixes a connected component of $\mathbf{R}^n \backslash L_\Delta$.*

(b) *A subgroup $\Sigma < \Gamma$ is admissible if and only if Σ is a cyclic extension of a strongly admissible group Δ, i.e., if Δ is normal in Σ and Σ/Δ is cyclic.*

Remark 2.5. This theorem is also valid for $n = 1, 2$ except in the case when n equals 2 and Γ is cyclic. In this case both Γ and $\mathbf{1}$ are strongly admissible and all other subgroups are weakly admissible.

As stated in the theorem, every admissible subgroup Σ has a strongly admissible subgroup Δ such that Σ/Δ is cyclic. However, this subgroup need not be unique. For example, consider the group $\Gamma = D_1$ acting on $\mathbf{R}^2$ by a reflection. Then both $\mathbf{1}$ and $\mathbf{D}_1$ are strongly admissible. Hence we may choose either $\mathbf{D}_1$ or $\mathbf{1}$ as a strongly admissible subgroup of $\mathbf{D}_1$. The cyclic group is then either $\mathbf{D}_1/\mathbf{D}_1 \cong \mathbf{Z}_1$ or $\mathbf{D}_1/\mathbf{1} \cong \mathbf{Z}_2$.

In what follows we will describe a way in which to find a unique subgroup Δ. That is, given a Σ-symmetric attractor A, we want to determine a specific strongly admissible normal subgroup of Σ which we are going to call the *associated group* of A. For this purpose we need to explain some notation and results obtained in [**14**], where the necessary conditions of Theorem 2.4 were proved.

To begin, we let L be a subset of $\mathbf{R}^n$ and define $\mathcal{P}_L$ as the set containing L and all the preimages of L under f, that is,

$$\mathcal{P}_L = \bigcup_{k=0}^{\infty} f^{-k}(L).$$

Because of $f^{-1}(\mathcal{P}_L) \subset \mathcal{P}_L$, we then have a mapping

$$f : \mathbf{R}^n \backslash \mathcal{P}_L \longrightarrow \mathbf{R}^n \backslash \mathcal{P}_L.$$

Connected components of $\mathbf{R}^n \backslash \mathcal{P}_L$ are mapped into connected components, since f is continuous. For an attractor, we have the following result for these components (the proof can be found in [**14**]).

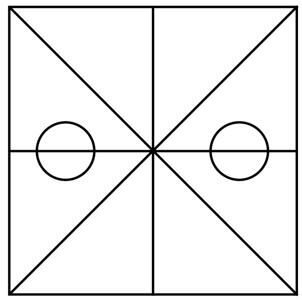


FIGURE 2. Sketch of a $\mathbf{D}_2$-symmetric attractor of a $\mathbf{D}_4$-equivariant system. The associated group Δ is generated by the reflection whose reflection hyperplane intersects the attractor.

Lemma 2.6. *Let $L \subset \mathbf{R}^n$ and, let A be an attractor of $f : \mathbf{R}^n \to \mathbf{R}^n$. Suppose that $A \cap L = \varnothing$. Then the following holds.*

(a) *The attractor A is covered by finitely many connected components $C_0, \ldots, C_{r-1}$ of $\mathbf{R}^n \backslash \mathcal{P}_L$.*

(b) *These components can be ordered such that $f(C_i) \subset C_{i+1 \bmod r}$ holds.*

As our set L, we choose the following set

$$L = \bigcup_{\substack{\tau \in \Gamma \text{ reflection,} \\ \mathrm{Fix}\,(\tau) \cap A = \varnothing}} \mathrm{Fix}\,(\tau)$$

and consider the connected components of $\mathbf{R}^n \backslash \mathcal{P}_L$. By Lemma 2.6, A is covered by finitely many of these components, which we denote by $C_0, \ldots, C_{r-1}$. Now let

$$\Delta = \{\delta \in \Sigma \mid \delta C_i = C_i \text{ for all } i \in \{0, \ldots, r-1\}\}. \tag{2.2}$$

In the proof of Theorem 4.10 in [**14**] it is shown that the group Δ as defined here is in fact a strongly admissible normal subgroup of Σ. Moreover, we even have $L_\Delta = L$ with this choice of Δ, see [**8**]. We define

Definition 2.7. Let A be a Σ-symmetric attractor, and suppose that Δ is constructed as in (2.2). Then we call Δ the *associated group* of A.

To illuminate the choice of the associated group, we consider the following example. Let $\Gamma = \mathbf{D}_4$, and suppose that A is a $\mathbf{D}_2$-symmetric attractor, outlined as shown in Figure 2. Of the copies of $\mathbf{D}_1$ contained in $\mathbf{D}_2$, just one may be chosen as the associated group Δ of $\mathbf{D}_2$, namely the one generated by the reflection whose reflection hyperplane intersects the attractor. This reflection is, apart from the identity, the only element in $\mathbf{D}_2$ fixing connected components of $\mathbf{R}^n \backslash L$, namely the ones in which the attractor is contained.

2.3 Admissible triples. We will now introduce our definition of a *symmetry increasing bifurcation* via a *collision of attractors*. To begin with, let us state some notations which we assume to be valid throughout the paper.

Notation 2.8. Let $\Gamma < \mathbf{O}(n)$ be a finite group. We assume that the group Δ is a strongly admissible subgroup of Γ; for a definition of a strongly admissible subgroup, see Definition 2.3, and for a classification of admissible subgroups, see Theorem 2.4. Moreover, let Σ be a cyclic extension of this group. We denote the order of Σ/Δ by p and choose an element $\rho \in \Sigma$ of a generator of Σ/Δ.

Now let κ be an element of $\Gamma \backslash \Delta$. We wish to consider collisions of attractors conjugated by κ. If κ is a reflection, we will allow $\kappa \in \Sigma$. Later, it will become apparent why we admit this case although it does not lead to an increase in symmetry, see Remark 4.31. If κ is not a reflection, we assume $\kappa \in \Gamma \backslash \Sigma$. Finally, we define L_Δ as in (2.1) and let D be a connected component of $\mathbf{R}^n \backslash L_\Delta$ which is fixed by Δ.

Definition 2.9. The Γ-equivariant mapping $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ undergoes a *symmetry increasing bifurcation via a collision of attractors* for $\lambda = \lambda_c$ if the following holds:

(C1) For a sequence $\lambda_j \nearrow \lambda_c$, $j \in \mathbf{N}$, the mapping $f(\cdot, \lambda_j)$ possesses Σ-symmetric attractors A_j. For all j, let Δ be the associated group of A_j.

(C2) There are sequences $x_j \in A_j \cap D$ and $y_j \in \kappa A_j \cap \kappa D$ such that x_j and y_j both converge to a periodic point x of $f(\cdot, \lambda_c)$ for $j \to \infty$.

(R) If κ is a reflection, let the isotropy group of x be $\{1, \kappa\}$.

(NR) If κ is not a reflection, we assume $x \in D \cap \kappa D$.

(C3) All A_j consist of precisely k connected components for some $k \in \mathbf{N}$.

(C4) There is a sequence $\tilde{\lambda}_j \searrow \lambda_c$ such that, for any $j \in \mathbf{N}$, there exists a $\tilde{\Sigma} \stackrel{\text{def}}{=} \langle \Sigma \cup \{\kappa\} \rangle$-symmetric attractor $\tilde{A}_j$ of $f(\cdot, \tilde{\lambda}_j)$.

If there exists a mapping f undergoing a symmetry increasing bifurcation, we say that the triple (Σ, Δ, κ) is *admissible.*

There are some technical assumptions in this definition which are clarified in the following remarks:

Remarks 2.10. (a) The assumption that the attractors collide at a *periodic point* x in (C2) can for our purposes be weakened to the assumption that the first k iterates of x have the same isotropy group as x itself, $k \in \mathbf{N}$ is defined in (C3). Note also that we do not require x to be the only point of collision.

(b) The assumptions (R) and (NR) can be understood as follows. If κ is a reflection and the attractors A_j are connected, A_j and κA_j must be separated by the reflection hyperplane $\operatorname{Fix}(\kappa)$, and hence we must have $x \in \operatorname{Fix}(\kappa)$. Of course, if the A_j's are not connected there might be other possible collisions which do not take place at $\operatorname{Fix}(\kappa)$, but we could then find an $s \in \mathbf{N}$ such that a collision in the mapping f^s would not be a collision of *conjugate* attractors, see [**12**].

We do not allow a larger isotropy group than $\{1, \kappa\}$ for x because after the collision we do not wish to observe an increase in symmetry beyond the symmetry group $\tilde{\Sigma} = \langle \Sigma \cup \{\kappa\} \rangle$. Observe that, from (R), it follows that

$$\dim(\operatorname{Fix}(\kappa) \cap \partial D) = n - 1.$$

This means that $\operatorname{Fix}(\kappa)$ is one of the reflection hyperplanes which are essentially forming the boundary of D.

If κ is not a reflection, then the collision should not take place at a reflection hyperplane, because again this could mean that after the collision there might be additional symmetries not contained in $\tilde{\Sigma} = \langle \Sigma \cup \{\kappa\} \rangle$. This leaves only the possibility of a collision inside D (one can show that all components of $\mathbf{R}^n \backslash L_\Delta$ which contain part of the A_j are fixed by Δ, hence D may be chosen accordingly). From the viewpoint of the attractors κA_j, the collision should then take place inside κD; hence, we ask that $x \in D \cap \kappa D$.

(c) The assumption (C3) is equivalent to the assumption that the number of connected components of the A_j is bounded. Considering the fact that an attractor which contains a periodic point of period m can have at most m connected components, see [**14**], this assumption is not too restrictive as well.

3. Necessary conditions. In this section we are going to derive necessary conditions for symmetry increasing bifurcations. As pointed out in the last section we have to distinguish two cases depending on whether κ is a reflection or not. In both cases we are going to find necessary conditions, which will later turn out to be sufficient as well.

3.1. κ is a reflection. First let us consider the case where κ is a reflection. Then the following holds.

Proposition 3.1. *Let $\kappa \in \Gamma \backslash \Delta$ be a reflection. If the triple (Σ, Δ, κ) is admissible, then it follows that*

$$\dim \left(\mathrm{Fix}\left(\sigma\kappa\sigma^{-1}\right) \cap \partial D\right) = n - 1 \tag{3.3}$$

for all $\sigma \in \Sigma$.

Proof. Suppose that the mapping f undergoes a symmetry increasing bifurcation. For each j, by Lemma 2.6 there exist connected components $C_0(\lambda_j), \dots, C_{r-1}(\lambda_j)$ of $\mathbf{R}^n \backslash \mathcal{P}_L$ covering A_j (by (C3) we may without loss of generality assume that r does not depend on j). We define $A_j^i = A_j \cap C_i(\lambda_j)$. The mapping $f(\cdot, \lambda_j)$ permutes the $C_i(\lambda_j)$ cyclically for fixed $j \in \mathbf{N}$, hence it permutes the $A_j^0, \dots, A_j^{r-1}$ as well.

Now consider the sequence $x_j \in A_j \cap D$, which exists by (C2). Without loss of generality we may assume that all elements of the sequence x_j are contained in A_j^0. As the $C_i(\lambda_j)$ cannot intersect the boundary of D, A_j^0 must be fully contained in D.

All iterates of x by $f(\cdot, \lambda_c)$ have the same isotropy $\{1, \kappa\}$, since this is true for x and x is a periodic point. As the A_j^i are being permuted cyclically by $f(\cdot, \lambda_j)$ we may find, in all A_j^i, sequences converging to

$\mathrm{Fix}\,(\kappa)$, in the point $f^i(x, \lambda_c)$. Apart from κ this point does not possess any other nontrivial isotropies.

Now let $\sigma \in \Sigma$. The A_j are Σ-symmetric, hence we may find a sequence in each A_j^i converging to a point with isotropy group $\{1, \sigma\kappa\sigma^{-1}\}$. In particular this holds for A_j^0. However, these components are fully contained in D, which proves the proposition. □

3.2 κ is not a reflection. We now turn to the case where κ is not a reflection.

Proposition 3.2. *Suppose that $\kappa \in \Gamma\backslash\Sigma$ is not a reflection. If the triple (Σ, Δ, κ) is admissible, then we have*

$$\sigma^{-1}\kappa\sigma D \cap D \neq \varnothing \tag{3.4}$$

for all $\sigma \in \Sigma$.

Proof. Suppose that the mapping f undergoes a symmetry increasing bifurcation. We take a closer look at the sequences $x_j \in A_j \cap D$ and $y_j \in \kappa A_j \cap \kappa D$, which exist by (C2). Both sequences converge to $x \in D \cap \kappa D$.

Again we use the sets $A_j^i = A_j \cap C_i(\lambda_j)$, which are permuted cyclically by f as stated in Lemma 2.6. Without loss of generality we may assume that x_j is contained in $A_j^0 \subset D$ for all j and $y_j \in \kappa A_j^l \subset \kappa D$ for some fixed $l \in \{0, \dots, r-1\}$.

For a given $\sigma \in \Sigma$, we may find a $k \in \{0, \dots, r-1\}$ such that, for all j, the equation $f^k(A_j^0, \lambda_j) = A_j^k = \sigma A_j^0 \subset \sigma D$ holds. We may conclude

$$\begin{aligned} f^k(x_j, \lambda_j) \in f^k(A_j^0, \lambda_j) &= \sigma A_j^0 \subset \sigma D \quad \text{and} \\ f^k(y_j, \lambda_j) \in \kappa f^k(A_j^l, \lambda_j) &= \kappa f^l(A_j^k, \lambda_j) \\ &= \kappa f^l(\sigma A_j^0, \lambda_j) \\ &= \kappa\sigma A_j^l \subset \kappa\sigma D. \end{aligned}$$

Now recall that $f^k(x, \lambda_c)$ is the limit of both sequences for $j \to \infty$, which implies $f^k(x, \lambda_c) \in \kappa\sigma\overline{D} \cap \sigma\overline{D}$. But $x \in D \cap \kappa D$; thus, x is not contained in any fixed point space of a reflection in K_Δ or in $K_{\kappa\Delta\kappa}^{-1}$.

Moreover, x is a periodic point; hence, the same is valid for all iterates of x under f. Thus we obtain $f^k(x, \lambda_c) \in \kappa\sigma D \cap \sigma D$, which proves the proposition. $\square$

4. Sufficient conditions. The proof that the conditions derived in the last section are sufficient as well requires much more work than the necessity. This is due to the fact that we have to construct mappings undergoing a symmetry increasing bifurcation. Since we rely heavily on the methods developed by Ashwin and Melbourne [**3**], we begin by giving an outline of their method of construction for symmetric attractors, which they use to prove Theorem 2.4.

