ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 3, Fall 1999

CONVEXITY AND SMOOTHNESS IN REAL INTERPOLATION FOR FAMILIES OF BANACH SPACES

L.Y. NIKOLOVA AND T. ZACHARIADES

ABSTRACT. We show how the uniform convexity and uniform nonsquare properties of an interpolation operator can be preserved by the induced operator between the real interpolation spaces for infinite families. Also we show how uniform smoothness is inherited by real interpolation spaces for infinite families.

Introduction. In [**9**] the authors studied how the properties of uniform convexity and uniform non-l_k^1 are inherited by the real interpolation spaces, which were constructed by Carro's K-method for infinite families.

In this paper we show how the uniform convexity and the uniform nonsquare properties of an interpolation operator can be preserved when we pass to the induced operator between the real interpolation spaces, which are constructed by Carro's K and J-methods for infinite families. These results extend some of the results of [**9**]. Let us note that the analogous problem has already been considered for the compact, weak compact and limited operators [**4, 3**]. Results of this type for finite families can be found in [**6, 7**] and other papers.

We also study how uniform smoothness is inherited by these interpolation spaces.

Preliminaries. Let D denote the unit disc $\{z \in C : |z| < 1\}$ and Γ its boundary. Let $\overline{A} = \{A(\gamma) : \gamma \in \Gamma, \mathcal{A}, \mathcal{U}\}$ be a complex interpolation family (i.f.) on Γ, with $\mathcal{U}$ as the containing Banach space and $\mathcal{A}$ as the log-intersection space, in the sense of [**8**]. More precisely, following [**8, 1, 2**] and [**5**], we assume

Received by the editors on July 26, 1997, and in revised form on October 14, 1997.
The research of the first author was partially supported by contract MM406/94 of Bulgarian Ministry of Education.

(i) the complex Banach spaces $A(\gamma)$ are continuously embedded in $\mathcal{U}(\|\cdot\|_\gamma$ will be the norm on $A(\gamma)$ and $\|\cdot\|_{\mathcal{U}}$ the norm on $\mathcal{U})$,

(ii) for every $x \in \cap_{\gamma\in\Gamma} A(\gamma)$, the function $\gamma \to \|x\|_\gamma$ is measurable on Γ,

(iii) if $\mathcal{A} = \{x \in A(\gamma),\ \text{a.e. } \gamma \in \Gamma : \int_\Gamma \log^+ \|x\|_\gamma \, d\gamma < +\infty\}$ is the log-intersection space, there exists a measurable function P on Γ such that $\int_\Gamma \log^+ P(\gamma)\, d\gamma < +\infty$ and, for every $x \in A$, $\|x\|_{\mathcal{U}} \leq P(\gamma)\|x\|_\gamma$ almost everywhere on Γ.

Let

$$\mathcal{L} = \{s : \Gamma \to R^+ \mid s \text{ is measurable and } \log s \in L^1(\Gamma)\},$$

and

$$\mathcal{C} = \left\{\alpha = \sum_{j=1}^n x_j \chi_E : n \in N, x_j \in \mathcal{A} \text{ for } j = 1, \dots, n \text{ and } (E_j)_{j=1}^n \right.$$
$$\left. \text{are pairwise disjoint measurable subsets of } \Gamma\right\}.$$

We denote by $\overline{\mathcal{C}}$ the set of all Bochner integrable functions $\alpha : \Gamma \to \mathcal{U}$ such that $\alpha(\gamma) \in A(\gamma)$ almost everywhere on Γ, and there exists a sequence $(\alpha_n)_n$ in $\mathcal{C}$ such that $\lim_n \|\alpha_n(\gamma) - \alpha(\gamma)\|_\gamma = 0$ almost everywhere on Γ. For every $\alpha \in \overline{\mathcal{C}}$ we define $f_\alpha : \Gamma \to R$, by $f_\alpha(\gamma) = \|\alpha(\gamma)\|_\gamma$ if $\alpha(\gamma) \in A(\gamma)$, and $f_\alpha(\gamma) = 0$ otherwise. It is clear that $\overline{\mathcal{C}}$ is a linear space, and if $\alpha_1, \alpha_2 \in \overline{\mathcal{C}}$ and $sf_{\alpha_1}, sf_{\alpha_2} \in L^q(\Gamma)$, then $sf_{\alpha_1+\alpha_2} \in L^q(\Gamma)$ for $1 \leq q \leq +\infty$ and $s \in \mathcal{L}$.

