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ON THE NONEXISTENCE OF COFREE FRÉCHET MODULES OVER LOCALLY MULTIPLICATIVELY-CONVEX FRÉCHET ALGEBRAS

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ABSTRACT. Suppose A is a locally multiplicatively-convex Fréchet algebra. It is proved that, if there exists at least one nonzero, cofree Fréchet A-module, then A is normable.

Topological homology is based on two fundamental concepts: projectivity and injectivity. These concepts can be defined in the context of locally convex modules over a locally convex algebra A, see [**3**]. If A is a Banach algebra, then many important statements about projective and injective modules, known from classical homological algebra, are valid in the categories of Banach A-modules. In particular, each Banach A-module has projective and injective resolutions, see [**3**]. However, if A is an arbitrary locally convex algebra, then only the notion of a projective A-module is rich in content. The main obstacle for the study of injective A-modules is the possible absence of so-called *cofree* objects in categories of locally convex A-modules. Even if A is a nonnormable Fréchet algebra, we do not have any information about injective Fréchet A-modules. In this connection, the following question is of interest. Do there exist nonzero, cofree Fréchet A-modules over an arbitrary Fréchet algebra A?

This problem is closely connected with some questions concerning injective Fréchet A-modules, for example, is it true that any A-module has an injective resolution? Is it true that each injective A-module is a retract of a cofree A-module? Finally, does there exist at least one nonzero, injective A-module? See [**4**]. If A is a Banach algebra and the A-modules which are under consideration are also Banach modules, then the answers to all these questions are affirmative, as well as in pure homological algebra. In particular, a cofree Banach A-module is topologically isomorphic to an A-module $\mathcal{B}(A, E)$ for some Banach space E (by $\mathcal{B}(A, E)$ we denote the A-module of all continuous linear

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maps from A to E with the operation $[a \cdot \varphi](b) = \varphi(ba)$, $a, b \in A$, $\varphi \in \mathcal{B}(A, E)$). If A is nonnormable, then the main difficulty is that the vector space $\mathcal{B}(A, E)$ does not possess any natural Fréchet topology and cannot be considered as an A-module with a jointly continuous action of A. This fact was noticed by Taylor in [**7**]; in the same work there was represented a theory of injective locally convex modules with hypocontinuous action of a locally convex algebra, A. However, all of the above-mentioned questions about injective and cofree Fréchet modules are open.[1]

In this paper we give a negative answer to the first of these questions under the additional assumption that the Fréchet algebra A is locally multiplicatively-convex, or, in other words, A is a metrizable Arens-Michael algebra. We recall that a locally convex algebra A is called *locally multiplicatively-convex* if its topology can be defined by a family of seminorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ having the property $\|ab\|_\nu \leq \|a\|_\nu \|b\|_\nu$ for all $a, b \in A$.

Let A be an arbitrary Fréchet algebra. In the sequel, by Fréchet A-modules we shall mean *left* Fréchet A-modules. *Morphism* means always 'continuous morphism,' *isomorphism* means 'topological isomorphism.'

Definition. A cofree Fréchet A-module over a Fréchet algebra A is a pair (X, E) where X is a Fréchet A-module and E is a Hausdorff quotient space of X, and the following condition holds. For any Fréchet A-module Y and a linear continuous map $\varphi : Y \to E$, there exists a unique A-module morphism $\psi : Y \to X$ such that $p\psi = \varphi$, where p is the quotient map of X onto E:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\psi"] \arrow[r, "\varphi"'] & E \end{tikzcd}$$

In such a situation we shall say that ψ *lifts* φ.

Remark 1. It can be proved that this definition is equivalent to the following one. A cofree Fréchet A-module is a pair (X, E), where X is a Fréchet A-module, E is a Fréchet space, and X represents the functor $Y \mapsto \mathcal{B}(Y, E)$.

Remark 2. It is easy to show that X is uniquely determined by the space E. This means that, if $(X_1, E, p_1 : X_1 \to E)$ and $(X_2, E, p_2 : X_2 \to E)$ are cofree Fréchet A-modules, then there exists an A-module isomorphism $\tau : X_1 \to X_2$ such that $p_2\tau = p_1$.