4.1. Equivariant dynamics on graphs. Ashwin and Melbourne use embedded Σ-invariant graphs to construct Γ equivariant mappings with Σ-symmetric attractors. In this subsection we give an outline of their method of construction. For a sketch of these methods we begin with a simple lemma. Its proof may be found in [**3**, Lemma 4.1].

Lemma 4.1. *Let Γ be a finite group acting on the topological spaces Y and Z. Suppose that X is a closed subset of Y such that $Y = \cup_{\gamma \in \Gamma} \gamma X$. Finally, suppose that $f : X \to Z$ is a continuous mapping satisfying $f(\gamma x) = \gamma f(x)$ whenever x and γx are both contained in X for some $\gamma \in \Gamma$. Then f can be uniquely extended to a continuous Γ-equivariant mapping $f : Y \to Z$.*

We now consider graphs invariant under the operation of a finite group. For elementary graph theory the reader is referred to [**5**]. The aim is to embed these graphs into $\mathbf{R}^n$ and to turn these embedded graphs into attractors. Each graph may be seen as a metric space by identifying each edge with the unit interval, see [**3**]. Subsequently, we will do so without further notice. However, we note that this leads to an ambiguity in the definition of a graph. On the one hand, a graph is a set of vertices and edges; on the other hand, we look at a set of points with a metric structure. It should be clear from the context which of these viewpoints we assume in the given setting.

Definition 4.2. Let Δ be a finite group. A graph G is called a

Δ-*graph*, if

(i) Δ acts isometrically on G,

(ii) the set of edges (or equivalently the set of vertices) of G is invariant under the action of Δ on G, and

(iii) Δ acts fixed point freely on the set of edges of G.

A subgraph $J \subset G$ of a Δ-graph G is called a *fundamental subgraph*, if $G = \cup_{\delta \in \Delta} \delta J$ holds and, moreover, for each edge $E \in J$ and every $\delta \in \Delta$ we have $\delta E \in J$ if and only if $\delta = \mathrm{id}$.

Assumption (iii) is equivalent to the existence of a fundamental subgraph, as can easily be shown. In particular, for every Δ-graph G there exists a fundamental subgraph.

The reason for introducing Δ-graphs is that we may define a Δ-equivariant dynamical system on such a graph such that the whole graph becomes an ω-limit set of this system. This is the contents of the following proposition by Adler and Flatto, as stated in the appendix of [**3**], see also [**1**].

Proposition 4.3. *Suppose that G is a finite graph with edges $E_1, \dots, E_m$, and let $g : G \to G$ be a continuous mapping with the following properties.*

(i) *For every j, $g(E_j)$ is the union of certain other E_i's.*

(ii) *Let $E_{ij} = E_i \cap g^{-1}(E_j)$. Then $g|_{E_{ij}}$ is an invertible C^2-map.*

(iii) *There exist $q \in \mathbf{N}$ and $\theta > 1$ such that $|(g^q)'| \geq \theta$ wherever it is defined.*

(iv) *For all j we have $\cup_{p \in \mathbf{N}} g^p(E_j) = G$.*

Then G is topologically transitive, periodic points are dense in G and there is sensitive dependence on initial conditions. Moreover, there exists a g-invariant ergodic Lebesgue-equivalent measure on G. If, instead of (iv)*, we have the stronger property*

(iv)$'$ *There exists $p \in \mathbf{N}$ such that $g^p(E_j) = G$ for all $j \in \{1, \dots, m\}$, then G is even topologically mixing.*

For a definition of the properties names in the definition, we refer the reader to [**3**]. We proceed by defining a Δ-equivariant dynamical system with the help of the previous proposition. The *degree* of a vertex is the number of edges emanating from this vertex. With this notation a *Eulerian* graph is defined as a graph having an even degree at each vertex. It is well known that, for any vertex of a Eulerian graph, there exists a path starting and ending at this vertex and passing through each edge of the graph precisely once.

Theorem 4.4. *Suppose that G is a Eulerian Δ-graph where each vertex has at least degree four. Then there exists a continuous Δ-equivariant dynamical system $g : G \to G$ with the following properties.*

(i) *G is topologically mixing; in particular, it is topologically transitive.*

(ii) *Periodic points are dense in G and there is sensitive dependence on initial conditions.*

(iii) *There is a unique g-invariant, and Lebesgue-equivalent, ergodic measure on G.*

(iv) *The vertices of G are fixed points of g.*

(v) *Let J be a fundamental subgraph of G, and let an arbitrary function $F : J \to G$ be given, assuming to each edge $E \in J$ an edge $F(E)$ of G. Suppose that x_E denotes the midpoint of the edge E; then g may be constructed such that $g(x_E) = x_{F(E)}$ for all $E \in J$.*

The proof of the first three conclusions may be found in [**3**, Theorem 4.3] (with the restriction on the degree of the vertices). We have added (iv) and (v) since we are going to need them for the subsequent sections. To prove these statements, it is necessary to give a short outline of the construction in [**3**].

Sketch of Proof. We choose an arbitrary fundamental subgraph J of G and remove an edge E from this subgraph. Since G is Eulerian, we may find a path in $G \backslash E$ connecting the vertices v and w which correspond to E, and passing through every edge of $G \backslash E$ precisely once. We let $g(v) = v$, $g(w) = w$ and $g(E) = G \backslash E$ such that, while $x \in E$ moves from v to w, the point $g(x)$ follows the path chosen above. In this way we may define g on every edge of the fundamental subgraph and extend g using Lemma 4.1 to obtain a Δ-equivariant mapping $g : G \to G$.

We are free to change the parametrization of g on the edges. This allows us to construct g in such a way that the requirements of Proposition 4.3 (with $q = 1$ and $p = 2$) are satisfied. From the construction it is immediately apparent that (iv) is also valid.

It remains to show (v). Let a fundamental subgraph J and a function F as in (v) be given. Since the construction of g on each edge of the fundamental subgraph is independent of the outer edges, it suffices to consider each edge separately. Suppose that some arbitrary edge $E \in J$ is given. There are two cases we have to consider.

If $F(E) = E$, we temporarily consider the midpoint $x_E \in E$ as a new vertex. Then E consists of two new edges which we treat precisely as described above when defining g. With this definition of g, x_E becomes a fixed point of g and hence we have $g(x_E) = x_E = x_{F(E)}$.

On the other hand, if $F(E) \neq E$, by the above definition of g the midpoint $x_{F(E)} \in F(E)$ lies in the image of E under g. Hence we may parametrize g on E such that $g(x_E) = x_{F(E)}$ is satisfied. However, when looking at the properties in Proposition 4.3, we have to pay special attention to (iii). No difficulty occurs if, while transversing the path, the edge $F(E)$ is not hit as the first or the last one (in the chosen metric on the graph all edges have the same length). But we had assumed previously that each of our vertices has at least degree four, which means that apart from E at least three other edges are connected with the vertices v and w. This implies that we may always choose a path from v to w which does transverse $F(E)$ as neither the first nor the last edge in the path. $\square$

Now we want to embed the Δ-graph into $\mathbf{R}^n$. It is therefore necessary to define the concepts of *embeddability* and *extendability* of a Δ-graph.

Definition 4.5. A Δ-*equivariant embedding* of a Δ-graph G into $\mathbf{R}^n$ is a continuous one-to-one mapping $e : G \to \mathbf{R}^n$ such that the following assumptions hold:

(i) e is Δ-equivariant.

(ii) $\gamma e(G) \cap e(G) = \varnothing$ for all $\gamma \in \Gamma \backslash \Delta$.

Using this concept we are in a position to embed the graph of a group Δ fixing a connected component of $\mathbf{R}^n \backslash L_\Delta$. The aim is to turn the image of this graph in $\mathbf{R}^n$ into a Δ-symmetric attractor. If Σ is a cyclic extension of such a group Δ, we are going to construct the attractor by *extending* the Δ-graph. The precise definition is as follows:

Definition 4.6. Let $\rho \in N(\Delta)\backslash\Delta$. A Δ-graph G is called ρ-*extendable* if there exists a Δ-equivariant isometry $h : G \to \rho G$. This isometry is called a ρ-extension. If G may be ρ-extended for any $\rho \in N(\Delta)\backslash\Delta$, we say that G is *extendable*.

We note that the graph ρG mentioned in this definition is defined formally as the set $\{\rho x \mid x \in G\}$. Since ρ is an element of the normalizer of Δ, we may define an operation of Δ on ρG by $\delta(\rho E) \stackrel{\text{def}}{=} \rho(\rho^{-1}\delta\rho E)$ for any edge $E \in G$ and any $\delta \in \Delta$.

The following Theorem 5.4 from [**3**] leads directly to the proof of Theorem 2.4.

Theorem 4.7. *If there exists a Eulerian, embeddable and extendable Δ-graph, then Δ is strongly admissible, and any cyclic extension of Δ is admissible.*

To prove Theorem 2.4, it remains to show that for any subgroup Δ of Γ fixing a connected component D of $\mathbf{R}^n\backslash L_\Delta$, we may construct a Eulerian, embeddable and extendable Δ-graph. For such Δ, the following lemma is implicit in [**3**, Section 6].

Lemma 4.8. *Let $n \geq 3$, and suppose that Δ fixes a connected component D of $\mathbf{R}^n\backslash L_\Delta$. Then there exists a Δ-graph and a Δ-equivariant embedding of this graph into D. Moreover, this Δ-graph is Eulerian and extendable.*

Remark 4.9. A short note on the construction of the graph and its embedding will make the arguments in the following sections more transparent. The difficulty of the construction lies in the fact that Δ itself may contain reflections. In this case the component D is separated into fundamental domains of the subgroup Δ_R of Δ generated by its reflections. The Δ-graph is then constructed in such a way that its embedding has a fundamental subgraph J contained entirely inside the closure of one of these fundamental domains, with its vertices lying inside the reflection hyperplanes of reflections in Δ which bound this fundamental domain. In this way, images of the edges of J under elements of Δ will intersect these reflection hyperplanes only in vertices, and hence the embedding remains one-to-one.

If Δ does not contain any reflections, the construction of the Δ-graph and its embedding is much less complicated. For our purposes it suffices to note that in this case there are no $(n-1)$-dimensional obstructions inside D, and hence each embedded edge may be connected with the boundary of D by a curve which does not intersect any fixed point space.

4.2 κ is a reflection. In this section we aim to construct symmetry increasing bifurcations for the case when κ is a reflection. Using the methods developed by Ashwin and Melbourne, as outlined in the preceding section, we are able to prove that the necessary condition found in Proposition 3.1 is indeed sufficient for symmetry increasing bifurcations.

Theorem 4.10. *We assume Notations* 2.8 *and let* $\kappa \in \Gamma\backslash\Delta$ *be a reflection. The triple* (Σ, Δ, κ) *is admissible if and only if*

$$\dim(\operatorname{Fix}(\sigma\kappa\sigma^{-1} \cap \partial D) = n - 1 \tag{4.5}$$

for all $\sigma \in \Sigma$.

Since we have shown the necessity of (4.5) in Proposition 3.1, it remains to construct a mapping undergoing a symmetry increasing bifurcation for a given triple (Σ, Δ, κ) satisfying (4.5).

Throughout the proof we shall assume $n \geq 3$. The cases $n = 1$ and $n = 2$ will be treated separately in Section 4.4.

An overview of the proof is as follows. The first step is to construct a suitable Δ-graph and a corresponding embedding e into $\mathbf{R}^n$; this we accomplish in Lemma 4.11. We then introduce a parameter λ into the embedding and, by varying this parameter, move the embedded edges of the graph towards the reflection hyperplanes of the elements $\sigma\kappa\sigma^{-1}$ such that they touch these spaces for $\lambda = 1$, see Lemma 4.12.

Precisely as in [**3**] we define the dynamics f on the embedded graph by first choosing a mapping g on the graph and then conjugating this mapping with the previously defined embedding e, that is, we let $f(\cdot,\lambda) = e(\cdot,\lambda)\circ g\circ e^{-1}(\cdot,\lambda)$. We then want to extend f Γ-equivariantly. However, for $\lambda = 1$ the embedded graphs touch conjugate embedded graphs, which corresponds to a collision of conjugate ω-limit sets in $\mathbf{R}^n$. At this critical point we have to take special care not to destroy the equivariance of the mapping f. This problem is an essential point in the proof, and we treat it in Lemma 4.16.

After the collision, for $\lambda > 1$, we have to change the dynamics on the embedded graph to obtain $\tilde{\Sigma}$-symmetric ω-limit sets instead of merely Σ-symmetric ones. We do this by enlarging the image of a particular embedded edge under the mapping f, such that it additionally contains small pieces of the conjugate graphs. To ensure the continuity of f, these pieces have to shrink when $\lambda \searrow 1$. For this purpose we introduce new edges into the embedded graph, see Lemma 4.18, whose length goes to zero as $\lambda \searrow 1$. We then expand the image of the above mentioned embedded edge under f onto these new edges. In Lemma 4.19 we show that, by using this method, we indeed obtain $\tilde{\Sigma}$-symmetric ω-limit sets after the collision. Afterwards we have to extend the mapping f equivariantly onto all of $\mathbf{R}^n$, such that the constructed ω-limit sets become attractors. This is accomplished in Lemma 4.20.

Now we turn to the proof. In the following lemma we guarantee the existence of a Δ-graph fulfilling the requirements we will need later on.

Lemma 4.11. *There exists a finite Eulerian Δ-graph $G(\Delta)$ with the following properties. The graph is extendable and embeddable into D, contains a fundamental subgraph numbering at least $(2p+1)$ edges, and all vertices of the graph have at least degree four.*

Proof. An embeddable and extendable Eulerian Δ-graph exists by Lemma 4.8. If this graph does not satisfy the properties of the lemma, for each edge E of the fundamental subgraph we introduce another edge E' connecting the same vertices as E. We extend this construction Δ-equivariantly. The new edges may be embedded into $\mathbf{R}^n$ without difficulty, since $n \geq 3$. Moreover, we may easily extend any ρ-extension h of the graph onto the new edges: for each "new" edge E' we simply let $h(E') = h(E)'$ where $h(E)'$ denotes the "new" edge corresponding to the "old" edge $h(E)$. This extension of h is clearly a ρ-extension of the new graph.