Now following [**1**] and [**2**], we define the K_q and J_q-functionals and the interpolation spaces $[A]^S_{\theta,p,q}$ and $(A)^S_{\theta,p,q}$, with respect to the i.f. $\overline{A}$.

Let $s \in \mathcal{L}$ and $1 \leq q \leq +\infty$.

We put

$$K_q(s) = \left\{x \in \mathcal{U} : \text{ there exists } \alpha \in \overline{\mathcal{C}} \text{ with } \right.$$
$$\left. x = \int_\Gamma \alpha(\gamma)\, d\gamma \text{ and } sf_a \in L^q(\Gamma)\right\}$$

and, for each $x \in K_q(s)$, we define the $K-q$-functional by

$$K_q(s,x) = \inf\left\{ \left(\int_\Gamma (s(\gamma) f_\alpha(\gamma))^q \, d\gamma \right)^{1/q} : \right.$$
$$\left. x = \int_\Gamma \alpha(\gamma)\, d\gamma \text{ and } s f_\alpha \in L^q(\Gamma) \right\}.$$

Also we put

$$J_q(s) = \left\{ x \in \mathcal{A} : \int_\Gamma (s(\gamma)\|x\|_\gamma)^q \, d\gamma < +\infty \right\},$$

and, for each $x \in J_q(s)$, we define the J_q-functional by

$$J_q(s,x) = \left(\int_\Gamma (s(\gamma)\|x\|_\gamma)^q \, d\gamma \right)^{1/q}.$$

If we have more than one i.f., then we write $K_{q,\overline{A}}(s)$, $J_{q,\overline{A}}(s)$, $K_{q,\overline{A}}(s,x)$ and $J_{q,\overline{A}}(s,x)$.

It is easy to see that the K_q-functional is a semi-norm and the J_q-functional is a norm.

For $s \in \mathcal{L}$ and $\theta \in D$ we write $s(\theta) = \exp \int_\Gamma \log s(\gamma) P_\theta(\gamma)\, d\gamma$ where P_θ is the Poisson kernel at the point θ.

Let $S \subset \mathcal{L}$ and $1 \le p,\ q \le \infty$, $\overline{A}$ be an i.f. and $\theta \in D$. The space $[A]^S_{\theta,p,q}$ consists of all $x \in \mathcal{U}$ for which $x \in K_q(s)$ for every $s \in S$ and $(K_q(s,x)/s(\theta))_{s\in S} \in l^p(S)$, endowed with the quasi-seminorm

$$\|x\|_{[A]^S_{\theta,p,q}} = \left(\sum_{s\in S} \left(\frac{K_q(s,x)}{s(\theta)} \right)^p \right)^{1/p}.$$

The space $(A)^S_{\theta,p,q}$ is the set of all elements $x \in \mathcal{U}$ such that there exists $u(s) \in J_q(s)$ for $s \in S$, such that $x = \sum_{s\in S} u(s)$ (in the $\mathcal{U}$-norm) and $(\sum_{s\in S}(J_q(s,u(s))/s(\theta))^p)^{1/p} < +\infty$. This space will be endowed with the quasi-semi-norm

$$\|x\|_{(A)^S_{\theta,p,q}} = \inf\left\{ \left(\sum_{s\in S} \left(\frac{J_q(s,u(s))}{s(\theta)} \right)^p \right)^{1/p} \right\}$$

where the infimum extends over all possible representations of x.

If S satisfies some conditions, see [**1, 2**], then the K_q-functional is a norm and the spaces $[A]^S_{\theta,p,q}$ and $(A)^S_{\theta,p,q}$ are interpolation Banach spaces. Under these conditions, S is a countable set. In the sequel we shall assume that the set S satisfies these conditions.

