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WEIGHTED NORM INEQUALITIES FOR MAXIMAL CONVOLUTION-TYPE OPERATORS

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1. Introduction. The classical approach to the study of convergence of approximate identity operators has strong connections to the weighted norm inequalities satisfied by the Hardy-Littlewood maximal function. In particular, if $\phi : \mathbf{R}^n \to \mathbf{R}_+$, $\|\phi\|_1 = 1$ and $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\varepsilon^{-1}x)$, then $\phi_\varepsilon * f \to f$ in L^p, $1 \leq p < \infty$. Further, if the associated maximal operator

$$T^* f(x) = \sup_{\varepsilon > 0} |\phi_\varepsilon * f(x)|$$

is dominated by the Hardy-Littlewood maximal function, then $\phi_\varepsilon * f(x) \to f(x)$ for almost every x.

In this paper we study the convergence questions of more general convolution-type operators:

$$T_\delta f(x) = \int_{\mathbf{R}^n} \phi_\delta(x,t) f(t)\, d\nu(t).$$

Here ν is a measure on $\mathbf{R}^n$ and $\{\phi_\delta : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_+\}$, $\delta \in \Gamma$ is an arbitrary collection of measurable functions. We give "weight" conditions on the measures μ, ν and specify the sequences $\delta_{ix} \subset \Gamma$ to obtain pointwise and norm-convergence with respect to the measure μ of the sequence

$$\{T_{\delta_{ix}} f(x)\}_{i \geq 1}.$$

In the Lebesgue measure case the proof of the convergence of $\{\phi_\varepsilon * f(x)\}$ for almost every x, as $\varepsilon \to 0$, proceeds by first showing that this is the case for $f \in C_c(\mathbf{R}^n)$, and then by proving that $\sup_{\varepsilon>0} |\phi_\varepsilon * f(x)| \leq c\|\phi\|_1 Mf(x)$, where Mf is the Hardy-Littlewood

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maximal function. The last inequality uses the facts that $\|\phi_\varepsilon\|_1 = \|\phi\|_1$ and the translation invariance of Lebesgue measure, something no longer available in general. Further, Mf has to be replaced by a maximal function that is more suitably adapted to our problem. Our approach will generalize a maximal operator introduced by Bagby [**1**] that involves "averages" over arbitrary sets instead of cubes or balls.

Section 2 introduces a class of general maximal operators and the associated Bagby type operators. Weighted weak-type norm inequalities for the new operators are obtained. Section 3 defines the Bagby maximal operator that we will use to obtain our convergence results. Both weak and strong-type norm inequalities are obtained for this operator. In Section 4 we introduce the convolution operators we wish to study and we examine the corresponding maximal convolution operator. In Section 5 pointwise convergence results are given and in Section 6 norm convergence is studied.

Throughout, we will assume that μ and ν are two nonnegative Borel measures on $\mathbf{R}^n$ such that $0 < \mu(Q)$, $\nu(Q) < \infty$ for every cube $Q \subset \mathbf{R}^n$. $\mathcal{B}$ will denote the σ-algebra of Borel measurable sets. All functions will be assumed Borel measurable and nonnegative. f_μ^* is the nonincreasing rearrangement of f with respect to the measure μ. $\|f\|_{p,q,\mu}$ denotes the Lorentz space "norm" of f:

$$\|f\|_{p,q,\mu} = \left(\frac{q}{p}\int_0^\infty [t^{1/p} f_\mu^*(t)]^q \frac{dt}{t}\right)^{1/q}, \quad 1 \le p < \infty,$$

and, when $q = \infty$,

$$\|f\|_{p,\infty,\mu} = \sup_{y>0} y[\mu\{f > y\}]^{1/p}.$$

Given nonnegative integer s, s' will denote the conjugate exponent to s defined by the equation $ss' = s + s'$.

2. Bagby's maximal function. Let $B_{r,x}$ denote the closed ball centered at x with radius $r > 0$. Consider the maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B_{r,x})^\sigma \nu(B_{r,x})^\rho} \int_{B_{r,x}} f(t)\, d\nu(t)$$

where $f \ge 0$, $0 < \sigma \le 1$, $0 \le \rho < 1$ and $\sigma + \rho = 1$.

Lemma 2.1. *If* $1 \le \sigma^{-1} < \infty$, *then*

$$\|Mf\|_{(1/\sigma),\infty,\mu} \le C\|f\|_{(1/\sigma),\nu}. \tag{2.1}$$

Proof. If we set $\phi_{B_{r,x}} = \chi_{B_{r,x}} \nu(B_{r,x})^{-\rho} \mu(B_{r,x})^{-\sigma}$, then

$$Mf(x) = \sup_{r>0} \int f\phi_{B_{r,x}}\, d\nu.$$

If $M^R f(x)$ is the above maximal function with the sup extended over all $r \le R$ and if $E = \{M^R f > y\}$, then using the Besicovitch covering theorem, we can find $\{B_{r_j}\}_{j\ge1}$, $r_j \le R$, such that $E \subset \cup B_{r_j}$, $\sum \chi_{B_{r_j}} \le c$, and

$$\int f\phi_{B_{r_j}}\, d\nu > y.$$

From this, we get

$$\begin{aligned}\mu(E) &\le \sum \mu(B_{r_j}) \le \frac{1}{y^{1/\sigma}} \sum \mu(B_{r_j}) \Big(\int_{B_{r_j}} f\phi_{B_{r_j}}\, d\nu \Big)^{1/\sigma} \\ &\le \frac{1}{y^{1/\sigma}} \sum \mu(B_{r_j}) \int_{B_{r_j}} f^{1/\sigma}\, d\nu \Big(\int_{B_{r_j}} \phi_{B_{r_j}}^{1/\rho}\, d\nu \Big)^{\rho/\sigma}.\end{aligned}$$

Since

$$\mu(B_{r_j}) \Big(\int_{B_{r_j}} \phi_{B_{r_j}}^{1/\rho}\, d\nu \Big)^{\rho/\sigma} = \mu(B_{r_j}) \Big(\frac{\nu(B_{r_j})}{\nu(B_{r_j})\mu(B_{r_j})^{\sigma/\rho}} \Big)^{\rho/\sigma} = 1,$$

we obtain

$$\mu(E) \le \frac{c}{y^{1/\sigma}} \int_{\mathbf{R}^n} f^{1/\sigma}\, d\nu.$$

The constant c is independent of R and hence the result follows if we let $R \to \infty$.

Remark. When $\sigma = 0$, $Mf(x) = \sup_r \nu(B_{r,x})^{-1} \int_{B_{r,x}} f\, d\nu$, and this is weak (1,1) with respect to the single measure ν.

Lemma 2.2. *Assume* $1 \le r < \sigma^{-1} < \infty$*. If* $\mu \le c\nu$*, then*

$$\|Mf\|_{r,\infty,\mu} \le C\|f\|_{r,\nu}.$$

Proof. We begin as in Lemma 2.1.

$$\begin{aligned}\mu(E) &\le \sum \mu(B_j) \le \frac{1}{y^r}\sum \mu(B_j)\left(\int_{B_j} f\phi_{B_j}\,d\nu\right)^r \\ &\le \frac{1}{y^r}\sum \mu(B_j)\left(\int_{B_j} f^r\,d\nu\right)\left(\int_{B_j}\phi_{B_j}^{r'}\right)^{r/r'}\end{aligned}$$

and

$$\begin{aligned}\mu(B_j)\left(\int_{B_j}\phi_{B_j}^{r'}\right)^{r/r'} &= \mu(B_j)\left(\frac{\nu(B_j)}{\nu(B_j)^{r'\rho}\mu(B_j)^{r'\sigma}}\right)^{r/r'} \\ &= \frac{\mu(B_j)^{1-r\sigma}}{\nu(B_j)^{(r'\rho-1)r/r'}} \le c.\end{aligned}$$

Remark. Under the hypothesis of Lemma 2.2,

$$\int_{\mathbf{R}^n}(Mf)^r\,d\mu \le c\int_{\mathbf{R}^n} f^r\,d\nu$$

if $1 < r < \sigma^{-1}$. This follows from the Marcinkiewicz interpolation theorem.