Now we have obtained a graph where the number of edges as well as the degree of the vertices has doubled. Since the former graph was Eulerian, each vertex of the new graph must have at least degree four. If the new fundamental subgraph still contains less than $(2p+1)$ edges, we repeat the construction until this requirement is satisfied. □

Let us now choose a graph $G(\Delta)$ satisfying the properties of the lemma. We take p edges $E_0, \dots, E_{p-1}$ from the fundamental subgraph J of this graph, and we call their midpoints $x_i \in E_i$ for all $i \in \{0, \dots, p-1\}$. Later on, we will choose an additional edge E_p.

The next step is to introduce a parameter into the embedding belonging to the graph $G(\Delta)$ and, by varying this parameter, to move the embedded edges such that for a given parameter value they touch the fixed point spaces $\mathrm{Fix}\,(\sigma\kappa\sigma^{-1})$, $\sigma \in \Sigma$.

Lemma 4.12. *There exists a continuous mapping $e : G(\Delta) \times [0,1] \to \mathbf{R}^n$ with the following properties.*

(a) *For any $\lambda \in [0,1)$, the mapping $e(\cdot, \lambda)$ is a Δ-equivariant embedding of $G(\Delta)$ into D.*

(b) *$e(\cdot, 1)$ is a Δ-equivariant one-to-one mapping of $G(\Delta)$ into $\overline{D}$ and, for each $i \in \{0, l \dots, p-1\}$, there is a $\sigma_i \in \bar{\rho}^i$, such that*

(i) $y_i \stackrel{\mathrm{def}}{=} e(x_i, 1) \in \mathrm{Fix}\,(\sigma_i\kappa\sigma_i^{-1}) \cap \partial D$,

(ii) *the Γ-group orbits of the y_i are pairwise disjoint and do not coincide with Γ-group orbits of the embedded vertices of $G(\Delta)$, and*

(iii) *each y_i possesses the isotropy $\{1, \sigma_i\kappa\sigma_i^{-1}\}$.*

Proof. By Lemma 4.11 we know that for $G(\Delta)$ there exists an embedding into D. We denote this embedding by $e(\cdot, 0) : G(\Delta) \to \mathbf{R}^n$.

The σ_i are chosen as follows. If Δ contains reflections, then denote the subgroup of Δ generated by its reflections by Δ_R. The connected component D is then separated into fundamental domains by the reflection hyperplanes of the elements in Δ_R, see Remark 4.9. By this remark, the embedding of the fundamental subgraph J is contained in the closure of one of these fundamental domains F. By (4.5) and, since elements of Δ_R act transitively on these fundamental domains, for each $i \in \{0, \dots, p-1\}$ the fundamental domain F is bounded by at least one reflection hyperplane $\mathrm{Fix}\,(\sigma_i \kappa \sigma_i^{-1})$ for some $\sigma_i \in \bar{\rho}^i$.

If Δ does not contain any reflections, we choose the $\sigma_i \in \bar{\rho}^i$ arbitrarily.

Now we claim that, for each $i \in \{0, \dots, p-1\}$, the embedded midpoint $e(x_i, 0)$ may be connected with the fixed point space $\mathrm{Fix}\,(\sigma_i \kappa \sigma_i^{-1})$ by a continuous curve which does not intersect any other edges or fixed point spaces. We may do this since we have chosen the σ_i precisely such that $e(x_i, 0)$ is not separated from $\mathrm{Fix}\,(\sigma_i \kappa \sigma_i^{-1})$ by any other reflection hyperplane. Moreover, the dimension of the space is at least three and the group Γ is finite, which means that we may move around other edges or fixed point spaces.

Using these considerations we now move the embedded edges E_i continuously towards the fixed point spaces $\mathrm{Fix}\,(\sigma_i \kappa \sigma_i^{-1})$. We do this by fixing the vertices and moving the midpoints $e(x_i, 0)$ of the embedded edges $e(E_i, 0)$ towards the fixed point spaces. The edges in $J \backslash \{E_0, \dots, E_{p-1}\}$ remain fixed. Since $n \geq 3$ and Γ is finite, we may do this in such a way that in orbit space, the embedded edges never intersect themselves or the images of other embedded edges. We also may avoid fixed point spaces of nontrivial groups. We will denote the homotopy which we have obtained in this way by $e : J \times [0,1] \to \mathbf{R}^n$.

Let e be parametrize in such a way that all embedded edges $e(E_i, \lambda)$ touch the corresponding fixed point spaces simultaneously for $\lambda = 1$ in precisely one point, which is $y_i = e(x_i, 1)$. We may assume that this point has $\sigma_i \kappa \sigma_i^{-1}$ as the only nontrival isotropy. Since we may perturb the y_i inside $\mathrm{Fix}\,(\sigma_i \kappa \sigma_i^{-1})$ and since Γ is finite, we may fulfill the assumptions (ii) and (iii) by a suitable choice of the y_i's.

Using Lemma 4.1, for each $\lambda \in [0,1]$ we extend the mapping $e(\cdot, \lambda)$ equivariantly. It remains to show that $e(\cdot, \lambda)$ is one-to-one. We fix $\lambda \in [0,1]$ and suppose for contradiction that $e(E, \lambda)$ and $e(\tilde{E}, \lambda)$ have a nonempty intersection, where $E, \tilde{E}$ are arbitrary edges in $G(\Delta)$. If there is no $\delta \in \Delta \backslash \mathbf{1}$ satisfying $\delta E = \tilde{E}$, then the embedded edges $e(E, \lambda)$ and $e(\tilde{E}, \lambda)$ are not conjugate to each other and hence they belong to different edges of the fundamental subgraph in orbit space. By our construction they cannot intersect each other.

Hence, suppose that $\delta E = \tilde{E}$ for some $\delta \in \Delta\backslash\mathbf{1}$. We then choose a parametrization s of $e(E, \lambda)$ and define the induced parametrization $\tilde{s}$ on $e(\tilde{E}, \lambda)$ by $\tilde{s}(t) = \delta s(t)$. From $s(t_1) = \tilde{s}(t_2)$ we compute $\delta\tilde{s}(t_2) = \delta s(t_1) = \tilde{s}(t_1)$. Then we have either $t_1 = t_2$ which means that $e(\tilde{E}, \lambda)$ intersects the fixed point space $\mathrm{Fix}\,(\delta)$, or $t_1 \neq t_2$, and in this case the curve $e(\tilde{E}, \lambda)$ intersects itself in orbit space. However, we had excluded both cases during the construction of the homotopy e. We may conclude that indeed $e(\cdot, \lambda)$ is one-to-one for all $\lambda \in [0, 1]$. $\square$

Up to now we have mainly considered the strongly admissible group Δ. Now we turn to the case $\Sigma \neq \Delta$, in which we need to extend the Δ-graph $G(\Delta)$ to a Σ-graph. As in [**3**], we define

$$G(\Sigma) = G(\Delta)\dot{\cup}\rho G(\Delta)\dot{\cup}\cdots\dot{\cup}\rho^{p-1}G(\Delta).$$

We apply Lemma 4.1 to the mapping e constructed in Lemma 4.12 to obtain a Σ-equivariant mapping $e : G(\Sigma) \times [0, 1] \to \mathbf{R}^n$. For each $\lambda < 1$ the mapping $e(\cdot, \lambda)$ is a Σ-equivariant embedding of $G(\Sigma)$. For $\lambda = 1$, there are two cases to be considered.

Remark 4.13. (i) $\kappa \notin \Sigma$. The σy_i are pairwise disjoint for all $\sigma \in \Sigma$ and $i \in \{0, \ldots, p-1\}$, since we had chosen the y_i to lie on different group orbits and to have $\sigma_i\kappa\sigma_i^{-1}$ as the only nontrivial isotropy. Hence the images $e(\rho^i G(\Delta), 1)$ are pairwise disjoint for all $i \in \{0, \ldots, p-1\}$, which implies that $e(\cdot, 1)$ is one-to-one.

(ii) $\kappa \in \Sigma$. Using $y_i = \sigma_i\kappa\sigma_i^{-1}y_i$ for all $i \in \{0, \ldots, p-1\}$, we arrive at

$$e(\sigma x_i, 1) = \sigma y_i = e(\sigma\sigma_i\kappa\sigma_i^{-1}x_i, 1) \quad \text{for all } \sigma \in \Sigma.$$

This means that $e(\cdot, 1)$ is no longer one-to-one. Actually, for each $\sigma \in \Sigma$ and $i \in \{0, \ldots, p-1\}$ there are precisely two preimages of σy_i, and the mapping is one-to-one on all other points of the graph. However, we may restrict the mapping e to a one-to-one mapping as follows. If $\sigma \in \bar{\rho}^k$, then $\sigma\sigma_i\kappa\sigma_i^{-1}$ is contained in $\bar{\rho}^{k+p/2}$, since by Lemma 3.2 in [**8**] in this case p is even and $\kappa \in \bar{\rho}^{p/2}$. We let

$$\hat{\Sigma} \stackrel{\mathrm{def}}{=} \{\rho^i\delta \mid p/2 \leq i < p, \delta \in \Delta\}$$

and denote by $\hat{e}(\cdot, 1)$ the restriction of $e(\cdot, 1)$ to

$$\hat{G}(\Sigma) \stackrel{\text{def}}{=} G(\Sigma) \backslash \{\sigma x_i \mid \sigma \in \hat{\Sigma}, i \in \{0, \dots, p-1\}\}.$$

Our considerations imply that $\hat{e}(\cdot, 1)$ is one-to-one.

We now construct the dynamics on the graph using Theorem 4.4. Precisely we choose a mapping g_Δ on $G(\Delta)$ satisfying the assumptions (i) to (v) of the theorem. In Lemma 4.11 $G(\Delta)$ was constructed in a way which allows us to apply this theorem.

Case 1. $\Sigma = \Delta$. We choose a function F as in (v) of Theorem 4.4, such that this function satisfies $F(E_0) = E_0$. On all other edges of the fundamental subgraph F may be defined arbitrarily. Hence, the mapping g_Δ then obtained by the theorem satisfies $g_\Delta(x_0) = x_0$. We let

$$f(z, \lambda) \stackrel{\text{def}}{=} e(g_\Delta(e^{-1}(z, \lambda)), \lambda)$$

for all $\lambda \in [0, 1]$ and $z \in e(G(\Delta), \lambda)$. By Lemma 4.12, the inverse of e on $e(G(\Delta), \lambda)$ exists by Remark 4.13 (i) for any fixed value of λ. Now f is topologically conjugate to g_Δ which implies that f satisfies all the properties of Theorem 4.4.

Case 2. $\Sigma \neq \Delta$. For each $i \in \{0, \dots, p-1\}$, the image of the graph $\rho G(\Delta)$ under $e(\cdot, 1)$ contains a point $\tilde{y}'_i$ inside $\mathrm{Fix}\,(\sigma_i \kappa \sigma_i^{-1})$. This is due to the construction of e. The set $e(G(\Delta), 1)$ intersects $\mathrm{Fix}\,(\sigma \kappa \sigma^{-1})$ for any $\sigma \in \Sigma$, and by equivariance the same holds for $e(\rho G(\Delta), 1) = \rho e(G(\Delta), 1)$. We denote the preimage of $\tilde{y}'_i$ under $e(\cdot, 1)$ by $\tilde{x}'_i \in \rho G(\Delta)$. If $\kappa \in \Sigma$, this preimage is not well defined, see Remark 4.13 (ii), so in this case we choose the preimage of $\tilde{y}'_i$ under the mapping $\hat{e}$ defined in this remark. Using the Σ-equivariance of e we see that $\tilde{x}'_i$ is the midpoint of an edge in $\rho G(\Delta)$.

Moreover, there exists a ρ-extension $h : G(\Delta) \to \rho G(\Delta)$ by Lemma 4.11. For each $i \in \{0, \dots, p-1\}$, we denote by x'_i the preimage of $\tilde{x}'_i$ under h. This point is again the midpoint of an edge $E'_i \in G(\Delta)$, since h is an isometry.

We now define a function F, later to be used when applying Theorem 4.4, on the edges of the fundamental subgraph J as follows. On the subset $\{E_0, \dots, E_{p-1}\}$ of J we let $F(E_i) = E_i'$. On all other edges F may be defined arbitrarily. However, for subsequent considerations we will need an additional property of F, which can also be satisfied in the strongly admissible case.

Lemma 4.14. *There exists an edge E_p in the fundamental subgraph J, $E_p \neq E_i$ for all $i \in \{0, \dots, p-1\}$, and the function F may be chosen such that*

$$F(E) \neq \delta E_p$$

for all $\delta \in \Delta$ and all edges $E \in J$.

Proof. By Lemma 4.11, the fundamental subgraph J contains at least $(2p+1)$ edges. From these we had chosen p edges E_i, $i = 0, \dots, p-1$, and at most p more edges of J are conjugate to the edges E_i', $i = 0, \dots, p-1$ defined above. Hence, there remains at least one edge E_p in the fundamental subgraph J which is not conjugate to any of the above mentioned edges. We have chosen F to satisfy $F(E_i) = E_i'$. On all other edges, we may define F arbitrarily. Hence, there is a choice of F which satisfies the assumption stated in the lemma. Note that the proof works as well in the strongly admissible case. $\square$

We will need this lemma later on for the construction of attractors after the collision. Now we proceed by choosing a mapping $g_\Delta : G(\Delta) \to G(\Delta)$ using Theorem 4.4 with the function F chosen above. Then g_Δ satisfies $g_\Delta(x_i) = x_i'$ for all $i \in \{0, \dots, p-1\}$.

For an overview of the correlations between the mappings e, g_Δ and h we include the following diagram:

$$\begin{array}{cccccccccc} e(G(\Delta),1) & \xrightarrow{e^{-1}} & G(\Delta) & \xrightarrow{g_\Delta} & G(\Delta) & \xrightarrow{h} & \rho G(\Delta) & \xrightarrow{e} & \rho e(G(\Delta),1) \\ y_i & \longmapsto & x_i & \longmapsto & x_i' & \longmapsto & \tilde{x}_i' & \longmapsto & \tilde{y}_i' \end{array} \tag{4.6}$$

We now extend the mapping $h \circ g_\Delta$ Σ-equivariantly to obtain a mapping $g_\Sigma : G(\Sigma) \to G(\Sigma)$ using Lemma 4.1. For g_Σ, the property (iv)$'$ in Proposition 4.3 does not hold anymore. Hence, $G(\Sigma)$ is not topologically mixing, but at least topologically transitive, since property (iv) of this proposition is still valid. This implies that all properties of Theorem 4.4 except (i) remain true for g_Σ.