Let $\overline{A} = \{A(\gamma), \gamma \in \Gamma, \mathcal{U}\}$ and $\overline{B} = \{B(\gamma), \gamma \in \Gamma, \mathcal{V}\}$ be two i.f. An interpolation operator $T : \overline{A} \to \overline{B}$ is a linear operator $T : \mathcal{U} \to \mathcal{V}$ such that $T(A_\gamma) \subset B_\gamma$ and there exists $M \in \mathcal{L}$ with $\|T_\gamma\| \leq M(\gamma)$ for $\gamma \in \Gamma$, where $T_\gamma = T|_{A_\gamma}$.

Let $T : \overline{A} \to \overline{B}$ be an interpolation operator with $\|T_\gamma\| \leq M(\gamma)$ for some $M \in \mathcal{L}$. If $\|M\|_\infty < +\infty$, respectively if $\{MS : s \in S\} \subset S$, then, for every $1 \leq p$, $q \leq \infty$, $S \subset \mathcal{L}$ and $\theta \in D$, we get $T([A]^S_{\theta,p,q}) \subset [B]^S_{\theta,p,q}$, $T((A)^S_{\theta,p,q}) \subset (B)^S_{\theta,p,q}$ and the operators $T : [A]^S_{\theta,p,q} \to [B]^S_{\theta,p,q}$ and $T : (A)^S_{\theta,p,q} \to (B)^S_{\theta,p,q}$ are bounded operators, with norm less than or equal to $\|M\|_\infty$, respectively $M(\theta)$, see [**1, 2**].

We denote by $T_{K^S_{\theta,p,q}}$, respectively $T_{J^S_{\theta,p,q}}$, the operator $T : [A]^S_{\theta,p,q} \to [B]^S_{\theta,p,q}$, respectively $T : (A)^S_{\theta,p,q} \to (B)^S_{\theta,p,q}$.

Let X, Y be two normed spaces and $T : X \to Y$ be a bounded operator. For every $0 \leq \varepsilon \leq 2$, we put

$$\delta_T(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in B_X \text{ and } \|T(x) - T(y)\| \geq \varepsilon\right\},$$

where B_X is the unit ball of X, and

$$\hat{\varepsilon}(T) = \sup\{0 \leq \varepsilon \leq 2\|T\| : \delta_T(\varepsilon) = 0\}.$$

T is said to be a uniformly convex operator if and only if $\delta_T(\varepsilon) > 0$ for every $0 < \varepsilon \leq 2$, that is, $\hat{\varepsilon}(T) = 0$.

T is said to be a uniformly nonsquare operator if and only if there exists $0 < \varepsilon < 2$ such that $\delta_T(\varepsilon) > 0$, that is, $\hat{\varepsilon}(T) < 2$.

If X is a normed space and $I : X \to X$ is the identity operator, then X is uniformly convex if and only if I is uniformly convex and X is uniformly nonsquare if and only if I is uniformly nonsquare.

Let X be a normed space. For $\tau > 0$, we put

$$\rho_X(\tau) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in X, \|x\| \leq 1, \|y\| \leq \tau\right\}.$$

X is said to be uniformly smooth if $\lim_{\tau\to 0}(\rho_X(\tau)/\tau) = 0$.

1. Convexity. Lemma 1.2 is the basic lemma for the results of this paper. This is a general version of Lemma 2 in [**9**]. Before we prove Lemma 1.2, we give the following definition.

Definition 1.1. A family $T = (T_\gamma)_{\gamma\in\Gamma}$ of bounded operators, such that $T_\gamma : X_\gamma \to Y_\gamma$ where X_γ, Y_γ are normed spaces, is said to be compatible if $T_{\gamma_1}(x) = T_{\gamma_2}(x)$ for any $\gamma_1, \gamma_2 \in \Gamma$ and $x \in X_{\gamma_1} \cap X_{\gamma_2}$.

Further, for any such family $T = (T_\gamma)_{\gamma\in\Gamma}$ we define

$$M_T = \operatorname*{ess\,sup}_{\gamma\in\Gamma} \|T_\gamma\|, \cdot m_T = \operatorname*{ess\,inf}_{\gamma\in\Gamma} \|T_\gamma\|$$

and

$$\varepsilon_T = \inf\{0 < \varepsilon \le 2m_T : \operatorname*{ess\,inf}_{\gamma\in\Gamma} \delta_{T_\gamma}(\varepsilon) > 0\}.$$

Remark. It is clear that if $T : \overline{A} \to \overline{B}$ is an interpolation operator and $T_\gamma = T|_{A_\gamma}$, then the family $(T_\gamma)_{\gamma\in\Gamma}$ is compatible. We denote the family $(T_\gamma)_{\gamma\in\Gamma}$ by the same symbol T.