One of the most important properties of cofree Fréchet A-modules is that each cofree Fréchet A-module is injective. We recall that a Fréchet A-module J is *injective* if, whenever Z is a Fréchet A-module and Y is a closed submodule that is complemented as a subspace of Z, then each A-module morphism $\alpha : Y \to J$ can be extended to an A-module morphism $\psi : Z \to J$, i.e., the following diagram is commutative:

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "\alpha"'] & Z \arrow[dl, dashed, "\psi"'] \\ J & \end{tikzcd}$$

Here i is the canonical embedding of Y into Z.

Proposition 1. *Suppose (X, E) is a cofree Fréchet A-module. Then X is injective.*

Proof. Suppose Z is an arbitrary Fréchet A-module, Y is a closed submodule, $i : Y \to Z$ is the canonical embedding, and $q : Z \to Y$ is a continuous linear map such that $qi = \mathbf{1}_Y$. Let $\alpha : Y \to X$ be an arbitrary A-module morphism. Consider the A-module morphism $\psi : Z \to X$ that lifts the linear continuous map $\varphi : Z \to E$, $\varphi = p\alpha q$. Since $p\psi = p\alpha q$ and $qi = \mathbf{1}_Y$, we see that $p\psi i = p\alpha$. In other words, ψi lifts the linear continuous map $p\alpha : Y \to E$. Since ψi is the unique morphism having such a property, we conclude that $\psi i = \alpha$. Hence, X is an injective A-module. $\square$

Suppose E is an arbitrary vector space and p is a seminorm on E. Recall that the *accompanying Banach space* E_p is the completion of the space $E/p^{-1}(0)$ with respect to the quotient norm of the seminorm p. If E is an l.c.s. whose topology is defined by a family of seminorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$, then the accompanying Banach spaces $\{E_\nu : \nu \in \Lambda\}$ form an inverse system. It is known that if E is complete, then there exists a topological isomorphism $E \cong \varprojlim E_\nu$, see [**6**, II.5.4]. The canonical projection $\varprojlim E_\nu \to E_\nu$ coincides with the quotient map $E \to E_\nu$. Therefore, the image of this projection is dense in E_ν, in other words, the inverse system $\{E_\nu : \nu \in \Lambda\}$ is reduced. If, in addition, E is an Arens-Michael algebra, i.e., a complete locally multiplicatively-convex algebra, and the seminorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ are submultiplicative, i.e., $\|ab\|_\nu \le \|a\|_\nu \|b\|_\nu$ for all $a, b \in A$, then each E_ν is a Banach algebra, and the above isomorphism is an isomorphism of algebras, see [**2**, 5.2.17].

Theorem. *Suppose A is a nonnormable locally multiplicatively-convex Fréchet algebra. Then there are no nonzero, cofree Fréchet A-modules.*

Proof. Assume the converse. Let (X, E) be a nonzero, cofree Fréchet A-module. First, we note that $E \neq 0$. Indeed, suppose that $E = 0$; then both the identity morphism $\mathbf{1}_X : X \to X$ and the zero morphism $0 : X \to X$ lift the map $0 : X \to E$. Hence, $\mathbf{1}_X = 0$ and $X = 0$.

Without loss of generality, we can assume that A and X are unital and consider only unital A-modules. Let us represent A as a reduced inverse limit, $\varprojlim(A_n, \tau_n^m)$, of Banach algebras. Since A_n is a Banach algebra, the dual space A_n^* can be considered as a left A_n-module with respect to the operation $[a_n \cdot f_n](b_n) = f_n(b_n a_n)$, $a_n, b_n \in A_n$, $f_n \in A_n^*$. Moreover, we can endow A_n^* with a left A-module structure by putting $a \cdot f_n = \tau_n(a) \cdot f_n$ for $a \in A$.

Take an arbitrary element $e_0 \in E$, $e_0 \neq 0$. For each n we define the linear continuous map $\varphi_n : A_n^* \to E$ by $\varphi_n(f) = f(1_n)e_0$, where $1_n = \tau_n(1)$ is the identity in the algebra A_n. Let $\psi_n : A_n^* \to X$ be the A-module morphism that lifts φ_n.