For $\kappa \notin \Sigma$, we let

$$f(z,\lambda) \stackrel{\text{def}}{=} e(g_\Sigma(e^{-1}(z,\lambda)),\lambda)$$

for all $\lambda \in [0,1]$ and all $z \in e(G(\Sigma,),\lambda)$ (note that by Remark 4.13 (i) e is one-to-one in this case). The mapping f is topologically conjugate to g_Σ, hence possessing all properties of g_Σ we have listed above. In particular, f is topologically transitive and has a Σ-symmetric ω-limit set.

For $\kappa \in \Sigma$, we are going to use the mapping $\hat{e}$ defined in Remark 4.13 (ii). We have to show that in this manner we obtain a well-defined mapping f:

Lemma 4.15. *Let $\kappa \in \Sigma$. Then the mapping*

$$f(z,\lambda) \stackrel{\text{def}}{=} \begin{cases} e(g_\Sigma(e^{-1}(z,\lambda)),\lambda) & \text{for } \lambda \in [0,1), z \in e(G(\Sigma),\lambda), \\ e(g_\Sigma(\hat{e}^{-1}(z,1)),\lambda) & \text{for } \lambda = 1,\ z \in e(G(\Sigma),1), \end{cases}$$

is well-defined and continuous.

Proof. By Remark 4.13 (ii) we already know that $\hat{e}$ is one-to-one and hence f is well defined. The continuity of f may only be violated in the points σy_i for $i \in \{0,\dots,p-1\}$ and $\sigma \in \Sigma$. Therefore, we have to check that whether we use σx_i or $\sigma\sigma_i\kappa\sigma_i^{-1}x_i$ as a preimage for σy_i under $e(\cdot,1)$, we get the same result when applying $e(\cdot,1)\circ g_\Sigma$ to these points. We compute, see (4.6),

$$e(g_\Sigma(\sigma x_i),1) = \sigma e(h(g_\Delta(x_i)),1) = \sigma e(h(x_i'),1) = \sigma e(\tilde{x}_i',1) = \sigma\tilde{y}_i'$$

and

$$e(g_\Sigma(\sigma\sigma_i\kappa\sigma_i^{-1}x_i),1) = \sigma\sigma_i\kappa\sigma_i^{-1}e(h(g_\Delta(x_i)),1) = \sigma\sigma_i\kappa\sigma_i^{-1}\tilde{y}_i' = \sigma\tilde{y}_i',$$

since the isotropy group of $\tilde{y}_i'$ is $\{1,\sigma_i\kappa\sigma_i^{-1}\}$. □

We have succeeded in defining the mapping f on the embedded Σ-graph for all possible cases. Now we wish to extend it to a Γ-equivariant mapping on

$$B \stackrel{\text{def}}{=} \{(x,\lambda) \mid \lambda \in [0,1], x \in \Gamma e(G(\Delta),\lambda)\}.$$

Lemma 4.16. *Let f be defined as above. Then f can be extended to a Γ-equivariant mapping $f : B \to \mathbf{R}^n$.*

Proof. For each $\lambda \in [0,1)$, we want to apply Lemma 4.1 to $f(\cdot,\lambda)$ to obtain an Γ-equivariant mapping on $\Gamma e(G(\Delta),\lambda)$. There are no obstructions for $\lambda \in [0,1)$. Here, $e(G(\Delta),\lambda)$ is fully contained in D and does not intersect conjugate sets. For $\lambda = 1$, however, we have to show $f(\gamma z, 1) = \gamma f(z,1)$, whenever there exists $\gamma \in \Gamma$ such that, for some z, both z and γz are contained in $e(G(\Sigma),1)$.

By definition, $f(\cdot,1)$ is Σ-equivariant on $e(G(\Sigma),1)$, and this set is contained in

$$(\Sigma D) \cup \bigcup_{\sigma\in\Sigma} \mathrm{Fix}\,(\sigma\kappa\sigma^{-1}).$$

Hence, we need only consider the case $z = \sigma y_i$ for some $i \in \{0,\dots,p-1\}$, $\sigma \in \Sigma$ and $\gamma \in \Gamma\backslash\Sigma$. By assumption we have $\gamma z \in e(G(\Sigma),1)$, implying that there must be a $\tilde\sigma \in \Sigma$ such that $\gamma\sigma y_i = \tilde\sigma y_i$ (we have used Lemma 4.12 (b) (ii) in this step).

Moreover, the element $\sigma_i\kappa\sigma_i^{-1}$ is the only nontrivial isotropy of y_i which allows us to conclude $\gamma\sigma = \tilde\sigma\sigma_i\kappa\sigma_i^{-1}$ (otherwise we would have $\gamma\sigma = \tilde\sigma$, which can only hold if γ is contained in Σ).

To verify the equation $\gamma f(\sigma y_i,1) = f(\gamma\sigma y_i,1)$, we begin by calculating the value of $f(y_i,1)$. We ask the reader to recall the definitions of x_i, x_i' and $\tilde x_i'$ as displayed in (4.6). Suppose that $\Sigma \neq \Delta$ and $\kappa \notin \Sigma$, then we compute

$$\begin{aligned} f(y_i,1) &= e(h\circ g_\Delta(e^{-1}(y_i,1)),1) = e(h\circ g_\Delta(x_i),1)\\ &= e(h(x_i'),1)\\ &= e(\tilde x_i',1) = \tilde y_i'. \end{aligned}$$

Using this consideration we conclude

$$\begin{aligned} f(\gamma z,1) &= f(\gamma\sigma y_i,1) = f(\tilde\sigma\sigma_i\kappa\sigma_i^{-1}y_i,1) = \tilde\sigma f(y_i,1) = \tilde\sigma\tilde y_i'\\ &= \tilde\sigma\sigma_i\kappa\sigma_i^{-1}\tilde y_i' = \gamma\sigma f(y_i,1) = \gamma f(\sigma y_i,1) = \gamma f(z,1), \end{aligned}$$

since we had chosen $\tilde y_i$ to have the isotropy $\sigma_i\kappa\sigma_i^{-1}$.

In the case $\kappa \in \Sigma$ there is nothing to show, since $\gamma\sigma = \tilde\sigma\sigma_i\kappa\sigma_i^{-1}$ implies $\gamma \in \Sigma$. Finally, if $\Sigma = \Delta$, we arrive at the same conclusion.

Note, however, that in this case we have $p = 1$, $f(y_0, 1) = y_0$, and there is no ρ-extension h, which simplifies the calculations.

We have shown that even for $\lambda = 1$ the Lemma 4.1 may be applied. Hence, for all $\lambda \in [0, 1]$ we obtain a Γ-equivariant mapping $f(\cdot, \lambda)$ on $\Gamma e(G(\Delta), \lambda)$. The mapping $f : B \to \mathbf{R}^n$ is even continuous, since the extension by Lemma 4.1 does not destroy the continuity. $\square$

The next step is to construct a $\tilde{\Sigma}$-symmetric ω-limit set for $\lambda > 1$. We remind the reader that we have defined $\tilde{\Sigma} = \langle \Sigma \cup \{\kappa\} \rangle$. From the following construction it will become apparent that $\tilde{\Sigma}$ is indeed an admissible subgroup of Γ with strongly admissible subgroup

$$\tilde{\Delta} = \langle \Delta \cup \{\sigma\kappa\sigma^{-1} \mid \sigma \in \Sigma\} \rangle. \tag{4.7}$$

Case 1. $\kappa \in \Sigma$. Here the construction of a $\tilde{\Sigma}$-symmetric ω-limit set turns out to be no problem at all. Since $\Sigma = \tilde{\Sigma}$, such a limit set already exists for $\lambda = 1$. To simplify notation, we let $G_\lambda \stackrel{\text{def}}{=} e(G(\Delta), \lambda)$ for $\lambda \in [0, 1)$ and $G_\lambda \stackrel{\text{def}}{=} e(G(\Delta), 1)$ for $\lambda \in [1, 2]$. Then we may extend f to the following set

$$\tilde{B} \stackrel{\text{def}}{=} \{(x, \lambda) \mid \lambda \in [0, 2], x \in \Gamma G_\lambda\}.$$

Lemma 4.17. *Let $\kappa \in \Sigma$. Then $f : B \to \mathbf{R}^n$ may be extended to a continuous Γ-equivariant mapping*

$$f : \tilde{B} \to \mathbf{R}^n$$

such that for each $\lambda > 1$ the mapping $f(\cdot, \lambda)$ possesses a $\tilde{\Sigma}$-symmetric ω-limit set.

Proof. We let $f(x, \lambda) \stackrel{\text{def}}{=} f(x, 1)$ for all $\lambda \in (1, 2]$ and $x \in \Gamma e(G(\Delta), 1)$. Since we have $\tilde{\Sigma} = \Sigma$, there is nothing to show. $\square$

Case 2. $\kappa \notin \Sigma$. In this case there is considerably more work to be done. On the one hand, for $\lambda = 1$ the ω-limit sets have already collided; thus, there exists an embedded $\tilde{\Sigma}$-invariant graph $\tilde{\Sigma}e(G(\Delta), 1)$ consisting of several ω-limit sets. Apparently this set presents itself as the one on which we should construct our $\tilde{\Sigma}$-symmetric ω-limit set. On the other side, to this end we need to modify the dynamics of f on this graph such that this set indeed is turned into the $\tilde{\Sigma}$-symmetric ω-limit set we are looking for.

We will achieve this goal by a slight modification of the embedded graph and the mapping f defined on it. The first step is to introduce addition "small" edges $W_i(\lambda)$ near the collision points y_i into the embedded graph for $\lambda > 1$. Afterwards, we define a mapping $f(\cdot, \lambda)$ on this slightly larger graph by modifying the image of the set $e(E_p, 1)$ under the mapping $f(\cdot, 1)$. This image will then additionally contain the edges $\sigma_i \kappa \sigma_i^{-1} W_i(\lambda)$ in the neighboring embedded graphs. We will then show that the dynamics of $f(\cdot, \lambda)$, when defined in this way, extends onto the whole $\tilde{\Sigma}$-symmetric graph in the sense that this set becomes an ω-limit set of f. To preserve continuity while extending f, the length of the additional edges must approach zero when λ decreases to one.

We introduce some additional notation first. Since we only work on the embedded graph from now on, we denote the embedded edges by $Z_i = e(E_i, 1)$ for all $i = 0, \ldots, p$, and we name the embedded Δ-graphs $G_\lambda = e(G(\Delta), \lambda)$ for all $\lambda \in [0, 1]$. If $z, \tilde{z}$ are two points inside an edge $Z \in G_1$, then denote by $[z, \tilde{z}]_Z$ the piece of the edge Z which is bounded by these points. For $\lambda > 1$ we intend to define a graph G_λ which essentially consists of the edges in G_1 plus some additional "small" edges $W_i(\lambda)$ near the collision points. To this end we begin by defining points $w_i(\lambda)$ on the edges Z_i of G_1, which will later be turned into the endpoints of our new edges $W_i(\lambda)$.

Lemma 4.18. *For any $i \in \{0, \ldots, p-1\}$, there exist continuous functions $w_i : [1, 2] \to Z_i$ with the following properties*:

(i) *For each $l \in \mathbf{N}$, $f^l(w_i(1 + 1/l), 1)$ is a vertex of G_1.*

(ii) *For each $j \le l$, the set $f^j([y_i, w_i(1 + 1/l)]_{Z_i}, 1)$ is fully contained in a single edge of G_1. This edge is not contained in the Δ-group orbit of the edge Z_p.*

(iii) $w_i(1) = y_i$.

Proof. We remind the reader of the function $F : J \to G(\Delta)$ defined to obtain the dynamics g_Δ on the graph $G(\Delta)$ with the help of Theorem 4.4. In Lemma 4.14 this function was chosen in such a way that for all edges $E \in J$ and all $\delta \in \Delta$ we have $F(E) \neq \delta E_p$. Since midpoints of edges are again mapped onto midpoints by f, this implies in particular that none of the points y_i is ever being mapped onto a point in Z_p or onto an edge in its group orbit. Hence, if we choose a point on Z_i close to y_i, for a large number of iterates this point is not going to be mapped onto ΔZ_p as well. This remains true at least for as long as the images of this point and of y_i under f^j remain in the same edge.

For some fixed $i \in \{0, \dots, p-1\}$ and $l \in \mathbf{N}$ there exists a point $z(l,i)$ in Z_i which is mapped onto a vertex of G_1 after precisely l iterations by $f(\cdot, 1)$ and whose image by $f^j(\cdot, 1)$ will remain inside the same edge of G_1 as that of y_i for any $j < l$. This can be seen as follows. We choose a piece $[y_i, z(l,i)]_{Z_i}$ so small that for all $j \le l$ the set $f^j([y_i, z(l,i)]_{Z_i}, 1)$ is contained in one single edge. Now we move $z(l,i)$ on Z_i away from y_i until $f^l(z(l,i), 1)$ hits a vertex. Then $f^l([y_i, z(l,i)]_{Z_i}, 1)$ is precisely one half of an edge (recall that y_i is always mapped onto a midpoint). Since f is expanding, all sections $f^j([y_i, z(l,i)]_{Z_i}, 1)$ are fully contained in one edge for any $j < l$.

There are two possibilities to choose $z(l,i)$, since it may be positioned on either side of y_i. We choose the $z(l,i)$ such that for fixed $i = 0, \dots, p-1$ and any l the point $z(l,i)$ lies on the same side of y_i, and we define $w_i(1 + 1/l) \stackrel{\text{def}}{=} z(l,i)$. With this choice we have satisfied (i) and (ii). We just remind the reader that by definition of the function F in Lemma 4.14 the second property in (ii) holds as well.

Now we have to extend the points $w_i(1+1/l)$ to continuous functions $w_i : [1,2] \to Z_i$. There is no difficulty in doing this, since the sequence $(w_i(1+1/l))_{l \in \mathbf{N}}$ converges monotonically on Z_i towards y_i for $l \nearrow \infty$. This is again due to the expanding properties of the function $f(\cdot, 1)$. This especially proves (iii), which completes the proof of the lemma. $\square$

Using the points $w_i(\lambda)$ we now define new graphs G_λ for $\lambda \in (1,2]$ as follows. Let G_λ contain all the edges and vertices of the graph G_1. Additionally, let the $w_i(\lambda)$ be the vertices, and we introduce additional edges $W_i(\lambda)$ connecting the vertices y_i and $w_i(\lambda)$ and having no other common points with the remaining edges of the graph or with their own group orbits. Moreover, we may choose the $W_i(\lambda)$ in such a way that, apart from y_i, they are fully contained inside D and such that their length goes to zero as λ decreases to one. This last request is possible since, by Lemma 4.18 (iii),the vertex $w_i(\lambda)$ converges to y_i. Finally we add the Δ-group orbits of the vertices $w_i(\lambda)$ and of the edges $W_i(\lambda)$ to the graph G_λ. In this way we obtain, for each $\lambda \in (1, 2]$, a Δ-graph in $\mathbf{R}^n$.