Lemma 1.2. *Let $T = (T_\gamma)_{\gamma\in\Gamma}$ be a compatible family such that $T_\gamma : X_\gamma \to Y_\gamma$ and $(X_\gamma, \|\cdot\|_\gamma)$, $(Y_\gamma, |\cdot|_\gamma)$ are normed spaces, with $0 < m_T \le M_T < +\infty$. Then, for every $1 < q < +\infty$ and for every $\varepsilon_T < \varepsilon \le 2m_T$, there exists $\delta(\varepsilon) > 0$ such that if f, g are two functions on Γ, with $f(\gamma), g(\gamma) \in X_\gamma$ almost everywhere on Γ, $\int_\Gamma \|f(\gamma)\|_\gamma^q \, d\gamma \le 1$, $\int_\Gamma \|g(\gamma)\|_\gamma^q \, d\gamma \le 1$ and $(\int_\Gamma |T_\gamma(f(\gamma)) - T_\gamma(g(\gamma))|_\gamma^q \, d\gamma)^{1/q} \ge \varepsilon$, then $(\int_\Gamma \|f(\gamma)) + g(\gamma)\|_\gamma^q \, d\gamma)^{1/q} \le 2(1 - \delta(\varepsilon))$.*

Proof. For every $0 < \varepsilon \le 2m_T$ we put $\delta_0(\varepsilon) = \operatorname{ess\,inf}_{\gamma\in\Gamma}\delta_{T_\gamma}(\varepsilon)$. Let $1 < q < +\infty$, $\varepsilon_T < \varepsilon \le 2m_T$ and f, g be as above. We choose $\varepsilon_T < \varepsilon_2 < \varepsilon_1 < \varepsilon$.

Case 1. $\|f(\gamma)\|_\gamma = \|g(\gamma)\|_\gamma$ *almost everywhere on* Γ.

The proof of this case is similar to the proof of Lemma 2 in [**9**].

We put $\varphi : \Gamma \to R$, with $\varphi(\gamma) = |T_\gamma(f(\gamma)) - T_\gamma(g(\gamma))|_\gamma$ almost everywhere on Γ, and let

$$\Gamma_1 = \left\{\gamma \in \Gamma : f(\gamma), g(\gamma) \in X_\gamma, f(\gamma) \neq 0 \text{ and } \frac{\varphi(\gamma)}{\|f(\gamma)\|_\gamma} \geq \varepsilon_2\right\}$$

and

$$\Gamma_2 = \{\gamma \in \Gamma : f(\gamma), g(\gamma) \in X_\gamma \text{ and } \gamma \notin \Gamma_1\}.$$

Then, for every $\gamma \in \Gamma_1$, we have

$$\|f(\gamma) + g(\gamma)\|_\gamma \leq 2\|f(\gamma)\|_\gamma(1 - \delta_0(\varepsilon_2)).$$

Let $h : \Gamma \to R$, with

$$h(\gamma) = \begin{cases} \|f(\gamma)\|_\gamma & \gamma \in \Gamma_2, \\ [1 - 2\delta_0(\varepsilon_2)] \cdot \|f(\gamma)\|_\gamma & \gamma \in \Gamma_1, \end{cases}$$

and $f' : \Gamma \to R$, with $f'(\gamma) = \|f(\gamma)\|_\gamma$. It is clear that $h, f' \in B_{L^q(\Gamma)}$. We obtain

$$\left(\int_\Gamma \|f(\gamma) + g(\gamma)\|_\gamma^q \, d\gamma\right)^{1/q} \leq \|f' + h\|_{L^q(\Gamma)}. \tag{2}$$

It is clear that

$$\|f' - h\|_{L^q(\Gamma)} = 2\delta_0(\varepsilon_2)\left(\int_{\Gamma_1} \|f(\gamma)\|_\gamma^q \, d\gamma\right)^{1/q}. \tag{3}$$