We will use inequality (2.1) to obtain weighted weak-type norm inequalities for the following Bagby maximal function.

Definition. For any numbers $1 \le p_1, p_2, p_3 \le \infty$ with $1/p_1 + 1/p_2 + 1/p_3 \le 1$, define the Bagby set function

$$m(E,x) = \sup_{r>0}\{\mu(B_{r,x})^{1/p_1}\nu(B_{r,x})^{1/p_2}\nu(E\backslash B_{r,x})^{1/p_3}\}$$

where $E \subset \mathcal{B}$, $x \in \mathbf{R}^n$. Further, define

$$Af(x) = \sup \frac{1}{m(E,x)} \int_E f\, d\nu,$$

where the sup is taken over all sets $E \in \mathcal{B}$ with $0 < \nu(E) < \infty$.

Throughout, we shall assume that Af is measurable, and we denote by $\mathcal{C}$ the collection of all measures ν that satisfy the continuity conditions: $\nu(B_{rx}) \to 0$ as $r \to 0$, $\nu(B_{rx})$ is continuous in r for every x.

Theorem 2.3. *Let $\nu \in \mathcal{C}$. Given $1 < p_1, p_3 < \infty$, $1 < p_2 \leq \infty$ with $1/p_1 + 1/p_2 + 1/p_3 \leq 1$,*

$$\|Af\|_{p_1,\infty,\mu} \leq C\|f\|_{\zeta,1,\nu},$$

where $\zeta = (1 - (1/p_2) - (1/p_3))^{-1}$.

Proof. Let $\gamma = s'(p_3 - s')^{-1}$ where s is chosen so that $1/p_1 + 1/p_2 + s'/p_3 = 1$. Further, let $\sigma = p_3(p_1p_3 - p_1s')^{-1}$ and $\rho = p_3(p_2p_3 - p_2s')^{-1}$. Note that $\sigma + \rho = 1$. For each $E \in \mathcal{B}$ with $\nu(E) > 0$, $\tau > 0$, $x \in \mathbf{R}^n$, define

$$r(E,\tau,x) = \sup\left\{ r : \tau^\gamma \geq \frac{\mu(B_{r,x})^{\sigma - 1/p_1}\nu(B_{r,x})^{\rho - 1/p_2}}{\nu(E \backslash B_{r,x})^{1/p_3}} \right\}.$$

Then $r' < r(E,\tau,x) < r''$ gives

(2.2)
$$\frac{\mu(B_{r',x})^{\sigma-1/p_1}\nu(B_{r',x})^{\rho-1/p_2}}{\nu(E\backslash B_{r',x})^{1/p_3}} \leq \tau^\gamma \leq \frac{\mu(B_{r'',x})^{\sigma-1/p_1}\nu(B_{r'',x})^{\rho-1/p_2}}{\nu(E\backslash B_{r'',x})^{1/p_3}}.$$

By the continuity properties of ν, for any $0 < \tau < \infty$,

$$\begin{aligned}\frac{1}{m(E,x)} \int_E f\, d\nu &\leq \frac{1}{m(E,x)} \int_{B_{r(E,\tau,x),x}} f\, d\nu \\ &\quad + \frac{1}{m(E,x)} \int_{E\backslash B_{r(E,\tau,x),x}} f\, d\nu \\ &= \lim_{r'\uparrow r(E,\tau,x)} \left(\frac{1}{m(E,x)} \int_{B_{r',x}} f\, d\nu \right) \\ &\quad + \lim_{r''\downarrow r(E,\tau,x)} \left(\frac{1}{m(E,x)} \int_{E\backslash B_{r'',x}} f\, d\nu \right) \\ &= A + B.\end{aligned}$$

To estimate A, use the first half of expression (2.2)

$$\begin{aligned}\frac{1}{m(E,x)}\int_{B_{r',x}} f\,d\nu &\le \frac{\mu(B_{r',x})^{\sigma}\nu(B_{r',x})^{\rho}}{m(E,x)}Mf(x)\\ &\le \frac{\mu(B_{r',x})^{\sigma-1/p_1}\nu(B_{r',x})^{\rho-1/p_2}}{\nu(E\backslash B_{r',x})^{1/p_3}}Mf(x)\\ &\le \tau^{\gamma}Mf(x).\end{aligned} \tag{2.3}$$

The second half of (2.2) implies that, for $r'' > r(E,\tau,x)$,

$$\tau^{-1} \ge \frac{\nu(E\backslash B_{r'',x})^{(1/s')-(1/p_3)}}{\mu(B_{r'',x})^{1/p_1}\nu(B_{r'',x})^{1/p_2}}.$$

Therefore,

$$\begin{aligned}\frac{1}{m(E,x)}\int_{E\backslash B_{r'',x}} f\,d\nu &\le \frac{1}{m(E,x)}\|f\|_{s,\nu}\nu(E\backslash B_{r'',x})^{1/s'}\\ &\le \frac{\nu(E\backslash B_{r'',x})^{(1/s')-(1/p_3)}}{\mu(B_{r'',x})^{(1/p_1)}\nu(B_{r'',x})^{1/p_2}}\|f\|_{s,\nu}\\ &\le \tau^{-1}\|f\|_{s,\nu}.\end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4) we have that, for any $0 < \tau < \infty$,

$$\frac{1}{m(E,x)}\int_E f\,d\nu \le \tau^{\gamma}Mf(x) + \tau^{-1}\|f\|_{s,\nu}. \tag{2.5}$$

Taking $\tau = c_\gamma Mf(x)^{-1/(\gamma+1)}\|f\|_{s,\nu}^{1/(\gamma+1)}$ so that the right side of the above inequality is minimal gives

$$Af(x) \le c_\gamma Mf(x)^{1/(\gamma+1)}\|f\|_{s,\nu}^{\gamma/(\gamma+1)}. \tag{2.6}$$

Therefore,

$$\begin{aligned}\mu\{x : Af(x) > y\} &\le \mu\left\{x : Mf(x) > \frac{y^{\gamma+1}}{c_\gamma\|f\|_{s,\nu}^{\gamma}}\right\}\\ &\le \frac{C_\sigma}{y^{(\gamma+1)/\sigma}}\|f\|_{s,\nu}^{\gamma/\sigma}\|f\|_{1/\sigma,\nu}^{1/\sigma}\\ &\le \frac{C_\sigma}{y^{p_1}}\|f\|_{s,\nu}^{\gamma/\sigma}\|f\|_{1/\sigma,\nu}^{1/\sigma}.\end{aligned} \tag{2.7}$$

Letting $f = \chi_B \in L^s(\nu)$ for any set B with $\nu(B) < \infty$ gives

$$\mu\{x : Af(x) > y\} \leq \frac{C_\sigma}{y^{p_1}} \nu(B)^{(s'p_1+sp_3)/sp_3}.$$

That is,

$$\|A\chi_B\|_{p_1,\infty,\mu} \leq C_\sigma^{p_1} \nu(B)^\zeta.$$

Since A is sublinear, we may apply the proof of Theorem 3.13 [**10**, p. 195] to get the desired result.

Remarks. Following the proof of the previous theorem, we use the notation $\sigma = p_3(p_1p_3 - p_1s')^{-1}$ where necessary.