Moreover, we let $\tilde{G}_\lambda = \tilde{\Delta} G_\lambda$, see the definition of $\tilde{\Delta}$ in (4.7). This graph is obviously a $\tilde{\Delta}$-graph and even connected, since G_λ is connected to the conjugate graphs $\sigma_i \kappa \sigma_i^{-1} G_\lambda$ via the points y_i, and together with the elements in Δ the elements $\sigma_i \kappa \sigma_i^{-1}$ generate the group $\tilde{\Delta}$.

We proceed by defining the dynamics on $\tilde{G}_\lambda$. To this end we modify $f(\cdot, 1)$ to a mapping $f(\cdot, \lambda)$ defined on ΓG_λ. This function will then have $\tilde{\Sigma}$-symmetric ω-limit sets for $\lambda > 1$, but only for certain parameter values. We define

$$\tilde{B} \stackrel{\text{def}}{=} \{(x, y) \mid \lambda \in [0, 2], x \in \Gamma G_\lambda\}$$

and obtain the following lemma.

Lemma 4.19. *Let $\kappa \notin \Sigma$. The mapping f can be extended to a continuous and Γ-equivariant mapping $f : \tilde{B} \to \mathbf{R}^n$. For each $l \in \mathbf{N}$, $f(\cdot, 1 + 1/l)$ possesses a $\tilde{\Sigma}$-symmetric ω-limit set.*

Proof. For $\lambda \in (1, 2]$ and $z \in \tilde{G}_\lambda \backslash (\tilde{\Delta}\{Z_p\})$ we define $f(z, \lambda) \stackrel{\text{def}}{=} f(z, 1)$, that is, on all edges which are not contained in the group orbit of Z_p we do not change the dynamics. In particular, all properties of Lemma 4.18 hold not only for $f(\cdot, 1)$ but also for $f(\cdot, \lambda)$, $\lambda > 1$. This is due to the fact that in (ii) of this lemma, the relevant points are never mapped onto points in the Δ-group orbit of Z_p by $f^j(\cdot, 1)$ for any $j < l$, and hence $f^j(\cdot, 1)$ and $f^j(\cdot, \lambda)$ are identical at these points.

We define f on the new edges $W_i(\lambda)$, $i = 0, \ldots, p-1$, as follows. Let the image of the edge $W_i(\lambda)$ under $f(\cdot, \lambda)$ be the same as that of the section $[y_i, w_i(\lambda)]_{Z_i}$. We may assume that the edges $W_i(\lambda)$ have been chosen so small that the expanding property of f as in Proposition 4.3 (iii) is not destroyed by this extension.

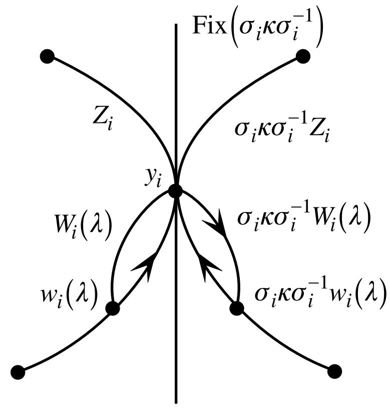


FIGURE 3. Part of the path in the image of the edge Z_p under $f(\cdot, \lambda)$.

On the edge Z_p we intend to modify the dynamics in such a way that all of $\tilde{\Sigma}G_\lambda$ is turned into an ω-limit set of $f(\cdot, \lambda)$. Let us assume first that $\Sigma = \Delta$ is strongly admissible. Then $f(Z_p, 1)$ describes a path containing all the edges in $G_1 \backslash \{Z_p\}$. Especially, it contains all the points y_i and in these points touches the conjugate graphs $\sigma_i\kappa\sigma_i^{-1}G_1$.

For $\lambda > 1$ let $f(Z_p, \lambda)$ be defined as the path described by $f(Z_p, 1)$. However, for every $i \in \{0, \ldots, p-1\}$ we replace the piece $[w_i(\lambda), y_i]_{Z_i}$ in the image of $f(Z_p, 1)$ by the path obtained by joining the sections $[w_i(\lambda), y_i]_{Z_i}$, $\sigma_i\kappa\sigma_i^{-1}W_i(\lambda)$ and $[\sigma_i\kappa\sigma_i^{-1}w_i(\lambda), y_i]_{\sigma_i\kappa\sigma_i^{-1}Z_i}$ in this order, see also Figure 3. At all other points $z \in Z_p$ we let $f(z, \lambda) = f(z, 1)$.

Hence the image of $f(Z_p, \lambda)$ contains small loops inside the conjugate graphs $\sigma_i\kappa\sigma_i^{-1}G_\lambda$. We claim that by defining $f(\cdot, \lambda)$ in this way, the whole graph $\tilde{G}_\lambda$ is turned into an ω-limit set. Note that f still expands sections of the graph, since $f(Z_p, \lambda)$ covers a longer path than $f(Z_p, 1)$.

Now we extend the construction of f $\tilde{\Delta}$-equivariantly for any $\lambda \in (1, 2]$. We may do this because $f(\cdot, 1)$ is Γ-equivariant and all changes in the mapping f while varying λ have been accomplished inside of D (recall that $Z_p \subset D$ and $W_i(\lambda)\backslash\{y_i\} \subset D$ by Lemma 4.18).

We claim that $f(\cdot, 1 + 1/l) : \tilde{G}_{1+1/l} \to \tilde{G}_{1+1/l}$ satisfies the properties of Proposition 4.3 for each fixed $l \in \mathbf{N}$. To prove the claim, we begin by defining new vertices of $G_{1+1/l}$. Let all midpoints of the former edges in

G_1 be additional vertices and, moreover, we consider all the images of the vertices $w_i(1+1/l)$ under $f^j(\cdot, 1+1/l)$, $j \in \mathbf{N}$, and their group orbits as additional vertices. By Lemma 4.18, after finitely many iterations these vertices are mapped onto "real" vertices of $G_{1+1/l}$; hence, with this method we only obtain a finite number of new vertices.

We have chosen these vertices mainly because now $f(\cdot, 1+1/l)$ maps vertices onto vertices. Hence, property (i) of Proposition 4.3 is satisfied. For a suitable choice of parametrization of f, the properties (ii) and (iii) may also be fulfilled. The critical property is (iv)$'$. We have to find $p \in \mathbf{N}$ such that $f^p(Z) = \tilde{G}_{1+1/l}$ for any edge $Z \in \tilde{G}_\lambda$.

For a given $l \in \mathbf{N}$, the image of each edge in $G_{1+1/l}$ after l iterations by $f(\cdot, 1+1/l)$ will at least contain a half edge of the graph G_1. Another iteration will then yield at least one "real" edge, and after $(l+4)$ iterations the image will cover all of $G_{1+1/l}$ except the group orbits of the edges $W_i(\lambda)$, and additionally the edges $\sigma\kappa\sigma^{-1}W_i(\lambda)$ of the conjugate graphs $\sigma\kappa\sigma^{-1}G_\lambda$ for all $\sigma \in \Sigma$. Now we apply the argument above to these small pieces in the conjugate graphs and conclude that these neighboring graphs will also be covered after $(l+4)$ more iterations. By this time we will also cover the edges $W_i(1+1/l)$ of the graph $G_{1+1/l}$, plus additional "small" edges (loops) of other conjugate graphs.

In the worst case $\tilde{G}_{1+1/l}$ consists of a finite number c of such conjugate components and with each $(l+4)$ iterations we reach just another one of these components. Then $p = c(l+4)$ is sufficient for our purposes. We have shown that the assumption (iv)$'$ of Proposition 4.3 is satisfied with this choice of p; hence, we may apply this proposition and conclude that the mapping $f(\cdot, 1+1/l)$ possesses a $\tilde{\Sigma}$-symmetric ω-limit set, namely $\tilde{G}_{1+1/l}$.

If Σ is weakly admissible, we proceed in a similar fashion as above. We only have to recall that the image of G_λ under $f(\cdot, \lambda)$ is contained in ρG_λ. Hence, we have to define the path $f(Z_p, \lambda)$ such that it contains the corresponding "small" edges inside the conjugate graphs neighboring ρG_λ instead of G_λ.

As was the case for $\lambda < 1$, in the nonstrongly admissible case the mapping will not satisfy property (iv)$'$ in Proposition 4.3. But in a similar manner as above, we may verify the properties (i)–(iv). In particular, for each $l \in \mathbf{N}$ the set $\tilde{\Sigma}G_{1+1/l}$ is topologically transitive and hence an ω-limit set.

Finally, we remark that f is continuous since, for $\lambda \searrow 1$, the additional edges $W_i(\lambda)$ shrink and all points on $W_i(\lambda)$ converge to y_i. □

Now we have constructed a mapping $f : \tilde{B} \to \mathbf{R}^n$ which, for all $\lambda < 1$, possesses Σ-symmetric ω-limit sets and, for at least a sequence of λ's greater than one, $\tilde{\Sigma}$-symmetric ones. The final step in the proof demands to extend this mapping equivariantly onto $\mathbf{R}^n \times [0, 2]$ such that these ω-limit sets are turned into attractors.

Lemma 4.20. *The mapping $f : \tilde{B} \to \mathbf{R}^n$ can be extended to a Γ-equivariant mapping $f : \mathbf{R}^n \times [0, 2] \to \mathbf{R}^n$ such that for all $\lambda < 1$, $f(\cdot, \lambda)$ possesses Σ-symmetric attractors and, for all $\lambda = 1 + 1/l$, $l \in \mathbf{N}$, $\tilde{\Sigma}$-symmetric ones.*

Proof. We begin by forming closed, tubular, Δ-symmetric neighborhoods $U_\lambda \subset D$ of $e(G(\Delta), \lambda)$ for each $\lambda \in [0, 1)$. We may assume that their boundaries vary continuously and converge to $U_1 \stackrel{\text{def}}{=} e(G(\Delta), 1)$ as λ increases to one.

On U_λ we define $f(\cdot, \lambda)$ precisely as in the proof of Theorem 5.4 in [**3**]. Roughly speaking this means that all points inside these neighborhoods are being mapped directly onto $e(G(\Sigma), \lambda)$.

Since we have $U_\lambda \subset D$, for fixed λ we may apply Lemma 4.1 to obtain a Γ-equivariant mapping on ΓU_λ. In particular, the set $e(G(\Sigma), \lambda)$ now becomes a Σ-symmetric attractor of $f(\cdot, \lambda)$.

For $\lambda \in (1, 2]$, we define closed $\tilde{\Delta}$-symmetric neighborhoods U_λ of $\tilde{G}_\lambda$ such that they are fully contained in $\tilde{\Delta}D$. As before we may construct these neighborhoods in such a way that their boundaries vary continuously and converge to $\tilde{\Delta}U_1$ as λ decreases to one. Just as before we may now extend the mapping $f(\cdot, \lambda)$ onto U_λ as in Theorem 5.4 of [**3**].

Afterwards, for each λ we extend f Γ-equivariantly onto ΓU_λ. In this manner we obtain, at least for all $\lambda = 1 + 1/l$, $l \in \mathbf{N}$, $\tilde{\Sigma}$-symmetric attractors $\tilde{\Sigma}G_\lambda$. For all remaining $\lambda > 1$ the sets $\tilde{\Sigma}G_\lambda$ are at least stable.

Now we have defined the mapping f on the set

$$U \stackrel{\text{def}}{=} \{(x,\lambda) \mid \lambda \in [0,2], x \in \Gamma U_\lambda\}.$$

On this set, f is continuous since the boundaries of the U_λ vary continuously.

Finally it remains to extend f onto all of $\mathbf{R}^n \times [0,2]$. The set U is closed; hence, we may use the Tietze extension theorem to extend f to a continuous mapping on $\mathbf{R}^n \times [0,2]$. Afterwards for each fixed $\lambda \in [0,2]$ we average f over the group to obtain a Γ-equivariant mapping $f(\cdot,\lambda)$. Since this process preserves the continuity of f, we have finally constructed the desired map. $\square$

Proof of Theorem 4.10. Choose an arbitrary sequence $\lambda_j \nearrow 1$. Then the mapping f constructed above satisfies (C1) in Definition 2.9 by Lemma 4.16 with the choice of $A_j = e(G(\Sigma),\lambda_j)$, $j \in \mathbf{N}$. By construction we have $f^p(e(G(\Delta),1),1) = e(G(\Delta),1)$. Since the set $e(G(\Delta),1) \cap \mathrm{Fix}\,(\kappa)$ contains only finitely many points, there must be a periodic point $x \in \mathrm{Fix}\,(\kappa) \cap \overline{D}$. Again, by construction, this point must have isotropy $\{1,\kappa\}$. Moreover, we will obviously find points $x_j \in A_j \cap D$ converging to x. If we additionally let $y_j = \kappa x_j$, then we have satisfied (C2)–(R) as well. Property (C3) is obvious, since each of the A_j's consists of precisely p connected components. Finally, we have taken care of the property (C4) in Lemma 4.17, respectively, 4.19. This shows that indeed the mapping f undergoes a symmetry increasing bifurcation via a collision of attractors. $\square$

4.3. κ is not a reflection. In this section we are going to construct symmetry increasing bifurcations for the case when κ is not a reflection. As described in Section 2.3, the collision should then take place inside the connected component D. In Proposition 3.2 we have found a necessary condition for such collisions. Now we are going to show that this condition is sufficient as well.

Theorem 4.21. *Assume the Notations* 2.8*, and suppose that* $\kappa \in \Gamma\backslash\Sigma$ *is not a reflection. Then the triple* (Σ,Δ,κ) *is admissible if and only if*

$$\sigma^{-1}\kappa\sigma D \cap D \neq \varnothing, \tag{4.8}$$

for all $\sigma \in \Sigma$.