For $n < m$, put $j_n^m = (\tau_n^m)^* : A_n^* \to A_m^*$. Let us show that

$\psi_{n+1} j_n^{n+1} = \psi_n$ for all n, i.e., the following diagram is commutative:

$$\begin{tikzcd} A_{n+1}^* \arrow[r, "\psi_{n+1}"] & X \arrow[d, "p"] \\ A_n^* \arrow[u, "j_n^{n+1}"] \arrow[ur, "\psi_n"] \arrow[r, "\varphi_n"'] & E \end{tikzcd}$$

Since ψ_n is the unique A-module morphism that lifts φ_n, we need only check that $p\psi_{n+1} j_n^{n+1} = \varphi_n$. We have

$$\begin{aligned} p\psi_{n+1} j_n^{n+1}(f) &= p\psi_{n+1}(f\tau_n^{n+1}) = \varphi_{n+1}(f\tau_n^{n+1}) \\ &= f(\tau_n^{n+1}(1_{n+1}))e_0 = f(1_n)e_0 = \varphi_n(f). \end{aligned}$$

Hence, $\psi_{n+1} j_n^{n+1} = \psi_n$, and we can consider the A-module morphism

$$\psi : \varinjlim (A_n^*, j_n^{n+1}) \longrightarrow X, \quad \psi = \varinjlim \psi_n.$$

Define the A-module morphism $\rho : X \to \mathcal{B}(A, E)$ by the rule $[\rho(x)](a) = p(a \cdot x)$. We endow the space $\mathcal{B}(A, E)$, with the topology of uniform convergence on bounded subsets of A. Let us prove the continuity of ρ relative to this topology. Suppose D is a bounded subset of A and U is a 0-neighborhood in E. A typical 0-neighborhood in $\mathcal{B}(A, E)$ has the form

$$M(D, U) = \{g \in \mathcal{B}(A, E) : g(D) \subset U\}.$$

Take such a 0-neighborhood $M(D, U) \subset \mathcal{B}(A, E)$; obviously, $W = p^{-1}(U)$ is a 0-neighborhood in X. Since D is bounded, we can take a 0-neighborhood $V \subset X$ such that $D \cdot V \subset W$. Indeed, if V_0 and V_1 are 0-neighborhoods in A and X, respectively, such that $V_0 \cdot V_1 \subset W$ and $\lambda > 0$ is such that $D \subset \lambda V_0$, then $D \cdot \lambda^{-1} V_1 \subset W$, and we can put $V = \lambda^{-1} V_1$. We have

$$[\rho(V)](D) = p(D \cdot V) \subset p(W) = U,$$

i.e., $\rho(V) \subset M(D, U)$. Hence, ρ is continuous.

Consider the following commutative diagram

$$\begin{tikzcd}
\varinjlim A_n^* \arrow[r, "\psi"] & X \arrow[r, "\rho"] \arrow[d, "p"'] & \mathcal{B}(A,E) \arrow[dl, "\tilde{p}"] \\
A_n^* \arrow[u, "j_n"] \arrow[r, "\varphi_n"'] & E &
\end{tikzcd}$$

Here $\tilde{p}(g) = g(1)$ for $g \in \mathcal{B}(A,E)$ and j_n is the natural map from A_n^* to $\varinjlim A_n^*$.

Put $\lambda = \rho\psi$. Since λ is an A-module morphism, we see that

$$[\lambda(j_n(f_n))](a) = f_n(\tau_n(a))e_0 \tag{1}$$

for all $n \in \mathbf{N}$ and $f_n \in A_n^*$; here $\tau_n : A \to A_n$ is the natural projection. Indeed,

$$\begin{aligned}
[\lambda(j_n(f_n))](a) &= [\lambda(j_n(f_n))](1 \cdot a) = [a \cdot \lambda(j_n(f_n))](1) \\
&= [\lambda(j_n(\tau_n(a) \cdot f_n))](1) = \varphi_n(\tau_n(a) \cdot f_n) \\
&= [\tau_n(a) \cdot f_n](1_n)e_0 = f_n(\tau_n(a))e_0.
\end{aligned}$$

Put $F = \operatorname{Im}\psi \subset X$. Note that $\operatorname{Im}\lambda$ is contained in the subspace

$$L = \{g \in \mathcal{B}(A,E) : \operatorname{Im} g \subset \mathbf{C}e_0\} \subset \mathcal{B}(A,E),$$

which is isomorphic to the strong dual space A^* (the isomorphism $i : A^* \to L$ is defined by the rule $f \mapsto f(\cdot)e_0$). We obtain the following commutative diagram:

$$\begin{tikzcd}
\varinjlim A_n^* \arrow[rr, "r"] \arrow[dr, "\alpha"'] & & A^* \\
& F \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Here $\alpha(x) = \psi(x)$, $\beta(y) = i^{-1}\rho(y)$, $r(x) = i^{-1}\lambda(x)$. It follows from (1) that

$$[r(j_n(f_n))](a) = f_n(\tau_n(a)) \tag{2}$$

for all $n \in \mathbf{N}$, $f_n \in A_n^*$. Combining this with [**6**, IV.4.4], we see that r is an algebraic isomorphism of the vector spaces $\varinjlim A_n^*$ and A^*. Hence, the maps α and β are also algebraic isomorphisms.