(i) In the proof of Theorem 2.3, $s = 1/\sigma$ implies that Af is of weak-type $(1/\sigma/, p_1)$, i.e.,

$$\mu\{x : Af(x) > y\} \leq \frac{C}{y^{p_1}} \|f\|_{1/\sigma,\nu}^{p_1}.$$

To see this, (2.7) gives

$$\begin{aligned}\mu\{x : Af(x) > y\} &\leq \frac{C}{y^{p_1}} \|f\|_{1/\sigma,\nu}^{(\gamma+1)/\sigma}\\ &= \frac{C}{y^{p_1}} \|f\|_{1/\sigma,\nu}^{p_1}.\end{aligned}$$

(ii) Under the above conditions, if $\mu \leq c\nu$ for some constant c, p_1, p_2, p_3 are such that $1 < p_2,\ p_3 < \infty$ and $1/p_1 + 1/p_2 + 1/p_3 + p_2/p_1p_3 < 1$, then we can deduce that

$$\|Af\|_{q_1,\mu} \leq C_{\sigma,\gamma} \|f\|_{q_2,\nu}$$

where $q_1 = \gamma(1 - 1/p_1 - 1/p_2 - 1/p_3)^{-1}$ and $q_2 = (1 - 1/p_1 - 1/p_2)(1 - 1/p_1 - 1/p_2 - 1/p_3)^{-1}$.

To see this, use the fact that $1 < s < \sigma^{-1}$ and Mf is strong (r, r) for any $1 < r < \sigma^{-1}$, by Lemma 2.2. Then, by (2.6),

$$\int_{\mathbf{R}^n} Af^{s(\gamma+1)}\, d\mu \leq C\|f\|_{s,\nu}^{s\gamma} \int_{\mathbf{R}^n} Mf^s\, d\mu \leq C\|f\|_{s,\nu}^{s(\gamma+1)}.$$

Calculating out the appropriate exponents gives the result.

(iii) If $\mu = \nu =$ Lebesgue measure, $p_2 = \infty$ and $p_3 = p_1'$, then A corresponds to Bagby's original maximal function [**1**]. He showed that Af is lower semi-continuous and that A need not be weak (p_1, p_1).

(iv) If $p_3 = \infty$, Af can be replaced by

$$Mf(x) = \sup \mu(B)^{-1/p_1} \nu(B)^{-1/p_1'} \int_B f \, d\nu$$

where the sup is taken over all balls B centered at x. This operator is weak-type (p_1, p_1) by Lemma 2.1.

With a slight modification, the proof of Theorem 2.3 yields certain strong-type norm inequalities for Af. Recall that, for $f \geq 0$,

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B_{r,x})^\sigma \nu(B_{r,x})^\rho} \int_{B_{r,x}} f(t) \, d\nu(t)$$

where $\sigma = p_3(p_1p_3 - p_1s')^{-1}$, $\rho = p_3(p_2p_3 - p_2s')^{-1}$.

Theorem 2.4. *If Mf satisfies the weighted strong-type norm inequality*

$$\|Mf\|_{p,\mu} \leq B\|f\|_{p,\nu} \tag{2.8}$$

for all $f \in L^p(\nu)$, $\nu \in \mathcal{C}$, $1 < p < \infty$, then for $r = pp_3(p_3 - p')^{-1}$,

$$\|Af\|_{r,\mu} \leq C_{p,p_3}\|f\|_{p,\nu}. \tag{2.9}$$

Specifically, $C_{p,p_3} = B^{1/r}[p'/(p_3 - p')]^{-p'/p_3}$.

Proof. Proceeding as in Theorem 2.3 with $s = p$ and $\gamma = p'/(p_3 - p')$, we choose $\tau = \gamma^{-1/(\gamma+1)}Mf(x)^{-1/(\gamma+1)}\|f\|_{p,\nu}^{-1/(\gamma+1)}$ so that the right side of inequality (2.5) is minimized. Thus,

$$\begin{aligned} Af(x) &\leq \gamma^{-\gamma/(\gamma+1)}Mf(x)^{1/(\gamma+1)}\|f\|_{p,\nu}^{\gamma/(\gamma+1)} \\ &= \left(\frac{p'}{p_3 - p'}\right)^{p'/p_3} Mf(x)^{1-p'/p_3}\|f\|_{p,\nu}^{p'/p_3}. \end{aligned}$$

Therefore, by (2.8),

$$\int_{\mathbf{R}^n} Af(x)^r \, d\mu(x) \le B\left(\frac{p'}{p_3 - p'}\right)^{-pp'/(p_3-p')} \|f\|_{p,\nu}^r,$$

and inequality (2.9) follows immediately.

Question. We note that $r \uparrow \infty$ as $p_3 \downarrow p'$. Similarly, $r \downarrow p$ as $p_3 \uparrow \infty$. Do these facts have any interesting consequences? For example, can we extract an $r = p$ result?

3. Convolution-type operators. Let $\{\phi_\delta : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_+\}$, $\delta \in \Gamma$, be a collection of Borel-measurable functions. For $f \ge 0$, we now study operators of the form

$$T_\delta f(x) = \int_{\mathbf{R}^n} f(t)\phi_\delta(x,t)\, d\nu(t)$$

for $f \ge 0$. We define the *essential least radial majorant* of ϕ_δ about x as

$$\tilde{\phi}_\delta(x,t) = \sup_{|\tau - x| \ge |t-x|} \phi_\delta(x,\tau).$$

Let E^δ_{xy} and $\tilde{E}^\delta_{xy}$ represent the level sets of height y for $\phi_\delta(x,\cdot)$ and $\tilde{\phi}_\delta(x,\cdot)$, respectively. That is,

$$E^\delta_{xy} = \{t : \phi_\delta(x,t) > y\}, \qquad \tilde{E}^\delta_{xy} = \{t : \tilde{\phi}_\delta(x,t) > y\}. \tag{3.1}$$

$M(x)$ is the function

$$M(x) = \sup_\delta \int_0^\infty m(E^\delta_{xy}, x)\, dy.$$

We define the maximal convolution operator with respect to the ϕ_δ's as

$$T^* f(x) = \sup_\delta T_\delta f(x).$$

Lemma 3.1. *If* $1 \le p_3 < \infty$, $1/p_1 + 1/p_2 + 1/p_3 \le 1$ *and* $1/p_1 + 1/p_2 + s'/p_3 = 1$,

$$T^* f(x) \le Af(x)M(x).$$

Further, define

$$\tilde{M}(x) = \sup_\delta \int_0^\infty \mu(\tilde{E}_{xy}^\delta)^{1/p_1} \nu(\tilde{E}_{xy}^\delta)^{1/p_2} \nu(E_{xy}^\delta)^{1/p_3}\, dy,$$

$$M_\mu(x) = \sup_\delta \int_0^\infty \tilde{\phi}_\delta(x,t)\, d\mu(t),$$

and $M_\nu(x)$ *is defined similar to* $M_\mu(x)$*. If, for each* x*, there is a* $0 < R(x) < \infty$ *so that* $\operatorname{supp}\phi_\delta(x,\cdot) \subset B_{R(x),x}$*, then*

$$\begin{aligned} Af(x)M(x) &\le Af(x)\tilde{M}(x) \\ &\le Af(x) M_\mu(x)^{1/p_1} M_\nu(x)^{1/p_2} \cdot \sup_\delta \|\phi_\delta(x,\cdot)\|_{s',1,\nu}^{s'/p_3}. \end{aligned} \tag{3.2}$$

Proof. If $E_x^\delta = \{(t,y) : \phi_\delta(x,t) > y\}$, writing $\phi_\delta(x,t) = \int_0^\infty \chi_{E_x^\delta}(t,y)\, dy$ gives

$$T_\delta f(x) = \int_0^\infty \int_{E_{xy}^\delta} f(t)\, d\nu(t)\, dy \le Af(x) \int_0^\infty m(E_{xy}^\delta, x)\, dy$$

which proves the first inequality.