If $\gamma \in \Gamma_2$, then $(1/\varepsilon_2)\varphi(\gamma) \leq \|f(\gamma)\|_\gamma$, and thus $(\int_{\Gamma_2} \varphi(\gamma)^q \, d\gamma)^{1/q} \leq \varepsilon_2$. So we get

$$\varepsilon \leq \left(\int_\Gamma \varphi(\gamma)^q \, d\gamma\right)^{1/q} \leq \varepsilon_2 + \left(\int_{\Gamma_1} \varphi(\gamma)^q \, d\gamma\right)^{1/q}.$$

From the above, it follows that

$$\varepsilon - \varepsilon_2 \leq \left(\int_{\Gamma_1} \varphi(\gamma)^q \, d\gamma\right)^{1/q} \leq 2M_T\left(\int_{\Gamma_1} \|f(\gamma)\|_\gamma^q \, d\gamma\right)^{1/q},$$

and so

$$\frac{\varepsilon-\varepsilon_2}{2M_T}\le\left(\int_{\Gamma_1}\|f(\gamma)\|_\gamma^q\,d\gamma\right)^{1/q}. \tag{4}$$

From (3) and (4), we obtain

$$\frac{\delta_0(\varepsilon_2)\cdot(\varepsilon-\varepsilon_2)}{M_T}\le\|f'-h\|_{L^q(\Gamma)}.$$

So

$$\|f'+h\|_{L^q(\Gamma)}\le 2\left(1-\delta_{L^q(\Gamma)}\left(\frac{\delta_0(\varepsilon_2)(\varepsilon-\varepsilon_2)}{M_T}\right)\right).$$

Thus, from (2), we take

$$\left(\int_\Gamma\|f(\gamma)+g(\gamma)\|_\gamma^q\,d\gamma\right)^{1/q}\le 2\left(1-\delta_{L^q(\Gamma)}\left(\frac{\delta_0(\varepsilon_2)(\varepsilon-\varepsilon_2)}{M_T}\right)\right).$$

Case 2. General case. We put $h:\Gamma\to\cup_{\gamma\in\Gamma}X_\gamma$, with

$$h(\gamma)=\begin{cases}(\|f(\gamma)\|_\gamma/\|g(\gamma)\|_\gamma)g(\gamma) & \text{if } g(\gamma)\in X_\gamma\\ & \text{and } g(\gamma)\neq 0\\ f(\gamma) & \text{if otherwise.}\end{cases}$$

Then $h(\gamma)\in X_\gamma$ almost everywhere on Γ, $\|h(\gamma)\|_\gamma=\|f(\gamma)\|_\gamma$ almost everywhere on Γ and $(\int_\Gamma\|h(\gamma)\|_\gamma^q\,d\gamma)^{1/q}\le 1$. We have

$$\begin{aligned}\left(\int_\Gamma\|f(\gamma)+g(\gamma)\|_\gamma^q\,d\gamma\right)^{1/q}&\le\left(\int_\Gamma\|f(\gamma)+h(\gamma)\|_\gamma^q\,d\gamma\right)^{1/q}\\&\quad+\left(\int_\Gamma\|g(\gamma)-h(\gamma)\|_\gamma^q\,d\gamma\right)^{1/q}\\&=\left(\int_\gamma\|f(\gamma)+h(\gamma)\|_\gamma^q\,d\gamma\right)^{1/q}\\&\quad+\left(\int_\Gamma|\,\|g(\gamma)\|_\gamma-\|f(\gamma)\|_\gamma|^q\,d\gamma\right)^{1/q}\\&=\left(\int_\Gamma\|f(\gamma)+h(\gamma)\|_\gamma^q\,d\gamma\right)^{1/q}\\&\quad+\|g'-f'\|_{L^q(\Gamma)},\end{aligned}\tag{5}$$

where $g' : \Gamma \to R$, with $g'(\gamma) = \|g(\gamma)\|_\gamma$ and $f' : \Gamma \to R$ with $f'(\gamma) = \|f(\gamma)\|_\gamma$. Also we have

$$\begin{aligned}\bigg(\int_\Gamma |T_\gamma(f(\gamma)) - T_\gamma(h(\gamma))|_\gamma^q \, d\gamma\bigg)^{1/q} \\ + \bigg(\int_\Gamma |T_\gamma(g(\gamma)) - T_\gamma(h(\gamma))|_\gamma^q \, d\gamma\bigg)^{1/q} \\ \geq \bigg(\int_\Gamma |T_\gamma(f(\gamma)) - T_\gamma(g(\gamma))|_\gamma^q \, d\gamma\bigg)^{1/q} \geq \varepsilon.\end{aligned}$$