The proofs of Theorems 4.10 and 4.21 are quite similar. In this section we therefore restrict ourselves to pointing out the differences between the two proofs instead of going through all of the proof of Theorem 4.10.

As in the preceding section, assume $n \geq 3$. The lower dimensions will be treated in Section 4.4. The case $n = 2$ is an exception and, even though the theorem is valid in this case, we ask the reader to take a look at Proposition 4.32 and the remarks preceding this proposition.

The main difference between the two proofs will be that, instead of a collision at a reflection hyperplane, we are going to construct a collision at a point with *trivial isotropy* lying inside of D. As a consequence of the trivial isotropy, we will have to make two embedded edges collide which are *not* conjugate to each other, instead of just moving one embedded edge towards a fixed point space and then obtaining the collision of two edges by equivariance. Due to this procedure we require a graph with a larger fundamental subgraph than previously.

Lemma 4.22. *There exists a finite Eulerian* Δ*-graph* $G(\Delta)$*. This graph is extendable and embeddable in* D*, possesses a fundamental subgraph with at least* $(4p+1)$ *edges and the degree of all vertices is at least four.*

The proof is essentially the same as that of Lemma 4.11, the only difference being that we need a larger number of edges in the fundamental subgraph. Now let $G(\Delta)$ be a graph satisfying the properties of Lemma 4.22. The next step is to choose an embedding and to introduce a parameter into this embedding. In this fashion we are going to move certain embedded edges of the graph towards other embedded edges of some conjugate graph. The following lemma corresponds to Lemma 4.12.

Lemma 4.23. *There exists a continuous mapping* $e : G(\Delta) \times [0,1] \to \mathbf{R}^n$ *with the following properties.*

(a) *For each* $\lambda \in [0,1)$*, the mapping* $e(\cdot,\lambda)$ *is a* Δ*-equivariant embedding of* $G(\Delta)$ *into* D*.*

(b) $e(\cdot,1)$ *is a* Δ*-equivariant one-to-one mapping into* D *and, for each* $i \in \{0,\dots,p-1\}$*, there exist edges* K_i *and* $\hat{K}_i$ *in* $G(\Delta)$ *with midpoints* x_i*, respectively* $\hat{x}_i$*, such that*

(i) $y_i \stackrel{\text{def}}{=} e(x_i,1) = \rho^{-i}\kappa\rho^i e(\hat{x}_i,1) \in e(G(\Delta),1) \cap \rho^{-i}\kappa\rho^i e(G(\Delta),1)$,

(ii) *the* Γ*-group orbits of the* y_i *are pairwise disjoint and do not coincide with the group orbits of the embedded vertices of* $G(\Delta)$,

(iii) *the edges* K_i *and* $\hat{K}_j$ *are pairwise conjugate to different edges of the fundamental subgraph* J *of* $G(\Delta)$ *and*

(iv) *each* y_i *possesses trivial isotropy.*

Proof. Let $e(\cdot,0) : G(\Delta) \to \mathbf{R}^n$ denote a Δ-equivariant embedding of $G(\Delta)$ into D. First we need to show that, for all $i \in \{0,\dots,p-1\}$, parts of the sets $\rho^{-i}\kappa\rho^i e(G(\Delta),0)$ are indeed contained inside of D, since we want to move edges of the embedded graph towards these conjugate graphs without leaving D.

Suppose first that Δ contains reflections and denote the subgroup of Δ generated by its reflections by Δ_R. As stated in Remark 4.9, the corresponding reflection hyperplanes separate D into fundamental domains. Similarly, κD is separated into fundamental domains by the hyperplanes of the reflections in the group $\kappa\Delta_R\kappa^{-1}$. Because of $\kappa D \cap D \neq \varnothing$ and since κ maps reflection hyperplanes onto reflection hyperplanes, there has to be at least one fundamental domain F_0 which D and κD have in common. Because of (4.8), we conclude the same for any $\sigma^{-1}\kappa\sigma D$. In particular, for any $i \in \{0,\dots,p-1\}$ we may find a fundamental domain $F_i \subset \rho^{-i}\kappa\rho^i D \cap D$. Since the elements of Δ_R, respectively $\rho^{-i}\Delta_R\rho^i$, act transitively on these fundamental domains, in each F_i we may find a whole fundamental subgraph of $e(G(\Delta),0)$, respectively of $\rho^{-i}\kappa\rho^i e(G(\Delta),0)$.

If Δ does not contain any reflections, by similar considerations as above, we must even have $\sigma^{-1}\kappa\sigma D = D$ for all $\sigma \in \Sigma$. This implies that, for each $i \in \{0,\dots,p-1\}$, the whole embedded graph $\rho^{-i}\kappa\rho^i e(G(\Delta),0)$ is fully contained in D.

We proceed by describing the construction in the case when Δ contains reflections. For each $i \in \{0,\dots,p-1\}$ we choose an edge $K_i \in G(\Delta)$, such that its embedding is contained in F_i. The F_i contain a fundamental subgraph of the embedded graph $\rho^{-i}\kappa\rho^i e(G(\Delta),0)$ as well. Hence, for each $i \in \{0, \dots, p-1\}$ we choose an additional edge $\hat{K}_i \in G(\Delta)$ with midpoint $\hat{x}_i$ such that $\rho^{-i}\kappa\rho^i e(\hat{K}_i, 0) \subset F_i$ holds. By Lemma 4.22 and, since each F_i contains a fundamental subgraph of $e(G(\Delta), 0)$, we may choose these edges such that the K_i and $\hat{K}_j$ are pairwise not conjugate to each other. We have satisfied (iii) of the lemma.

Since we have $n \geq 3$, we may connect the embedded midpoints $e(x_i, 0)$ with the points $\rho^{-i}\kappa\rho^i e(\hat{x}_i, 0)$ without intersecting fixed point spaces or other embedded edges. Hence, we are now able to move the embedded edges K_i continuously towards the edges $\rho^{-i}\kappa\rho^i e(\hat{K}_i, 0)$. The latter edges remain fixed in this process, just as all other edges of the fundamental subgraph. While varying λ, we keep the embedded vertices fixed and move $e(x_i, \lambda)$ towards $\rho^{-i}\kappa\rho^i e(\hat{x}_i, \lambda) = \rho^{-i}\kappa\rho^i e(\hat{x}_i, 0)$ such that we touch this embedded edge for $\lambda = 1$. As noted previously, we may do this without intersecting fixed point spaces or other embedded edges. We may also avoid that the embedded edges intersect themselves in orbit space. The homotopy obtained in this fashion is denoted by $e : J \times [0,1] \to \mathbf{R}^n$. For $\lambda = 1$, all embedded edges K_i simultaneously touch the corresponding edges in precisely one point $y_i = e(x_i, 1) = \rho^{-i}\kappa\rho^i e(\hat{x}_i, 1)$, as stated in (i). By construction this point has trivial isotropy, which proves (iv). Obviously we may also satisfy (ii). Using Lemma 4.1, for each fixed $\lambda \in [0,1]$ the mapping $e(\cdot, \lambda)$ can be extended equivariantly onto $G(\Delta)$. Just as in the proof of Lemma 4.12 we see that e is one-to-one.

If Δ does not contain any reflections, we do not have to concern ourselves with obstructions posed by reflection hyperplanes inside of D. As mentioned above we then even have $\sigma^{-1}\kappa\sigma D = D$ for all $\sigma \in \Sigma$. Hence we may choose the K_i and $\hat{K}_i$ with less restrictions, but nevertheless proceed in a similar fashion as above.

Now, for $\Sigma \neq \Delta$, let

$$G(\Sigma) = G(\Delta) \dot\cup \rho G(\Delta) \dot\cup \cdots \dot\cup \rho^{p-1} G(\Delta).$$

Using Lemma 4.1 we extend e to a Σ-equivariant mapping $e : G(\Sigma) \times [0,1] \to \mathbf{R}^n$. For each $\lambda < 1$, the mapping $e(\cdot, \lambda)$ is a Σ-equivariant embedding of $G(\Sigma)$. Since $\kappa \notin \Sigma$, the mapping $e(\cdot, 1)$ is at least one-to-one if not a Σ-equivariant embedding.

Now we construct the dynamical system on the graph. We again use Theorem 4.4 to obtain a mapping g_Δ on $G(\Delta)$, satisfying the properties (i)–(v) of the theorem. In Lemma 4.23 we have constructed $G(\Delta)$ such that we may apply this theorem. However, we again need a special choice of the function F mentioned in (v) of this theorem.

Case 1. $\Sigma = \Delta$. We define F by $F(K_0) = K_0$ and $F(\hat{K}_0) = \hat{K}_0$ and choose arbitrary values for F on all other edges of the fundamental subgraph. Hence, the mapping g_Δ obtained by applying Theorem 4.4 satisfies $g_\Delta(x_0) = x_0$ and $g_\Delta(\hat{x}_0) = \hat{x}_0$. We let

$$f(z, \lambda) = e(g_\Delta(e^{-1}(z, \lambda)), \lambda)$$

for all $\lambda \in [0, 1]$ and $z \in e(G(\Delta), \lambda)$.

Case 2. $\Sigma \neq \Delta$. By Lemma 4.22 there exists a ρ-extension $h : G(\Delta) \to \rho G(\Delta)$. Since ρx_i is contained in $\rho G(\Delta)$ for each i, there exists $u_i \in G(\Delta)$ satisfying $h(u_i) = \rho x_i$. Similarly, for $\hat{x}_i$ there exists $\hat{u}_i$ satisfying $h(\hat{u}_i) = \rho \hat{x}_i$. Now h is an isometry and hence the u_i and $\hat{u}_i$ are midpoints of certain edges L_i, respectively $\hat{L}_i$, in $G(\Delta)$.

We define the function F mentioned in (v) of Theorem 4.4 on the subset $\tilde{J} = \{K_0, \ldots, K_{p-1}\} \cup \{\hat{K}_0, \ldots, \hat{K}_{p-1}\}$ of a fundamental subgraph of $G(\Delta)$ by letting $F(K_i) = L_{(i+1) \bmod p}$ and $F(\hat{K}_i) = \hat{L}_{(i+1) \bmod p}$. We extend $\tilde{J}$ to a fundamental subgraph J and define F arbitrarily on the remaining edges of this graph. However, for later use we again need a certain property of F stated in the next lemma.

Lemma 4.24. *There exists an edge K_p of the fundamental subgraph J, $K_p \neq K_i$ for all $i \in \{0, \ldots, p-1\}$, and the function F can be chosen such that all edges $K \in J$ satisfy*

$$F(K) \neq \delta K_p \quad \textit{for all } \delta \in \Delta.$$

Proof. By Lemma 4.22, J contains at least $(4p+1)$ edges. Of these edges, $2p$ are the K_i and $\hat{K}_i$, $i = 0, \ldots, p-1$, and at most $2p$ additional edges are conjugate to the edges L_i and $\hat{L}_i$, $i = 0, \ldots, p-1$. Hence, there remains at least one other edge K_p in J. Since we had defined

$F(K_i) = L_{(i+1) \bmod p}$ and $F(\hat{K}_i) = \hat{L}_{(i+1) \bmod p}$ and since F may be chosen arbitrarily on all other edges, we can satisfy the lemma with an appropriate choice for F. The argument remains true if $\Sigma = \Delta$. ◻

Now choose a dynamical system $g_\Delta : G(\Delta) \to G(\Delta)$ using Theorem 4.4. With our particular choice of F, this mapping satisfies $g_\Delta(x_i) = u_{(i+1) \bmod p}$ and $g_\Delta(\hat{x}_i) = \hat{u}_{(i+1) \bmod p}$ for all $i \in \{0, \dots, p-1\}$. We give a graphical overview of the definitions of the $x_i, \hat{x}_i, u_i$ and $\hat{u}_i$, for simplicity we write $(i+1)$ instead of $((i+1) \bmod p)$,

(4.9)

$$\begin{array}{ccccccccc} e(G(\Delta),1) & \xrightarrow{e^{-1}} & G(\Delta) & \xrightarrow{g_\Delta} & G(\Delta) & \xrightarrow{h} & \rho G(\Delta) & \xrightarrow{e} & \rho e(G(\Delta),1) \\ y_i & \longmapsto & x_i & \longmapsto & u_{i+1} & \longmapsto & \rho x_{i+1} & \longmapsto & \rho y_{i+1} \\ \rho^{-i}\kappa^{-1}\rho^i y_i & \longmapsto & \hat{x}_i & \longmapsto & \hat{u}_{i+1} & \longmapsto & \rho \hat{x}_{i+1} & \longmapsto & \rho^{-i}\kappa^{-1}\rho^{i+1} y_{i+1}. \end{array}$$

Using Lemma 4.1 we extend the mapping $h \circ g_\Delta$ Σ-equivariantly to a mapping $g_\Sigma : G(\Sigma) \to G(\Sigma)$ and define

$$f(z,\lambda) = e(g_\Sigma(e^{-1}(z,\lambda)),\lambda)$$

for all $\lambda \in [0,1]$ and all $z \in e(G(\Sigma),\lambda)$. This mapping is well defined since e is one-to-one. We now aim to extend f Γ-equivariantly. Let

$$B \stackrel{\text{def}}{=} \{(x,\lambda) \mid \lambda \in [0,1], x \in \Gamma e(G(\Delta),\lambda)\}.$$

Then, analogously to Lemma 4.16, we obtain

Lemma 4.25. *The mapping f can be extended to a continuous, Γ-equivariant mapping $f : B \to \mathbf{R}^n$.*

Proof. The idea, once again, is to apply Lemma 4.1 to extend f Γ-equivariantly. The difficulties lie once more in the case $\lambda = 1$, for all $\lambda < 1$ there are no obstructions. We have to show that $f(\gamma z, 1) = \gamma f(z,1)$ whenever there is $\gamma \in \Gamma$ and $z, \gamma z \in \Sigma U_1$.