Further, assume that, for each x, $R(x)$ exists as stated. We define

$$r(y) = r_x^\delta(y) = \sup\{r : \nu(E_{xy}^\delta \backslash B_{r,x}) > 0\} \le R(x) < \infty.$$

Then

$$\mu(B_{r,x})^{1/p_1} \nu(B_{r,x})^{1/p_2} \nu(E_{xy}^\delta \backslash B_{r,x})^{1/p_3} \le \mu(B_{r(y),x})^{1/p_1} \nu(B_{r(y),x})^{1/p_2} \nu(E_{xy}^\delta)^{1/p_3}$$

so that

$$\begin{aligned} \int_0^\infty m(E_{xy}^\delta, x)\, dy &\le \int_0^\infty \mu(B_{r(y),x})^{1/p_1} \nu(B_{r(y),x})^{1/p_2} \nu(E_{xy}^\delta)^{1/p_3}\, dy \\ &= \int_0^\infty \mu(\tilde{E}_{xy}^\delta)^{1/p_1} \nu(\tilde{E}_{xy}^\delta)^{1/p_2} \nu(E_{xy}^\delta)^{1/p_3}\, dy \\ &\le \left(\int_0^\infty \mu(\tilde{E}_{xy}^\delta)\, dy \right)^{1/p_1} \left(\int_0^\infty \nu(\tilde{E}_{xy}^\delta)\, dy \right)^{1/p_2} \\ &\quad \cdot \left(\int_0^\infty \nu(E_{xy}^\delta)^{1/s'}\, dy \right)^{s'/p_3} \\ &= \left(\int_{\mathbf{R}^n} \tilde{\phi}_\delta(x,t)\, d\mu(t) \right)^{1/p_1} \left(\int_{\mathbf{R}^n} \tilde{\phi}_\delta(x,t)\, d\nu(t) \right)^{1/p_2} \\ &\quad \cdot \|\phi_\delta(x,\cdot)\|_{s',1,\nu}^{s'/p_3}. \end{aligned}$$

We thus obtain the chain of inequalities (3.2).

Theorem 3.2. *Let* $1 \le p_1,\ p_3 < \infty$, $1/p_1 + 1/p_2 + 1/p_3 \le 1$. *If* $M(x) < \infty$ *for* μ*-almost every* x *and* $\nu \in \mathcal{C}$, *then there are sets* $A_1 \subset A_2 \subset A_3 \subset \cdots$, $\mu(\mathbf{R}^n \setminus \cup A_j) = 0$, *such that*

$$\|\chi_{A_j} T^* f\|_{p_1,\infty,\mu} \le C_j \|f\|_{\zeta,1,\nu}$$

where $\zeta = (1 - (1/p_2) - (1/p_3))^{-1}$. C_j *depends only on* A_j, p_1, p_2 *and* p_3.

Proof. By Lemma 3.1, $T^* f(x) \le A f(x) M(x)$. Letting $A_j = \{x : M(x) \le j\}$ and using Theorem 2.3 gives the result.

If $p_2 = \infty$, we get the following

Corollary 3.3. *Let* $\nu \in \mathcal{C}$. *If* $1 \le p_1, p_3 < \infty$ *are such that* $1/p_1 + 1/p_3 \le 1$ *and*

$$\sup_\delta \int_0^\infty \mu(\tilde{E}^\delta_{xy})^{1/p_1} \nu(E^\delta_{xy})^{1/p_3}\, dy = K(x) < \infty, \quad \mu\text{-almost every } x,$$

then

$$\|\chi_{A_j} T^* f\|_{p_1,\infty,\mu} \le C_j \|f\|_{p_3',1,\nu}$$

where $A_j = \{x : K(x) \le j\}$.

Remarks. (i) If $p_3 = \infty$, we can use the fact $\phi_\delta(x,t) \le \tilde{\phi}_\delta(x,t)$ to deduce that

$$T^* f(x) \le M f(x) M_\mu(x)^{1/p_1} M_\nu(x)^{1/p_1'} \tag{3.3}$$

where $Mf(x) = \sup \mu(B)^{-1/p_1} \nu(B)^{-1/p_1'} \int_B f\, d\nu$, the sup being taken over all balls B centered at x. The proof of Theorem 3.2 can be altered to produce

$$\|\chi_{A_j} T^* f\|_{p_1,\infty,\mu} \le C_{jp_1} \|f\|_{p_1,\nu}$$

where $A_j = \{x : M_\mu(x) \le j, M_\nu(x) \le j\}$.

(ii) Condition (3.3) of the previous remark is related to the A_p classes. If $d\mu = u\,dx$, $d\nu = v^{1-p'}\,dx$, $\phi = \chi_{B_{1,0}}$, $\phi_\varepsilon(x,t) = \varepsilon^{-n}\phi(\varepsilon^{-1}(x-t))$, then

$$T^*(fv^{p'-1})(x) = c_n M_{HL}f(x)$$

where $M_{HL}f(x)$ is the Hardy-Littlewood maximal function. If (u,v) satisfies the following "weak A_p" condition

$$\sup_{r>0}\left(\frac{1}{|B_{rx}|}\int_{B_{rx}} u\,dt\right)\left(\frac{1}{|B_{rx}|}\int_{B_{rx}} v^{1-p'}\,dt\right)^{p-1} < \infty, \quad \text{a.e. } x,$$

then weak-type norm inequalities can be obtained for $M_{HL}f(x)$ restricted to the increasing sets A_j; by the previous remark,

$$u\{x \in A_j : M_{HL}f(x) > y\} \le C_j y^{-p}\int f^p v\,dt.$$

(iii) Let α_j be a sequence of positive real numbers with $\alpha_j \uparrow \infty$. Let I_j be a sequence of disjoint intervals in $\mathbf{R}^1$ such that if $u_0 = \sum \alpha_j \chi_{I_j}$, then $M_{HL}u_0(x) = \alpha_j$ for $x \in I_j$. If $v \equiv 1$, $(u_0+v, v) \in$ weak $A_2 \backslash A_2$.

The classical approach to the subject of approximate identities, as discussed in [**8**] or [**11**], for example, requires that the least decreasing radial majorant of the kernel function be integrable. We will show now that Corollary 3.3 allows us to replace $\|\tilde{\phi}\|_1 < \infty$ by the weaker condition

$$\int_0^\infty |\{x : \tilde{\phi}(x) > y\}|^{1/p_0} |\{x : \phi(x) > y\}|^{1/p_0'}\,dy < \infty \tag{3.4}$$

for some $1 < p_0 < \infty$.

Corollary 3.4. *Let* $\phi : \mathbf{R}^n \to \mathbf{R}_+$ *be in* $L^1(\mathbf{R}^n)$. *If* $T_\delta f(x) = \phi_\delta * f(x)$, $\delta > 0$, $\phi_\delta(t) = \delta^{-n}\phi(\delta^{-1}t)$, *and* $T^*f(x) = \sup_{\delta>0}\phi_\delta * f(x)$, *then* (3.4) *implies*

(a) $|\{x : T^*f(x) > y\}| \le (c/y^{p_0})\|f\|_{p_0,1}^{p_0}$,

(b) $\|T^*f\|_p \le c_p\|f\|_p$, $p_0 < p \le \infty$.

Proof. The sets $\tilde{E}_{xy}^\delta$, E_{xy}^δ are as in (3.1) with $\phi_\delta(x,t) = \phi_\delta(x-t)$. Then, for every $\delta > 0$,

$$\int_0^\infty |\tilde{E}_{xy}^\delta|^{1/p_0} |E_{xy}^\delta|^{1/p_0'}\,dy = (3.4).$$

Hence, $A_j = \mathbf{R}^n$ and Corollary 3.3 gives (a); (b) follows by interpolation.

Remark. A result similar to Corollary 3.4 can be proved by denominating T^*f by the pth power Hardy-Littlewood maximal function

$$M_p f(x) = \sup_{r>0} \left(\frac{1}{|B_{rx}|} \int_{B_{Rx}} f^p \, dt \right)^{1/p}.$$

Proceeding as in Lemma 3.1,

$$\begin{aligned} \phi_\varepsilon * f(x) &= \int \phi_\varepsilon(x-t) f(t)\, dt \\ &= \int_0^\infty \int_{\{t:\phi_\varepsilon(x-t)>y\}} f(t)\, dt\, dy \\ &\leq \int_0^\infty |\{\tilde{\phi}_\varepsilon > y\}|^{1/p} |\{\phi_\varepsilon > y\}|^{1/p'}\, dy \cdot M_\phi f(x), \end{aligned}$$

where

$$\begin{aligned} M_\phi f(x) &= \sup_{\varepsilon, y} \frac{1}{|\{\tilde{\phi}_\varepsilon > y\}|^{1/p} |\{\phi_\varepsilon > y\}|^{1/p'}} \int_{\{\phi_\varepsilon > y\}} f(t)\, dt \\ &\leq \sup \frac{1}{|\{\tilde{\phi}_\varepsilon > y\}|^{1/p}} \left(\int_{\{\phi_\varepsilon > y\}} f^p(t)\, dt \right)^{1/p} \\ &\leq \sup \left(\frac{1}{|\{\tilde{\phi}_\varepsilon > y\}|} \int_{\{\tilde{\phi}_\varepsilon > y\}} f^p(t)\, dt \right)^{1/p} \\ &\leq M_p f(x). \end{aligned}$$

Since M_p is weak-type (p,p), so is T^*.