So

$$\begin{aligned}(6)\quad \bigg(\int_\Gamma |T_\gamma(f(\gamma)) - T_\gamma(h(\gamma))|_\gamma^q \, d\gamma\bigg)^{1/q} \\ &\geq \varepsilon - \bigg(\int_\Gamma |T_\gamma(g(\gamma)) - T_\gamma(h(\gamma))|_\gamma^q \, d\gamma\bigg)^{1/q} \\ &= \varepsilon - \bigg(\int_\Gamma |T_\gamma(g(\gamma) - h(\gamma))|_\gamma^q \, d\gamma\bigg)^{1/q} \\ &\geq \varepsilon - M_T\bigg(\int_\Gamma \|g(\gamma) - h(\gamma)\|_\gamma^q \, d\gamma\bigg)^{1/q} \\ &= \varepsilon - M_T \cdot \|g' - f'\|_{L^q(\Gamma)}.\end{aligned}$$

We choose $n \in N$ large enough such that

$$\varepsilon_1 < \varepsilon - \frac{\varepsilon}{n} \quad \text{and} \quad \frac{\varepsilon}{2M_T n} < \delta_{L^q(\Gamma)}\bigg(\frac{\delta_0(\varepsilon_2) \cdot (\varepsilon_1 - \varepsilon_2)}{M_T}\bigg).$$

If $M_T \cdot \|g' - f'\|_{L^q(\Gamma)} \leq (\varepsilon/n)$ then, from (6), we take

$$\bigg(\int_\Gamma |T_\gamma(f(\gamma)) - T_\gamma(h(\gamma))|_\gamma^q \, d\gamma\bigg)^{1/q} \geq \varepsilon - \frac{\varepsilon}{n} > \varepsilon_1,$$

and so, from Case 1, we obtain

$$\bigg(\int_\Gamma \|f(\gamma) + h(\gamma)\|_\gamma^q \, d\gamma\bigg)^{1/q} \leq 2\bigg(1 - \delta_{L^q(\Gamma)}\bigg(\frac{\delta_0(\varepsilon_2)(\varepsilon_1 - \varepsilon_2)}{M_T}\bigg)\bigg).$$

Thus, from (5), we have

$$\left(\int_\Gamma \|f(\gamma)+g(\gamma)\|_\gamma^q\,d\gamma\right)^{1/q} \\ \leq 2\left[1-\left(\delta_{L^q(\Gamma)}\left(\frac{\delta_0(\varepsilon_2)\cdot(\varepsilon_1-\varepsilon_2)}{M_T}\right)-\frac{\varepsilon}{2M_T n}\right)\right].$$

We set

$$\delta'(\varepsilon)=\delta_{L^q(\Gamma)}\left(\frac{\delta_0(\varepsilon_1)\cdot(\varepsilon_1-\varepsilon_2)}{M_T}\right)-\frac{\varepsilon}{2M_T n}.$$

From the properties of n, we have that $\delta'(\varepsilon)>0$. If $M_T\cdot\|g'-f'\|_{L^q(\Gamma)}>(\varepsilon/n)$, then

$$\|g'+f'\|_{L^q(\Gamma)}\leq 2\left(1-\delta_{L^q(\Gamma)}\left(\frac{\varepsilon}{nM_T}\right)\right),$$

and so

$$\left(\int_\Gamma \|f(\gamma)+g(\gamma)\|_\gamma^q\,d\gamma\right)^{1/q}\leq \|f'+g'\|_{L^q(\Gamma)} \\ \leq 2\left(1-\delta_{L^q(\Gamma)}\left(\frac{\varepsilon}{nM_T}\right)\right).$$

We put

$$\delta(\varepsilon)=\min\left[\delta_{L^q(\gamma)}\left(\frac{\delta_0(\varepsilon_2)\cdot(\varepsilon_1-\varepsilon_2)}{M_T}\right)-\frac{\varepsilon}{2M_T n},\delta_{L^q(\Gamma)}\left(\frac{\varepsilon}{nM_T}\right)\right].$$

Remarks 1. It is clear that Lemma 1.2 is true, not only for the space Γ, but for any measure space. The discrete variant of Lemma 1.2 is the following.