Suppose first that Σ is not strongly admissible. The mapping $f(\cdot, 1)$ is Σ-equivariant and contained in the set $e(G(\Sigma),1) \subset \Sigma D$. Hence we only have to consider the cases $z = \sigma y_i$ for $\sigma \in \Sigma$ and $i \in \{0, \dots, p-1\}$ since in precisely these points the embedded graph touches conjugate graphs. Moreover, we may restrict to $\gamma \in \Gamma\backslash\Sigma$ because f is Σ-equivariant. Suppose $\gamma z = \gamma\sigma y_i$ is contained in $e(G(\Sigma),1)$. By Lemma 4.12 we have

$$\begin{aligned} &e(x_i,1) = y_i \quad \text{and} \\ &e(\hat{x}_i,1) = \rho^{-i}\kappa^{-1}\rho^i y_i. \end{aligned}$$

Now $\gamma\sigma y_i$ is contained in $e(G(\Sigma),1)$ if and only if the preimage of $\gamma\sigma y_i$ under $e(\cdot,1)$ is defined and hence contained in $G(\Sigma)$. The first equation yields $\gamma\sigma x_i \in G(\Sigma)$ implying $\gamma \in \Sigma$, which we had excluded previously. Hence, only the second equation remains and here we find that $\gamma\sigma y_i \in e(G(\Sigma),1)$ if we have $\gamma\sigma\rho^{-i}\kappa\rho^i\hat{x}_i \in G(\Sigma)$ or, equivalently, $\gamma\sigma\rho^{-i}\kappa\rho^i = \tilde{\sigma} \in \Sigma$.

With these considerations in mind we check the equation $f(\gamma z,1) = \gamma f(z,1)$. Using the definitions of $x_i, \hat{x}_i, u_i$ and $\hat{u}_i$ as shown in (4.9), we obtain

$$\begin{aligned} f(z,1) = f(\sigma y_i,1) &= \sigma f(y_i,1) = \sigma e(h \circ g_\Delta(x_i),1) \\ &= \sigma e(h(u_{(i+1)\bmod p}),1) = \sigma e(\rho x_{(i+1)\bmod p},1) \\ &= \sigma\rho y_{(i+1)\bmod p}, \end{aligned}$$

and hence

$$\begin{aligned} f(\gamma z,1) &= f(\gamma\sigma y_i,1) = f(\tilde{\sigma}\rho^{-i}\kappa^{-1}\rho^i y_i,1) \\ &= \tilde{\sigma} f(\rho^{-i}\kappa^{-1}\rho^i y_i,1) = \tilde{\sigma} e(h \circ g_\Delta(\hat{x}_i),1) \\ &= \tilde{\sigma} e(h(\hat{u}_{(i+1)\bmod p}),1) = \tilde{\sigma} e(\rho\hat{x}_{(i+1)\bmod p},1) \\ &= \tilde{\sigma}\rho\rho^{-(i+1)}\kappa^{-1}\rho^{(i+1)} y_{(i+1)\bmod p} \\ &= \tilde{\sigma}\rho^{-i}\kappa^{-1}\rho^i\rho y_{(i+1)\bmod p} \\ &= \gamma\sigma\rho y_{(i+1)\bmod p} = \gamma f(z,1). \end{aligned}$$

Thus, for $\Sigma \neq \Delta$, we may, by the above calculation, extend the map f Γ-equivariantly using Lemma 4.1. For $\Sigma = \Delta$, we only need to consider $z = \sigma y_0$. Using

$$\begin{aligned} e(x_0,1) &= y_0 \\ e(\hat{x}_0,1) &= \kappa^{-1} y_0, \end{aligned}$$

we obtain, applying similar arguments as above, $\gamma\sigma\kappa = \tilde{\sigma}$ for some $\tilde{\sigma} \in \Sigma$. Analogously, we compute

$$f(z,1) = f(\sigma y_0,1) = \sigma f(y_0,1) = \sigma e(g_\Delta(x_0),1) = \sigma e(x_0,1) = \sigma y_0$$

and hence, for γz,

$$\begin{aligned} f(\gamma z, 1) &= f(\gamma\sigma y_0, 1) = f(\tilde{\sigma}\kappa^{-1}y_0, 1) = \tilde{\sigma}f(\kappa^{-1}y_0, 1) \\ &= \tilde{\sigma}e(g_\Delta(\hat{x}_0), 1) = \tilde{\sigma}e(\hat{x}_0, 1) \\ &= \tilde{\sigma}\kappa^{-1}y_0 = \gamma\sigma y_0 = \gamma f(z, 1). \end{aligned}$$

This shows that in the strongly admissible case as well we can apply Lemma 4.1 to obtain a Γ-equivariant mapping f on B. □

Finally, for $\lambda > 1$ the mapping f has to be extended such that we can observe a $\tilde{\Sigma} = \langle \Sigma \cup \{\kappa\}\rangle$-symmetric attractor. The construction of these attractors can be carried out using almost precisely the same arguments as in the preceding section. Hence we refrain from stating all the arguments again and restrict ourselves to citing the final result. The proof can be done by showing lemmas analogous to Lemmas 4.18, 4.19 and 4.20. In the verification of the first one of these lemmas, we have to use Lemma 4.24 instead of Lemma 4.14. For a more detailed argument we refer the reader to [**12**]. Using this procedure we arrive at

Lemma 4.26. *The mapping $f : B \to \mathbf{R}^n$ can be extended to a Γ-equivariant mapping $f : \mathbf{R}^n \times [0, 2] \to \mathbf{R}^n$ such that, for all $\lambda < 1$, f possesses Σ-symmetric attractors and for all $\lambda = 1 + 1/l$, $l \in \mathbf{N}$, $\tilde{\Sigma}$-symmetric ones.*

Finally we are in the position to prove the main result of this section.

Proof of Theorem 4.21. Given any sequence $\lambda_j \nearrow 1$, $\lambda > 0$, the mapping f satisfies (C1) in Definition 2.9 when choosing $A_j = e(G(\Sigma), \lambda_j)$. The set $e(G(\Delta), 1) \cap \kappa e(G(\Delta), 1)$ contains only finitely many points. Since $f^p(\cdot, 1)$ fixes this set, it must contain a periodic point x which, by construction, is lying inside $D \cap \kappa D$. Obviously, we may also find sequences $x_j \in A_j \cap D$ and $y_j \in \kappa A_j \cap D$ converging towards x. This proves (C2)–(NR). Since the A_j each consist of precisely p connected components, (C3) is also true. Finally, (C4) has been shown in Lemma 4.26, at least for a sequence of parameter values $\lambda = 1 + 1/l$ converging towards the critical value $\lambda_c = 1$. This proves that indeed the triple (Σ, Δ, κ) is admissible. □

Both Theorems 4.10 and 4.21, respectively their conditions (4.5) and (4.8), can be reduced in particular cases. Suppose that κ commutes with all $\sigma \in \Sigma$. Then these conditions reduce to

$$\dim(\mathrm{Fix}(\kappa) \cap \partial D) = n - 1$$

if κ is a reflection and

$$\kappa D \cap D \neq \varnothing$$

otherwise. We have already stated these assumptions in Notations 2.8. Hence we obtain the following corollary, see Theorem 5.2 in [**8**].

Corollary 4.27. *Suppose κ commutes with all $\sigma \in \Sigma$. Then the triple (Σ, Δ, κ) is admissible.*

In particular, for $\kappa \in \Sigma$, according to Lemma 3.2 in [**8**], κ always commutes with the elements of Σ. Hence, a triple (Σ, Δ, κ) for $\kappa \in \Sigma$ is always admissible.

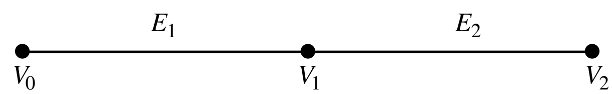
If Σ is strongly admissible, the conditions of the theorems can be reduced in the same manner. This is due to the fact that a strongly admissible subgroup $\Sigma = \Delta$ fixes D and hence its boundary as well. We obtain, see Corollary 5.9 in [**8**],

Corollary 4.28. *Suppose Σ is strongly admissible. Then the triple (Σ, Σ, κ) is admissible.*

4.4. Dimensions one and two. For a full proof of Theorems 4.10 and 4.21, we have to deliver constructions for the admissible triples in the dimensions one and two. These will be treated now on a case-by-case basis.

To begin with, consider a one-dimensional mapping $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$. We denote the elements of $\mathbf{O}(1)$ by 1 and κ. The strongly admissible subgroups are $\mathbf{1}$ and $\mathbf{Z}_2$, and there are precisely two admissible triples.

Proposition 4.29. *Let $n = 1$. Then both triples $(\mathbf{1}, \mathbf{1}, \kappa)$ and $(\mathbf{Z}_2, \mathbf{1}, \kappa)$ are admissible.*



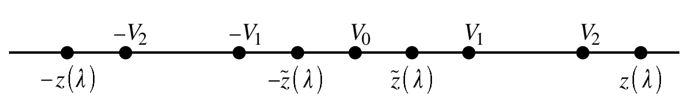
Proof. We only give a sketch of the proof; for details the reader is referred to [**12**]. Generally, the methods of Theorem 4.10 can be applied in this case as well; we only have to consider the special geometry of $\mathbf{R}$. Note, for example, that a graph embedded in $\mathbf{R}$ can never by Eulerian. We therefore choose a graph $G(\mathbf{1})$ consisting of two edges E_1 and E_2 and three vertices V_0, V_1 and V_2, see the figure above.

We let the dynamics $g : G(\mathbf{1}) \to G(\mathbf{1})$ on this graph be given as follows: Let the edge E_1 be mapped onto all of the graph $G(\mathbf{1})$ and similarly for the edge E_2, only here we change the direction of the path. In particular, for the vertices of the graph we have $g(V_0) = V_0$, $g(V_1) = V_2$ and $g(V_2) = V_0$. Note that V_0 is the only vertex of $G(\mathbf{1})$ which is fixed by g. The assumptions of Proposition 4.3 apply to g with $q = p = 1$.

We embed $G(\mathbf{1})$ on the positive real axis and then move the embedded vertex V_0 towards zero. Using methods from the proof of Theorem 4.10, we are able to construct the desired equivariant mapping f up to the point of collision. At this point there are no obstructions to equivariance, since the embedded vertex V_0 is a fixed point of g.

For $\lambda \geq 1$, we introduce a new embedded vertex $z(\lambda)$ satisfying $z(1) = e(V_2, 1)$ for $\lambda = 1$ and then increasing monotonically. An additional vertex $\tilde{z}(\lambda)$ satisfying $\tilde{z}(1) = 0$ is introduced close to zero, see the following figure.

We define $f(\cdot, \lambda)$ for $\lambda > 1$ in the following way. If $Z_i = e(V_i, 1)$



denote the embedded vertices, we choose f such that the vertices are being mapped as follows:

$$\begin{aligned} f(Z_0,\lambda) &= Z_0 = 0 \\ f(Z_1,\lambda) &= z(\lambda) \\ f(Z_2,\lambda) &= Z_0 = 0 \\ f(z(\lambda),\lambda) &= -\tilde{z}(\lambda). \end{aligned}$$

Since we have $z(1) = Z_2$ and $\tilde{z}(1) = Z_0$, we may extend f continuously and equivariantly. Using an appropriate parametrization we can show, in a similar fashion as in the proof of Lemma 4.19 that, for a sequence of parameter values $\lambda_j \searrow 1$ the mapping f defined in this way possesses a $\mathbf{Z}_2$-symmetric ω-limit set $[-z(\lambda), z(\lambda)]$. Note, however, that on the interval $[Z_2, z(\lambda)]$ the mapping f is not expanding and hence in Proposition 4.3 we have to use $q = 2$. Finally, we may extend f equivariantly to a mapping $f : \mathbf{R} \times [0,2] \to \mathbf{R}$ as in the proof of Lemma 4.20 such that the set $[e(V_0,\lambda), e(V_2,\lambda)]$ for $\lambda < 1$, respectively $[-z(\lambda), z(\lambda)]$ for $\lambda = \lambda_j > 1$, become attractors. This shows that the triple $(\mathbf{1}, \mathbf{1}, \kappa)$ is admissible.

To verify admissibility of the triple $(\mathbf{Z}_2, \mathbf{1}, \kappa)$ we merely multiply the mapping f constructed above with -1. $\square$

We can now prove Theorems 4.10 and 4.21 for $n = 1$ using the preceding result: since κ is a reflection, there is nothing to show for Theorem 4.21, and Theorem 4.10 states that both triples $(\mathbf{1}, \mathbf{1}, \kappa)$ and $(\mathbf{Z}_2, \mathbf{1}, \kappa)$ are admissible. For an example of such a one-dimensional symmetry increasing bifurcation, see the odd logistic mapping as discussed in [**6**].

Now we turn to dimension two. The finite subgroups of $\mathbf{O}(2)$ up to conjugacy are $\mathbf{Z}_m$ and $\mathbf{D}_m$ for $m \in \mathbf{N}$. We begin by verifying Theorem 4.10. In this case we only need to consider $\mathbf{D}_m$ since $\mathbf{Z}_m$ does not contain any reflections. The strongly admissible subgroups of $\mathbf{D}_m$ are $\mathbf{D}_m$, $\mathbf{D}_1$ and $\mathbf{1}$, the weakly admissible are $\mathbf{Z}_k$, $2 \leq k \mid m$ and $\mathbf{D}_2$ if m is even, see Theorem 7.2 in [**3**]. We note that we distinguish between $\mathbf{D}_1$ and $\mathbf{Z}_2$. The first group is generated by a reflection, the last one by a rotation. Only the first one is strongly admissible. The following proposition yields the classification of the two-dimensional case when κ is a reflection. It can easily be checked that the proposition agrees with the contents of Theorem 4.10.

Proposition 4.30. *Let $n = 2$, and let $\kappa \in \Gamma \backslash \Delta$ be a reflection in $\mathbf{D}_m$. Then the triples*

Admissible Triple	Symmetry Group after the Collision
$(\mathbf{1}, \mathbf{1}, \kappa)$	$\mathbf{D}_1$
$(\mathbf{D}_1, \mathbf{1}, \kappa)$	$\mathbf{D}_1$
$(\mathbf{D}_1, \mathbf{D}_1, \kappa)$	$\mathbf{D}_m$

are admissible. If m is even, additionally the triples

Admissible Triple	Symmetry Group after the Collision
$(\mathbf{Z}_2, \mathbf{1}, \kappa)$	$\mathbf{D}_2$
$(\mathbf{D}_1, \mathbf{1}, \kappa)$	$\mathbf{D}_2$
$(\mathbf{D}_2, \mathbf{D}_1, \kappa)$	$\mathbf{D}_m$

are admissible. The triples $(\mathbf{Z}_k, \mathbf{1}, \kappa)$ for $2 < k \leq m$ are not admissible, as well as $(\mathbf{D}_1, \mathbf{1}, \kappa)$ if the symmetry group after the collision is $\mathbf{D}_m$, that is, if we have $\langle \mathbf{D}_1 \cup \{\kappa\} \rangle = \mathbf{D}_m$.