We will now show that $\tilde{\phi} \in L^1$ fails rather markedly to be a necessary condition for a norm inequality of T^*f. We do this for $n = 1$.

Theorem 3.5. *Assume that $\phi_0 \geq 0$ is a nonnegative decreasing function supported in $[0,1]$ such that $\phi_0(t) \to \infty$ as $t \to 0$ and $\phi_0(1) = 0$. There is an $E \subset [0,1]$ such that if $\phi(t) = \chi_E(t)\phi_0(t)$ and $\phi_\varepsilon(t) = \varepsilon^{-1}\phi(\varepsilon^{-1}t)$, then*

(a) $\phi \in L^1$,
(b) $L_y = |\{t : \tilde{\phi}(t) > y\}| \cdot |\{t : \phi_0(t) > y\}|^{-1} \to 1$ *as* $y \to \infty$,
(c) $\|T^* f\|_p \le C_p \|f\|_p$ *for all* $p > 1$, *where* $T^* f(x) = \sup_{\varepsilon > 0} \phi_\varepsilon * f(x)$.

Proof. Find $0 < \theta_n < \infty$ so that
(i) $\sum_n \phi_0(1/n)\theta_n < \infty$,
(ii) $\{I_n = [n^{-1}, n^{-1} + \theta_n]\}_{n \ge 2}$ *is a disjoint collection*,
(iii) $0 < \theta_n \le e^{-\phi_0(1/n)} - e^{-\phi_0(1/(n+1))}$.

Let $E = \cup_{n \ge 2} I_n$. We prove each point in the statement separately.

(a) $\int_0^1 \phi(t)\, dt = \sum_{n \ge 2} \int_{1/n}^{n^{-1}+\theta_n} \phi_0(t)\, dt \le \sum \phi_0(n^{-1})\theta_n$. Using (i) above gives the desired result.

(b) For fixed y, find n so that $\phi_0(1/n) \le y < \phi_0(1/(n+1))$. Then, since $\tilde{\phi}(t) \le \phi_0(t)$ almost everywhere,

$$L_y = \frac{|\{t : \tilde{\phi}(t) > y\}|}{|\{t : \phi_0(t) > y\}|} \le \frac{|\{t : \phi_0(t) > y\}|}{|\{t : \phi_0(t) > y\}|} = 1.$$

On the other hand,

$$\begin{aligned} L_y = \frac{|\{t : \tilde{\phi}(t) > y\}|}{|\{t : \phi_0(t) > y\}|} &\ge \frac{1}{n+1} \frac{1}{\{t : \phi_0(t) > y\}|} \\ &= \frac{1}{|t : \phi_0(t) > y\}|} \frac{n}{n+1} \frac{1}{n} \\ &\ge \frac{1}{|\{t : \phi_0(t) > y\}|} \frac{n}{n+1} |\{t : \phi_0(t) > y\}| \\ &= \frac{n}{n+1} \longrightarrow 1 \quad \text{as } y \to \infty. \end{aligned}$$

That is, $L_y \to 1$ as $y \to \infty$.

(c) Let $1 < p_0 < \infty$ be given. For each $\varepsilon > 0$, define $\tilde{E}_y^\varepsilon = \{t : \tilde{\phi}_\varepsilon(t) > y\}$ and $E_y^\varepsilon = \{t : \phi_\varepsilon(t) > y\}$. Put $\tilde{M}_\varepsilon = \int_0^\infty |\tilde{E}_y^\varepsilon|^{1/p_0} |E_y^\varepsilon|^{1/p_0'}\, dy$. For each y, find n so that $\phi_0(1/(n+1)) > \varepsilon y \ge \phi_0(1/n)$. Then

$$|\tilde{E}_y^\varepsilon| = \varepsilon |\{\tau : \tilde{\phi}(\tau) > \varepsilon y\}| \le \varepsilon |\{\tau : \phi_0(\tau) > \varepsilon y\}| \le \varepsilon \tag{3.5}$$

and

$$
\begin{aligned}
|E_y^\varepsilon| &= \varepsilon|\{\tau : \phi(\tau) > \varepsilon y\}| \le \varepsilon \sum_{j\ge n+1} \theta_j \\
&\le \varepsilon \sum_{j\ge n+1} \left(e^{-\phi_0(1/j)} - e^{-\phi_0(1/(j+1))}\right) \\
&= \varepsilon e^{-\phi_0(1/(n+1))} \le \varepsilon e^{-\varepsilon y}.
\end{aligned} \tag{3.6}
$$

Now, writing $\tilde{M}_\varepsilon = \int_0^\infty |\tilde{E}_y^\varepsilon|^{1/p_0}|E_y^\varepsilon|^{1/p_0'}\,dy = \int_0^{1/\varepsilon} + \int_{1/\varepsilon}^\infty = A_\varepsilon + B_\varepsilon$ allows us to make the following estimates using (3.5) and (3.6)

$$
\begin{aligned}
A_\varepsilon &= \int_0^{1/\varepsilon} |\tilde{E}_y^\varepsilon|^{1/p_0}|E_y^\varepsilon|^{1/p_0'}\,dy \\
&\le \varepsilon^{1/p_0}\varepsilon^{1/p_0'} \int_0^{1/\varepsilon} |\{\tau : \phi(\tau) > \varepsilon y\}|^{1/p_0'}\,dy \\
&= \int_0^1 |\{\tau : \phi(\tau) > y\}|^{1/p_0'}\,dy \le \|\phi\|_1^{1/p_0'}, \\
B_\varepsilon &= \int_{1/\varepsilon}^\infty |\tilde{E}_y^\varepsilon|^{1/p_0}|E_y^\varepsilon|^{1/p_0'}\,dy \\
&\le \varepsilon^{1/p_0}\varepsilon^{1/p_0'} \int_{1/\varepsilon}^\infty e^{-\varepsilon y/p_0'}\,dy \\
&= \int_1^\infty e^{-y/p_0'}\,dy = p_0' e^{-1/p_0'}.
\end{aligned}
$$

So, for large enough j, the sets A_j appearing in Corollary 3.3 equal $\mathbf{R}$. Hence, $\|T^*f\|_{p_0,\infty} \le C\|f\|_{p_0,1}$. Since the above argument holds for any $1 < p_0 < \infty$, interpolation gives the result.

Remark. Under the conditions of the theorem, we see that T^* is an operator that is strong (p,p) for every $1 < p < \infty$ but not necessarily weak $(1,1)$. This follows directly from Theorem 10.4.1 or Theorem 10.4.5 in [**3**].

4. Pointwise convergence. With each x in some subset $E \subset \mathbf{R}^n$, we associate a sequence $\{\delta_{ix}\}_{i\ge1} \subset \Gamma$, and we ask the question: given measures μ and ν, when do we get μ-almost everywhere convergence of the sequence $T_{\delta_{ix}}f(x)$ for appropriate $f \ge 0$?

For each function $h : \mathbf{R}^n \to \mathbf{R}_+$, define its exceptional set for convergence as

$$E_{h\nu} = \{x \in E : \overline{\lim_{i\to\infty}}\, T_{\delta_{ix}} h(x) - \underline{\lim_{i\to\infty}}\, T_{\delta_{ix}} h(x) > 0\}.$$

Since $E_{h\nu}$ need not be measurable, we will make use of μ^*, the outer measure induced by μ.