To continue the proof of the theorem, we need several lemmas. In what follows, by Z^* we shall denote the strong dual of an l.c.s. Z.

Lemma 1. *Let $E = \varinjlim(E_\alpha, j_\beta^\alpha)$ be a direct limit of l.c.s's (here α runs over some directed set $\mathcal{A}$). Consider the linear map $s : E^* \to \varprojlim(E_\alpha^*, (j_\beta^\alpha)^*)$, $s : f \mapsto \{j_\alpha^*(f) : \alpha \in \mathcal{A}\}$. Then s is a continuous algebraic isomorphism.*

Proof. It is well known that s is bijective, see [**1**, 26.1.2]. The continuity of s obviously follows from the definition of a topology on $\varprojlim(E_\alpha^*, (j_\beta^\alpha)^*)$. □

Corollary. *Suppose $E = \varinjlim E_n$ is the direct limit of a sequence of Banach spaces. Then $s : E^* \to \varprojlim E_n^*$ is a topological isomorphism.*

Proof. Since E is a (DF)-space, as the direct limit of a sequence of (DF)-spaces, we see that its strong dual E^* is a Fréchet space. Further, the space $\varprojlim E_n^*$ is also a Fréchet space (as the inverse limit of a sequence of Banach spaces). Now the assertion follows from Lemma 1 and the closed graph theorem. □

The following lemma, as well as the previous one, is surely well-known for specialists in the theory of topological vector spaces. Unfortunately, we could not find an exact reference, and so we give a proof.

Lemma 2. *Suppose that a Fréchet space E is represented as the reduced inverse limit of the accompanying Banach spaces: $E = \varprojlim(E_n, \tau_n^m)$. Then the continuous linear map $\theta : E^{**} \to \varprojlim(E_n^{**}, (\tau_n^m)^{**})$, $\theta(\alpha) = \{\tau_n^{**}(\alpha) : n \in \mathbf{N}\}$ is a linear topological embedding of E^{**} into $\varprojlim(E_n^{**}, (\tau_n^m)^{**})$.*

Proof. Let us first introduce some notation. Given a set $C \subset E$, by C° we denote the polar of C with respect to the duality $\langle E, E^*\rangle$. Further, given a set $D \subset E^*$, we write $D^\bullet$ for the polar of D with respect to the duality $\langle E^*, E^{**}\rangle$.

Suppose $\{U_n : n \in \mathbf{N}\}$ is a basis of convex circled 0-neighborhoods in E. Then $\{U_n^{\circ\bullet} : n \in \mathbf{N}\}$ is a basis of convex circled 0-neighborhoods in E^{**}. Without loss of generality, we can assume that $U_n \supset U_{n+1}$ for all n. Hence, $U_n^{\circ\bullet} \supset U_{n+1}^{\circ\bullet}$ for all n. Let $\|\cdot\|_n$ be the Minkowski functional of U_n. To simplify notation, we also denote the Minkowski functional of the set $U_n^{\circ\bullet}$ by the same symbol $\|\cdot\|_n$. Since E^{**} is a Fréchet space, see [**6**, IV.6.5], we see that it is isomorphic to the reduced inverse limit of the sequence of the accompanying Banach spaces $(E^{**})_n$. Denote the corresponding connecting maps by $\sigma_n^m : (E^{**})_m \to (E^{**})_n$, $m \geq n$, and denote the natural projections $E^{**} \to (E^{**})_n$ by σ_n. For each $n \in \mathbf{N}$ consider the linear continuous map $\tau_n^{**} : E^{**} \to E_n^{**}$. We have

$$\begin{aligned} \|\tau_n^{**}(\alpha)\| &= \sup\{|\tau_n^{**}(\alpha)f_n| : f_n \in \tau_n(U_n)^\circ\} \\ &= \sup\{|\alpha(f_n\tau_n)| : f_n \in \tau_n(U_n)^\circ\}; \\ \|\alpha\|_n &= \sup\{|\alpha(f)| : f \in U_n^\circ\}, \end{aligned} \tag{3}$$