Remark 4.31. We take a closer look at the group $\mathbf{D}_1$. A transition from $\mathbf{D}_1$ to $\mathbf{D}_m$-symmetry is possible only if $\mathbf{D}_1$ itself is the associated group of the attractor. In other words, the attractor has to intersect the reflection hyperplane corresponding to the reflection in $\mathbf{D}_1$. If this is not the case, that is, if $\mathbf{1}$ is the associated group, then the attractor may first collide with itself at this reflection hyperplane, and only afterwards a transition to full $\mathbf{D}_m$-symmetry becomes possible. This is the reason why we have admitted the case $\kappa \in \Sigma$ if κ is a reflection.

Proof. We do not need to show that $(\mathbf{Z}_k, \mathbf{1}, \kappa)$ for $k > 2$ and $(\mathbf{D}_1, \mathbf{1}, \kappa)$, $\kappa \notin \mathbf{D}_1$, are not admissible. This follows from Proposition 3.1 where we have made no restriction to the dimension n.

To show admissibility of the remaining triples, we may even apply the methods of Theorem 4.10 directly. We only have to make sure that appropriate graphs exist, can be embedded in $\mathbf{R}^2$ and can then be deformed such that certain edges collide with reflection hyperplanes without intersecting other edges. This may easily be seen using a sketch for each of the triples. $\square$

Now we turn to the case when κ is not a reflection. Then we need not consider $\mathbf{D}_m$, since in this group there does not exist any triple (Σ, Δ, κ) where κ is not a reflection and satisfies $\kappa D \cap D \neq \varnothing$.

Hence we turn our attention to $\mathbf{Z}_m$. This group is an exception concerning admissibility. From [**3**] we know that $\mathbf{Z}_m$ and $\mathbf{1}$ are strongly admissible and $\mathbf{Z}_k$, $1 < k < m$ divides m, weakly admissible. If we applied the criterion for strongly admissible subgroups as in Theorem 2.4, then we would obtain that all subgroups must be strongly admissible since $\mathbf{Z}_m$ does not contain any reflections. However, any connected $\mathbf{Z}_k$-symmetric attractor, $k > 1$, has to contain a closed curve around zero or zero itself, which already makes it fully symmetric by Proposition 2.2.

This special property of $\mathbf{Z}_m$ affects our problem as well. First note that the construction of the associated group does not work in this case, since it always yields the full symmetry group of the attractor which may not be strongly admissible. However, we may always choose $\Delta = \mathbf{1}$.

If we apply Theorem 4.21 to the triples $(\Sigma, \mathbf{1}, \kappa)$, $\Sigma < \mathbf{Z}_m$, we conclude that all triples must be admissible, since $\mathbf{Z}_m$ does not contain reflections and we therefore have $D = \mathbf{R}^2$. The following proposition shows that indeed this statement is true. In the proof, however, we have to pay special attention to the structure of the group. Suppose we look at $\Gamma = \mathbf{Z}_4$ and want to observe a transition from trivial to $\mathbf{Z}_2$-symmetry. Then a connected attractor with trivial symmetry would first collide with its direct neighbor yielding $\mathbf{Z}_4$-symmetry instead of merely $\mathbf{Z}_2$. We may circumvent this problem by choosing an attractor with two connected components. Hence, to prove the following proposition we have to construct attractors with a larger number of connected components than usually.

Proposition 4.32. *Let* $\Gamma = \mathbf{Z}_m$, $m \in \mathbf{N}$, *then all triples* $(\mathbf{Z}_k, \mathbf{1}, \kappa)$ *where* $k < m$ *divides* m *and* $\kappa \in \mathbf{Z}_m \backslash \mathbf{Z}_k$ *are admissible.*

Proof. Again we restrict to a sketch of the proof. The full proof can be found in [**12**]. Let ρ be a generator of $\mathbf{Z}_m$, and let $\mathbf{Z}_l$, $l = kp$, be the group generated by $\mathbf{Z}_k$ and κ. Without loss of generality we may assume $\kappa = \rho^{m/l}$, since this element generates $\mathbf{Z}_l$.

We choose some half line T emanating from zero and consider the set $\mathbf{Z}_m T$. This set separates $\mathbf{R}^2$ into connected components, which can be seen as "fundamental domains" for the group $\mathbf{Z}_m$. We take one of these components, F say, and some appropriate $\mathbf{1}$-graph $G(\mathbf{1})$. Now we embed this graph p times into F using the embeddings $e_i(\cdot, 0) : G(\mathbf{1}) \to F$, $i = 0, \dots, p-1$, such that the images of the embeddings do not intersect. We denote the embedded graphs by $E_i \stackrel{\mathrm{def}}{=} e_i(G(\mathbf{1}), 0)$ and let $E = \cup_{i=0}^{p-1} E_i$.

Now choose a dynamical system $g : G(\mathbf{1}) \to G(\mathbf{1})$ with the properties of Theorem 4.4. Define $f(\cdot, 0)$ for $z \in E_i$ by

$$f(z, 0) \stackrel{\mathrm{def}}{=} \kappa e_{(i+1) \bmod p}(g(e_i^{-1}(z, 0)), 0)$$

for all $i = 0, \dots, p-1$ and extend $\mathbf{Z}_m$-equivariantly onto $\mathbf{Z}_m E$. Then one can check that the set

$$A = \bigcup_{j=0}^{l-1} f^j(E_0, 0)$$

becomes an f-invariant $\mathbf{Z}_k$-symmetric ω-limit set of $f(\cdot, 0)$ and f satisfies the properties (i)–(iv) of Proposition 4.3.

We proceed by introducing a parameter into the embeddings, such that the set A collides with its conjugate sets $\kappa^i A$ for $i = 1, \dots, l-1$ (without intersecting any other conjugate limit sets). To this end, we deform the embeddings such that for $\lambda = 1$, some arbitrary chosen vertex $V \in G(\mathbf{1})$ is mapped onto some fixed $y \in F$ by all $e_i(\cdot, 1)$. Without loss of generality we may assume that the embeddings of $G(\mathbf{1})$ have been chosen such that such a collision is possible without intersecting other edges during the deformation process. We extend f by defining

$$f(z, \lambda) \stackrel{\mathrm{def}}{=} \kappa e_{(i+1) \bmod p}(g(e_i^{-1}(z, \lambda)), \lambda)$$

for all $z \in e_i(G(\mathbf{1}), \lambda)$, $i = 0, \dots, p-1$, and extend f equivariantly. The mapping $f(\cdot, 1)$ is well defined in y, since V is a fixed point of g.

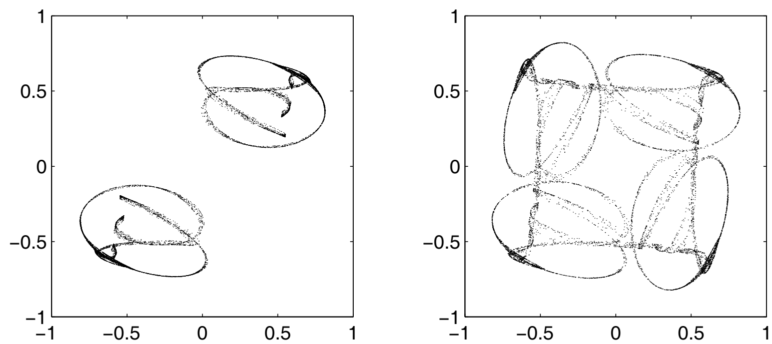


FIGURE 4. A symmetry increasing bifurcation.

After the collision we extend f in much the same way as in the proof of Lemma 4.19. Then we may construct attractors and finally obtain the desired mapping $f : \mathbf{R}^2 \times [0, 2] \to \mathbf{R}^2$ just as in Lemma 4.26. $\square$

Example 4.33. Figure 4 shows a symmetry increasing bifurcation for a cyclic symmetry group.

In this example we have $\Gamma = \mathbf{Z}_4$, and the corresponding admissible triple is $(\mathbf{Z}_2, \mathbf{1}, \kappa)$ with $\langle \mathbf{Z}_2 \cup \{\kappa\} \rangle = \mathbf{Z}_4$. As a dynamical system, we have chosen the mapping

$$f(z, \lambda) = (\alpha u + \beta v + \lambda + i\omega)z + \gamma \bar{z}^{m-1},$$

the parameters are $m = 4$, $\alpha = 2.0$, $\beta = 0.0$, $\gamma = 1.0$, $\omega = 0.1$, and the collision takes place between $\lambda = -1.78$ and $\lambda = -1.79$. For additional numerical examples in $\mathbf{R}^2$, we refer the reader to [**8**].

4.5. An example: The tetrahedral group. In this section we apply our results by classifying all admissible triples of the group $\mathbf{T} \oplus \mathbf{Z}_2^c < \mathbf{O}(3)$, which is the symmetry group of the tetrahedron $(\mathbf{T})$ plus the group $\mathbf{Z}_2^c$ generated by $-id$. This example was already discussed in [**8**], Example 5.10. Now, however, we are able to give a full classification using both Theorems 4.10 and 4.21.

We use the notation as in [**10**]. Then, by Table 1 in [**3**] we have the following list of admissible subgroups of $\mathbf{T} \oplus \mathbf{Z}_2^c$.

strongly admissible	$\mathbf{T} \oplus \mathbf{Z}_2^c$, $\mathbf{D}_2 \oplus \mathbf{Z}_2^2$, $\mathbf{Z}_3$, $\mathbf{D}_2^z$, $\mathbf{Z}_2^-$, $\mathbf{1}$
weakly admissible	$\mathbf{Z}_3 \oplus \mathbf{Z}_2^c$, $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$, $\mathbf{Z}_2^c$, $\mathbf{Z}_2$
inadmissible	$\mathbf{T}, \mathbf{D}_2$

We also add a list of inclusions between the different subgroups, see [**3**, Proposition 8.3].

$$
\begin{aligned}
\mathbf{T} \oplus \mathbf{Z}_2^c &\supset \mathbf{T}, \mathbf{D}_2 \oplus \mathbf{Z}_2^c, \mathbf{Z}_3 \oplus \mathbf{Z}_2^c \\
\mathbf{T} &\supset \mathbf{Z}_3, \mathbf{D}_2 \\
\mathbf{D}_2 \oplus \mathbf{Z}_2^c &\supset \mathbf{D}_2, \mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \mathbf{D}_2^z \\
\mathbf{Z}_3 \oplus \mathbf{Z}_2^c &\supset \mathbf{Z}_3, \mathbf{Z}_2^c \\
\mathbf{D}_2 &\supset \mathbf{Z}_2 \\
\mathbf{Z}_2 \oplus \mathbf{Z}_2^c &\supset \mathbf{Z}_2, \mathbf{Z}_2^c, \mathbf{Z}_2^- \\
\mathbf{D}_2^z &\supset \mathbf{Z}_2, \mathbf{Z}_2^- \\
\mathbf{Z}_2, \mathbf{Z}_2^c, \mathbf{Z}_2^-, \mathbf{Z}_3 &\supset \mathbf{1}.
\end{aligned}
$$

Applying Theorem 4.10 we can conclude, as in [**8**], which of the triples are admissible when κ is a reflection. The admissible triples are listed in Table 1.

TABLE 1. Admissible triples by Theorem 4.10.

	Admissible Triple (Σ, Δ, κ)	Resulting Group
1.	$(\mathbf{1}, \mathbf{1}, \kappa)$	$\mathbf{Z}_2^-$
2.	$(\mathbf{Z}_2^-, \mathbf{1}, k)$	$\mathbf{D}_2^z$
3.	$(\mathbf{Z}_2^-, \mathbf{Z}_2^-, \kappa)$	$\mathbf{D}_2^z$
4.	$(\mathbf{Z}_2, \mathbf{1}, \kappa)$	$\mathbf{D}_2^z$
5.	$(\mathbf{Z}_2, \mathbf{1}, \kappa)$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$
6.	$(\mathbf{Z}_2^c, \mathbf{1}, \kappa)$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$
7.	$(\mathbf{D}_2^z, \mathbf{Z}_2^-, \kappa)$	$\mathbf{D}_2 \oplus \mathbf{Z}_2^c$
8.	$(\mathbf{D}_2^z, \mathbf{D}_2^z, \kappa)$	$\mathbf{D}_2 \oplus \mathbf{Z}_2^c$
9.	$(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \mathbf{Z}_2^-, \kappa)$	$\mathbf{D}_2 \oplus \mathbf{Z}_2^c$
10.	$(\mathbf{Z}_3, \mathbf{Z}_3, \kappa)$	$\mathbf{T} \oplus \mathbf{Z}_2^c$
11.	$(\mathbf{Z}_3 \oplus \mathbf{Z}_2^c, \mathbf{Z}_3, \kappa)$	$\mathbf{T} \oplus \mathbf{Z}_2^c$

If κ is contained in Σ, then the following triples are admissible:

$$(\mathbf{Z}_2^{-1}, \mathbf{1}, \kappa), (\mathbf{D}_2^z, \mathbf{Z}_2^-, \kappa) \text{ and } (\mathbf{D}_2 \oplus \mathbf{Z}_2^c, \mathbf{D}_2^z, \kappa).$$

Now we apply Theorem 4.21 to look at the cases where κ is not a reflection. Then we compute that the triples in Table 2 are admissible. Note that in 1, 2 and 5 we even have $\sigma^{-1}\kappa\sigma D = D$, whereas for 3 and 4 only $\sigma^{-1}\kappa\sigma D \cap D \neq \varnothing$ is true.

TABLE 2. Admissible triples by Theorem 4.21.

	Admissible Triple (Σ, Δ, κ)	Resulting Group
1.	$(\mathbf{1}, \mathbf{1}, \kappa)$	$\mathbf{Z}_3$
2.	$(\mathbf{Z}_2^c, \mathbf{1}, \kappa)$	$\mathbf{Z}_3 \oplus \mathbf{Z}_2^c$
3.	$(\mathbf{Z}_2^-, \mathbf{Z}_2^-, \kappa)$	$\mathbf{T} \oplus \mathbf{Z}_2^c$
4.	$(\mathbf{D}_2 \oplus \mathbf{Z}_2^2, \mathbf{D}_2^z, \kappa)$	$\mathbf{T} \oplus \mathbf{Z}_2^c$
5.	$(\mathbf{D}_2 \oplus \mathbf{Z}_2^c, \mathbf{D}_2 \oplus \mathbf{Z}_2^c, \kappa)$	$\mathbf{T} \oplus \mathbf{Z}_2^c$

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