Theorem 4.1. *Assume $1/p_1 + 1/p_2 + s'/p_3 = 1$, $1 < p_1, p_2, p_3, s \leq \infty$, and let $\zeta = (1 - (1/p_2) - (1/p_3))^{-1}$. Let $f \geq 0$ in $L(\zeta, 1, \nu)$, $\varepsilon > 0$ and $\lambda < \mu^*(E_{f\nu})$ be given. Then if $M(x) < \infty$, μ-almost every x and $\nu \in \mathcal{C}$, there is a $g = g_{\varepsilon,\lambda,f,\nu} \in C_c(\mathbf{R}^n)$ such that*

$$\lambda \leq \mu^*(E_{g\nu}) + \varepsilon.$$

Proof. With $A_j = \{x : m(x) \leq j\}$ as in Theorem 3.2, let

$$E_{fj} = \{x \in E \cap A_j : \overline{\lim}\, T_{\delta_{ix}} f(x) - \underline{\lim}\, T_{\delta_{ix}} f(x) > 1/j\}.$$

Then $E_{f1} \subset E_{f2} \subset \cdots$.

We claim now that there is a j_0 such that $\lambda < \mu^*(E_{fj_0})$. If, for every j, $\mu^*(E_{fj}) \leq \lambda$, then $\mu^*(\cup E_{fj}) \leq \lambda$ and hence $\mu^*(E_{f\nu}) - \mu^*(\cup E_{fj}) > 0$. Since $\mu^*(E_{f\nu}) \leq \mu^*(\cup E_{fj}) + \mu^*(E_{f\nu} \backslash \cup E_{fj})$ and since $\mu^*(\cup E_{fj}) \leq \lambda < \infty$, we have $\mu^*(E_{f\nu} \backslash \cup E_{fj}) > 0$. However, $E_{f\nu} \backslash \cup E_{fj} \subset \mathbf{R}^n \backslash \cup A_j$ and $\mu(\mathbf{R}^n \backslash \cup A_j) = 0$.

For every $g \in C_c(\mathbf{R}^n)$,

$$E_{fj_0} \subset \{x \in E \cap A_{j_0} : 2T^*(f - g)(x) > (2j_0)^{-1}\} \cup E_{g\nu}.$$

Therefore, by Theorem 3.2,

$$\begin{aligned}\mu^*(E_{fj_0}) &\leq \mu\{x \in A_{j_0} : A|f - g|(x) > (4j_0)^{-2}\} + \mu^*(E_{g\nu}) \\ &\leq (4j_0^2)^{p_1} C \|f - g\|_{\zeta,1,\nu}^{p_1} + \mu^*(E_{g\nu}).\end{aligned}$$

Since $C_c(\mathbf{R}^n)$ is dense in $L(\zeta, 1, \nu)$, we choose g appropriately to give the result.

Theorem 4.1 allows us to attain the μ-almost everywhere convergence result for the appropriately defined sequences $\{\delta_{ix}\}$.

Definition. We call a sequence $\{\delta_{ix}\} \subset \Gamma$ a (ν, x)-sequence, provided

(i) $\int_{\mathbf{R}^n} \phi_{\delta_{ix}}(x,t)\, d\nu(t)$ converges as $i \to \infty$,

(ii) $\int_{|x-t|\geq\eta} \phi_{\delta_{ix}}(x,t)\, d\nu(t) \to 0$ as $i \to \infty$ for all $\eta > 0$.

Further, define the existence set E:

$$E = \{x \in \mathbf{R}^n : \exists (\nu, x)\text{-sequence } \{\delta_{ix}\}\}.$$

Theorem 4.2. *Assume* $1/p_1 + 1/p_2 + s'/p_3 = 1$, $1 < p_1, p_2, p_3$, $s \leq \infty$. *Let* $\{\phi_\delta : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_+\}$, $\delta \in \Gamma$ *be a collection of Borel measurable functions. Assume that* $M(x) < \infty$ *for* μ*-almost every* x *and* $\nu \in \mathcal{C}$. *Then, for* $f \in L(\zeta, 1, \nu)$, *the sequence* $\{T_{\delta_{ix}} f(x)\}_{i\geq 1}$ *converges for* μ*-almost every* $x \in E$.

Proof. We will show that, for $g \in C_c(\mathbf{R}^n)$ and $x \in E$, $T_{\delta_{ix}} g(x) \to L(x)g(x)$ where $L(x) = \lim_{i\to\infty} \int \phi_{\delta_{ix}}(x,t)\, d\nu(t)$. Theorem 4.1 then proves the result.

By the definition of (ν, x)-sequence, for each $x \in E$,

$$\int_{|x-t|<1} \phi_{\delta_{ix}}(x,t)\, d\nu(t) \longrightarrow L(x) \quad \text{as } i \to \infty.$$

Fix $x \in E$. For each $\varepsilon > 0$, there is an $N_\varepsilon \in \mathbf{N}$ so that $i \geq N_\varepsilon$ implies

$$\left| \int_{|x-t|<1} \phi_{\delta_{ix}}(x,t)\, d\nu(t) - L(x) \right| < \varepsilon.$$

Therefore,

$$\int_{|x-t|<1} \phi_{\delta_{ix}}(x,t)\, d\nu(t) \leq \max_{i=1,\dots,N_\varepsilon} \left\{ \int_{|x-t|<1} \phi_{\delta_{ix}}(x,t)\, d\nu(t), L(x) + \varepsilon \right\} = \mathcal{M}_x(\varepsilon).$$

Specifically taking $\varepsilon = 1$,

$$\int_{|x-t|<1} \phi_{\delta_{ix}}(x,t)\, d\nu(t) \le \mathcal{M}_x(1) < \infty, \quad \forall\, i.$$

Let $0 < \varepsilon < 1$ be given. Find $1 > \eta > 0$ so that $|x - t| < \eta \Rightarrow |g(x) - g(t)| < \varepsilon$. Then,

$$\begin{aligned} |T_{\delta_{ix}} g(x) - L(x)g(x)| &\le \varepsilon \int_{|x-t|<\eta} \phi_{\delta_{ix}}(x,t)\, d\nu(t) \\ &\quad + \int_{|x-t|\ge\eta} \phi_{\delta_{ix}}(x,t) |g(t) - g(x)|\, d\nu(t) \\ &\quad + |g(x)| \bigg| \int_{\mathbf{R}^n} \phi_{\delta_{ix}}(x,t)\, d\nu(t) - L(x) \bigg| \\ &\le \varepsilon \mathcal{M}_x(1) + 2\|g\|_{\infty,\nu} \int_{|x-t|\ge\eta} \phi_{\delta_{ix}}(x,t)\, d\nu(t) \\ &\quad + |g(x)| \bigg| \int_{\mathbf{R}^n} \phi_{\delta_{ix}}(x,t)\, d\nu(t) - L(x) \bigg|. \end{aligned}$$

Applying the definition of a (ν, x)-sequence finishes the proof.

The proof of Theorem 4.2 shows that, for $g \in C_c(\mathbf{R}^n)$, $T_{\delta_{ix}} g(x) \to L(x)g(x)$ where $L(x) = \lim \int \phi_{\delta_{ix}}(x,t)\, d\nu(t)$. We now show that, in general, the limit of the sequence $\{T_{\delta_{ix}} f(x)\}_{i\ge1}$ is $L(x)f(x)$.