for all $\alpha \in E^{**}$. It follows from (3) that $\|\tau_n^{**}(\alpha)\| \leq \|\alpha\|_n$. Let us prove the opposite inequality. Take any $f \in U_n^\circ$, then $|f(x)| \leq \|x\|_n$ for all $x \in E$, and f induces the functional $f_n \in E_n^*$ by the rule $f_n(\tau_n(x)) = f(x)$, $x \in E$. It follows that, for each $f \in U_n^\circ$, there exists $f_n \in \tau_n(U_n)^\circ$ such that $f = f_n\tau_n$. Combining this with (3), we obtain the inequality $\|\tau_n^{**}(\alpha)\| \geq \|\alpha\|_n$. Thus, we have $\|\tau_n^{**}(\alpha)\| = \|\alpha\|_n$. For each n the map τ_n^{**} induces the linear continuous map $\theta_n : (E^{**})_n \to E_n^{**}$, $\theta_n(\sigma_n(\alpha)) = \tau_n^{**}(\alpha)$. Since $\|\tau_n^{**}(\alpha)\| = \|\alpha\|_n$, we see that θ_n is an isometric embedding of $(E^{**})_n$ into E_n^{**}. Consider now the map $\theta : E^{**} \to \varprojlim E_n^{**}$, which is defined in the statement of the lemma. It is evident that $\theta = \varprojlim \theta_n$, and since all θ_n are isometric embeddings, it follows that the map θ is topologically injective. $\square$

Remark 3. It can easily be checked that the same proof is valid for an arbitrary complete barreled l.c.s. E.

Let us continue the proof of the main theorem. Recall that we have obtained the following commutative diagram:

$$\begin{tikzcd} \varinjlim A_n^* \arrow[rr, "r"] \arrow[dr, "\alpha"'] & & A^* \\ & F \arrow[ur, "\beta"'] & \end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd} & \varprojlim A_n^{**} & \\ (\varinjlim A_n^*)^* \arrow[ur, "s"] & & A^{**} \arrow[ul, "\theta"'] \arrow[ll, "r^*"'] \arrow[dl, "\beta^*"] \\ & F^* \arrow[ul, "\alpha^*"] & \end{tikzcd}$$

Here s is the isomorphism defined in Lemma 1 and θ is the embedding defined in Lemma 2. To prove that this diagram is commutative, we need only check that $\theta = sr^*$. Recall that we have obtained the equality (2): $[r(j_n(f_n))](a) = f_n(\tau_n(a))$ for all $a \in A$, $f_n \in A_n^*$. This means that $rj_n = \tau_n^*$. We have

$$\theta(\eta) = \{\tau_n^{**}(\eta)\} = \{\eta\tau_n^*\} = \{\eta r j_n\} = \{j_n^*(\eta r)\} = \{j_n^*(r^*(\eta))\} = s(r^*(\eta))$$

for all $\eta \in A^{**}$, i.e., $\theta = sr^*$. Hence, the latter diagram is commutative.

Since s is a topological isomorphism and θ is a topological embedding, we see that the map β^* is also a topological embedding. Hence, the spaces A^{**} and $A \subset A^{**}$ are topologically isomorphic to the corresponding subspaces of F^*. Further, the strong dual F^* of the metrizable l.c.s. F has a fundamental sequence of bounded sets; this means that there exists a sequence of bounded sets $\{B_n\}$ such that each bounded set $B \subset F^*$ is contained in some B_n, see [**6**, IV.5.2]. Hence, $A \subset F^*$ also has such a sequence $\{D_n\} : D_n = B_n \cap A$. Let U_n be a basis of convex 0-neighborhoods in A such that $U_n \supset U_{n+1}$ for all $n \in \mathbf{N}$. By assumption, A is nonnormable; hence, U_n is not contained in D_n for any n, see [**6**, II.2.1]. Therefore, we can take a point $x_n \in U_n \backslash D_n$

for each n. Evidently, the sequence $\{x_n\}$ converges to 0; hence, $\{x_n\}$ is a bounded set. On the other hand, $\{x_n\}$ is not contained in D_k for any k and thus cannot be bounded. This contradiction proves the theorem. $\square$

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ENDNOTE

1. Some of these questions were recently answered in the negative in author's paper [**5**].

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