Theorem 4.3. *Assume that $M(x) < \infty$, μ-almost everywhere, and that*

$$L(x) = \lim_{i\to\infty} \int \phi_{\delta_{ix}}(x,t)\, d\nu(t)$$

is measurable. Then, for $f \in L(\zeta, 1, \nu)$, $\nu \in \mathcal{C}$, where $\zeta = (1 - (1/p_2) - (1/p_3))^{-1}$,

$$T_{\delta_{ix}} f(x) \longrightarrow L(x)f(x) \quad \mu\text{-almost every } x \in E.$$

Proof. Case 1. Take $f \in L(\zeta, 1, \mu + \nu)$. Then

$$T_{\delta_{ix}} f(x) \to L(x)f(x) \quad \mu\text{-almost every } x \in E. \tag{4.1}$$

To see this, define $A_j^* = \{x \in A_j \cap E : |L(x)| \le j\}$ where $A_j = \{x : M(x) \le j\}$ as in Theorem 3.2. For $x \in A_j^*$ and any $g \in C_c(\mathbf{R}^n)$,

$$\begin{aligned}|T_{\delta_{ix}} f(x) - L(x)f(x)| &\le T_{\delta_{ix}}|f-g|(x) + |T_{\delta_{ix}} g(x) - L(x)g(x)| \\ &\quad + |L(x)|\,|g(x)-f(x)| \\ &\le jA(|f-g|)(x) + |T_{\delta_{ix}} g(x) - L(x)g(x)| \\ &\quad + j|g(x)-f(x)|.\end{aligned}$$

By the proof of Theorem 4.2,

$$\overline{\lim_{i\to\infty}}\, T_{\delta_{ix}} f(x) - L(x)f(x)| \le jA(|f-g|)(x) + j|g(x)-f(x)|.$$

Define $E_j = \{x \in A_j^* : \overline{\lim}\, |T_{\delta_{ix}} f(x) - L(x)f(x)| > j^{-1}\}$. We will show that $\mu(E_j) = 0$ for each j which proves (4.1). Note that $E_j \subset E_{j+1} \subset \cdots$ and

$$\begin{aligned}\mu(E_j) &\le \mu\{x \in A_j^* : A(|f-g|)(x) > (2j^2)^{-1}\} \\ &\quad + \mu\{x \in A_j^* : |g(x)-f(x)| > (2j^2)^{-1}\}.\end{aligned}$$

By Theorem 2.3,

$$\mu\{x \in A_j^* : A(|f-g|)(x) > (2j^2)^{-1}\} \le C(2j^2)^{p_1}\|f-g\|_{\zeta,1,\nu}^{p_1}$$

which can be made as small as we please by choosing g appropriately since $C_c(\mathbf{R}^n)$ is dense in $L(\zeta, 1, \nu)$. Similarly, by Chebychev's inequality,

$$\begin{aligned}&\mu\{x \in A_j^* \cap B : |g(x)-f(x)| > (2j)^{-1}\} \\ &\qquad \le (2j)^\zeta \int |f-g|^\zeta\, d\mu \le (2j)^\zeta \|f-g\|_{\zeta,1,\nu}^\zeta.\end{aligned}$$

Case 2. $f \in L(\zeta,1,\nu)$, $f = 0$ on $E \subset \mathbf{R}^n$, implies that

$$T_{\delta_{ix}} f(x) \longrightarrow 0 \quad \mu\text{-almost every } x \in E.$$

To prove this, take any compact set $K \subset E$ and show that $T_{\delta_{ix}} f(x) \to 0$ for μ-almost every $x \in K$. Let $G = \mathbf{R}^n \backslash K$, and let g be any arbitrary function in $C_c(G)$. For $x \in K$,

$$|T_{\delta_{ix}} f(x)| \le |T_{\delta_{ix}}(f-g)(x)| + |T_{\delta_{ix}} g(x)|.$$

Since $g = 0$ on K, $T_{\delta_{ix}} g(x) \to 0$. Therefore,

$$\overline{\lim}\, |T_{\delta_{ix}} f(x)| \le \overline{\lim}\, |T_{\delta_{ix}} (f - g)(x)|.$$

If A_j^* is defined as in Case 1, we can argue as before to show that

$$\begin{aligned}\mu\{x \in K \cap A_j^* : \overline{\lim}\, |T_{\delta_{ix}} f(x)| > j^{-1}\} &\le (c_j \cdot j)^\zeta \int_G |f - g| \, d\nu \\ &\le (c_j \cdot j)^\zeta \|f - g\|_{\zeta,1,\nu}^\zeta.\end{aligned}$$

Case 3. In general, for $f \in L(\zeta, 1, \nu)$, it suffices to show that $T_{\delta_{ix}} f(x) \to L(x) f(x)$ for μ-almost every $x \in B_k$ where

$$B_k = \{x : L(x) \le j, f(x) \le j, |x| \le j\}.$$

To do this, let $f_k = f \chi_{B_k}$ and $\tilde{f}_k = f \chi_{\mathbf{R}^n \setminus B_k}$. Then, for μ-almost every $x \in B_k$, Case 1 gives

$$T_{\delta_{ix}} f_k(x) \longrightarrow L(x) f_k(x) = L(x) f(x).$$

By Case 2,

$$T_{\delta_{ix}} \tilde{f}_k(x) \longrightarrow 0.$$

5. Norm convergence. In the previous section μ-almost everywhere convergence was obtained as a consequence of the weighted weak-type norm inequalities satisfied by Af. This section will use Theorem 2.4 to study the norm convergence properties of convolution operators. We will therefore require conditions under which Mf satisfies strong-type norm inequalities

$$\|Mf\|_{p,\mu} \le C \|f\|_{p,\nu}. \tag{5.1}$$

Simple examples show that (5.1) is not always satisfied. For instance, let $\nu =$ Lebesgue measure and $\mu = \nu + \delta_0$ where δ_0 is the Dirac delta concentrated at 0. Take any sequence $\{f_n\}$ in L^p with $f_n(0) \uparrow \infty$ and $\|f_n\|_p = 1$. Inequality (5.1) implies

$$\int M f_n(x)^p \, dx + f_n(0) \le C,$$

which is clearly false.

However, as a consequence of the following rearrangement inequality of Leckband and Neugebauer [**6**], we can find a suitable condition on the measures μ and ν for (5.1) to hold.

Theorem 5.1. *If, for each ball B in $\mathbf{R}^n$, there is a Borel measurable function $\phi_B : \mathbf{R}^n \to [0,\infty)$ with $\operatorname{supp}\phi_B \subset B$, then the maximal function*

$$\mathcal{M}f(x) = \sup_{r>0} \int_{\mathbf{R}^n} \phi_{B_{rx}} f\, d\nu$$

satisfies the rearrangement inequality

$$(\mathcal{M}f)^*_\mu(\xi) \le A \int_0^\infty \Phi(t) f^*_\nu(t\xi)\, dt.$$

A depends only on the dimension n and

$$\Phi(t) = \Phi_{\mu,\nu}(t) = \sup_B \{\mu(B)\phi^*_{B,\nu}(\mu(B)t)\}.$$

By Minkowski's integral inequality,

$$\|\mathcal{M}f\|_{p,\mu} \le C\|f\|_{p,\nu}$$

whenever $\Phi \in L(p',1)$. Therefore, a suitable condition on the measures μ and ν for (5.1) to hold is

$$\Phi(t) = \sup_{\mathcal{A}_t} [\mu(B)\nu(B)^{-1}]^{p_3/(p_2p_3 - p_2p')} \in L(p',1)$$

where $\mathcal{A}_t$ is the collection of balls B with $\nu(B) \ge t\mu(B)$. When $\mu = \nu$, for example, the condition is satisfied for every $p > 1$.

We shall assume that, if $\{\delta_{ix}\} \subset \Gamma$ is a (ν, x)-sequence for μ-almost every $x \in E$, then, for each i, $T_{\delta_{ix}} f(x)$ is measurable as a function of x on E.

Theorem 5.2. *Assume $M(x) < \infty$ μ-almost everywhere and $\Phi \in L(p',1)$, $1 < p' < p_3$. Let $A_j = \{x \in E : M(x) \le j\}$. Then, for every $f \in L^p(\nu)$, $\nu \in \mathcal{C}$, $1 < p < \infty$,*

$$\{\chi_{A_j} T_{\delta_{ix}} f(x)\}$$

converges in $L^r(\mu)$ *for* $r = pp_3(p_3 - p')^{-1}$.

Proof. Since $M(x) < \infty$ μ-almost everywhere, $\chi_{A_j} T_{\delta_{ix}} f(x) \to L(x)f(x)$, μ-almost everywhere. Also,

$$|\chi_{A_j} T_{\delta_{ix}} f(x) - L(x)f(x)|^r \leq 2^r T^* f(x)^r.$$

By Lemma 3.1 and inequality (2.9) of Theorem 2.4, we may apply the Lebesgue dominated convergence theorem.

The question of $L^1(\mu)$ convergence is more complicated. It does not appear that we can appeal to any of our pointwise convergence results since none apply to $L^1(\mu)$ functions. Define the nested sequence of sets A_j^* as

$$A_j^* = \left\{ x \in E : \sup_i \int \phi_{\delta_{ix}}(x,t)\, d\nu(t) \leq j \right\}.$$

Since $L(x) < \infty$ on E by definition, $\mu(E \setminus \cup A_j^*) = 0$. For each pair of nonnegative integers i and j, we define

$$H_{ij}(t) = \int_{A_j^*} \phi_{\delta_{ix}}(x,t)\, d\mu(x).$$

Theorem 5.3. *If* $\|H_{ij}\|_{\infty,\nu} \leq c_j < \infty$, $j = 1, 2, \ldots$, *then, for every* $f \in L^1(\mu + \nu)$,

$$T_{\delta_{ix}} f(x) \longrightarrow L(x)f(x) \quad \text{in } L^1(A_j^*, \mu).$$

Proof. Case 1. Assume $f \in C_c(\mathbf{R}^n)$. Let $F_0 = \operatorname{supp} f$, $F_k = \{x : \operatorname{dist}(x, F_0) \leq k\}$, $M = \sup f(x)$. Since each F_k is compact, $\mu(F_k) + \nu(F_k) < \infty$, $k \in \mathbf{N}$. Let $\varepsilon > 0$ be given.

Find k so large that

$$P_0 = \int_{A_j^* \setminus F_k} T_{\delta_{ix}} f(x)\, d\mu(x) < \varepsilon/6.$$

To do this, consider the integral

$$\begin{aligned}\int_{A_j^*\setminus F_0} T_{\delta_{ix}} f(x)\,d\mu(x) &= \int_{A_j^*\setminus F_0}\int_{F_0} \phi_{\delta_{ix}}(x,t)f(t)\,d\nu(t)\,d\mu(x)\\ &\le \int_{F_0} f(t)\int_{A_j^*}\phi_{\delta_{ix}}(x,t)\,d\mu(x)\,d\nu(t)\\ &\le M\cdot c_j\cdot \nu(F_0) < \infty.\end{aligned}$$

The finiteness of this integral together with the fact that $F_k \uparrow \mathbf{R}^n$ as $k\uparrow\infty$ provides the existence of such a k.

Next, choose $\eta > 0$ so that $|f(x)-f(t)| < \varepsilon[6j\mu(F_k)]^{-1}$ whenever $|x-t|\le\eta$. Define the sets S_m and T_m as

$$\begin{aligned}S_m &= \left\{x\in E : \int_{|x-t|>\eta}\phi_{\delta_{ix}}(x,t)\,d\nu(t) < \varepsilon[12M\mu(F_k)]^{-1}, i\ge m\right\},\\ T_m &= \left\{x\in E : \left|\int \phi_{\delta_{ix}}(x,t)\,d\nu(t) - L(x)\right| < \varepsilon[6M\mu(F_k)]^{-1}, i\ge m\right\}.\end{aligned}$$

We consider the following integrals:

$$\begin{aligned}P_1 &= \int_{A_j^*\cap F_k\cap S_m}\int_{|x-t|>\eta}\phi_{\delta_{ix}}(x,t)f(t)\,d\nu(t)\,d\mu(x),\\ P_2 &= \int_{A_j^*\cap F_k\setminus S_m}\int_{|x-t|>\eta}\phi_{\delta_{ix}}(x,t)f(t)\,d\nu(t)\,d\mu(x),\\ P_3 &= \int_{A_j^*\cap F_k}\int_{|x-t|\le\eta}\phi_{\delta_{ix}}(x,t)f(t)\,d\nu(t)\,d\mu(x),\\ P_4 &= \int_{A_j^*\cap F_k\cap T_m} f(x)\left|\int\phi_{\delta_{i_x}}(x,t)\,d\nu(t) - L(x)\right|d\mu(x),\\ P_5 &= \int_{A_j^*\cap F_k\setminus T_m} f(x)\left|\int\phi_{\delta_{ix}}(x,t)\,d\nu(t) - L(x)\right|d\mu(x).\end{aligned}$$

Then

$$\int_{A_j^*}|T_{\delta_{ix}}f(x) - L(x)f(x)|\,d\mu(x) \le P_0 + P_1 + P_2 + P_3 + P_4 + P_5.$$

By the definition of S_m, $P_1 \le \varepsilon/6$. $P_2 \le 2\cdot M\cdot j\mu(F_k\backslash S_m)$ which can be made less than $\varepsilon/6$ since $\mu(\mathbf{R}^n\backslash \cup S_m) = 0$. $P_3 \le \varepsilon/6$ by the definition of η. $P_4 \le \varepsilon/6$ by the definition of T_m, and, since $\mu(\mathbf{R}^n\backslash \cup T_m) = 0$, P_5 can be made less than $\varepsilon/6$.

Case 2. Assume now that $f \in L^1(\mu+\nu)$. For any $g \in C_c(\mathbf{R}^n)$,

$$\begin{aligned}\int_{A_j^*} |T_{\delta_{ix}} f(x) - L(x)f(x)|\,d\mu(x) &\le \int_{A_j^*}\int \phi_{\delta_{ix}}|f(t)-g(t)|\,d\nu(t)\,d\mu(x)\\ &\quad + \int_{A_j^*} |T_{\delta_{ix}} g(x) - L(x)g(x)|\,d\mu(x)\\ &\quad + \int_{A_j^*} L(x)|g(x)-f(x)|\,d\mu(x)\\ &= Q_1+Q_2+Q_3.\end{aligned}$$

Clearly, $Q_3 \le j\int|f-g|\,d\nu$. Fubini's theorem and the uniform boundedness condition on the H_{ij}'s gives $Q_1 \le c_j \int |f-g|\,d\mu$. Both of these terms can be made as small as we like by choosing g appropriately. Q_2 is controlled by Case 1.

Simple examples show that, in general, the L^1 convergence result cannot be extended to functions in $L^1(\nu)$. Let $\{\beta_n\}$, $n \ge 0$ be a sequence of nonnegative numbers with $\sum \beta_n \le 1$. Define

$$v(t) = \sum_{i\ge 0} \beta_i \chi_{(i,i+1]}(t) + \sum_{i\ge 0} \beta_i \chi_{(-i-1,-i]}(t),$$

and let δ_0 represent the Dirac delta measure concentrated at 0. Define the measure ν and μ by $d\nu(t) = v(t)\,dt + \delta_0$ and $\mu =$ Lebesgue measure. Put $\phi_i(x,t) = i^{-1}\chi_{(x,x+i)}(t)$. Clearly, $\int \phi_i(x,t)\,d\nu(t) \to L(x) \equiv 0$ as $i\to\infty$ and, as such, $\delta_{ix} = i$ is a (ν,x)-sequence for each x. Also, it is clear that $A_j^* = \mathbf{R}$ for $j \ge 3$. Therefore, for $j\ge 3$,

$$\int_{A_j^*} \phi_i(x,t)\,d\mu(x) = i^{-1}\int_{t-i}^{t} dx = 1.$$

For $f(t) \equiv 1 \in L^1(\nu)\backslash L^1(\mu)$ and $j \geq 3$,

$$\begin{aligned}\int_{A_j^*} |T_{\delta_{ix}} f(x) - L(x) f(x)| \, d\mu(x) &= \int_{\mathbf{R}} \int_{\mathbf{R}} \phi_i(x,t) \, d\nu(t) \, dx \\ &\geq \int_{-i}^{0} \phi_i(x,0) \, dx \\ &= 1.\end{aligned}$$

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