Let $T=(T_n)_{n\in/N}$ be a compatible sequence, where $T_n: X_n\to Y_n$ and $(X_n,\|\cdot\|_n)_{n\in/N}$, $(Y_n,|\cdot|_n)_{n\in/N}$ are normed spaces, with $0<\inf\|T_n\|\leq\sup_n\|T_n\|<+\infty$. Then, for every $1<q<+\infty$ and for every $\varepsilon_T<\varepsilon\leq 2\inf_{n\in/N}\|T_n\|$ there exists $\delta(\varepsilon)>0$ such that, if $x=(x_n)_{n\in N}$, $z=(z_n)_{n\in N}$ with $x_n,z_n\in X_n$, $n\in N$, $\sum_{n\in N}\|x_n\|_n^q\leq 1$, $\sum_{n\in N}\|z_n\|_n^q\leq 1$ and

$$\left(\sum_{n\in N}|T_n(x_n)-T_n(z_n)|_n^q\right)^{1/q}\geq\varepsilon,$$

then

$$\left(\sum_{n\in/N}\|x_n+z_n\|_n^q\right)^{1/q}\leq 2(1-\delta(\varepsilon)).$$

2. Let $T=(T_\gamma)_{\gamma\in\Gamma}$ be as in Lemma 1.2. For every $1<q<+\infty$ and $\varepsilon_T<\varepsilon\leq 2m_T$, we denote

$$\delta_{T,q}(\varepsilon)=\inf\left\{1-\frac{(\int_\Gamma\|f(\gamma)+g(\gamma)\|^q\ d\gamma)^{1/q}}{2}):f,g\text{ as in Lemma 1.2}\right\}.$$

If $\varepsilon=\varepsilon_T$ we put $\delta_{T,q}(\varepsilon)=0$. Lemma 1.2 says that $\delta_{T,q}(\varepsilon)>0$ for every $\varepsilon_T<\varepsilon\leq 2m_T$.

Theorem 1.3. *Let $\overline{A}=(A_\gamma,\|\cdot\|_\gamma)_{\gamma\in\Gamma}$, $\overline{B}=(B_\gamma|\cdot|_\gamma)_{\gamma\in\Gamma}$ be two i.f. and $T:\overline{A}\to\overline{B}$ be an interpolation operator with $m_T>0$ and $\|T_\gamma\|\leq M(\gamma)$ for some $M\in\mathcal{L}$ and $\|M\|_\infty<+\infty$. Then, for every $1<p$, $q<+\infty$, $S\subset\mathcal{L}$ and $\theta\in D$,*

(i) *the operator $T_{K^S_{\theta,p,q}}$ is a bounded operator with $\hat{\varepsilon}(T_{K^S_{\theta,p,q}})\leq\varepsilon_T$, and*

(ii) *the operator $T_{J^S_{\theta,p,q}}$ is a bounded operator with $\hat{\varepsilon}(T_{J^S_{\theta,p,q}})\leq\varepsilon_T$.*

Proof. Let $1<p$, $q<+\infty$, $S\subset\mathcal{L}$ and $\theta\in D$. We put $T_K=T_{K^S_{\theta,p,q}}$ and $T_J=T_{J^S_{\theta,p,q}}$.

(i) From [**2**] we have $\|T_K\|\leq\|M\|_\infty$.

From Lemma 1.2 we obtain that, for every $\varepsilon_T<\varepsilon\leq 2m_T$, there exists $\delta(\varepsilon)>0$ such that $K_q(s,x_1+x_2)\leq 2(1-\delta(\varepsilon))$, for every $s\in S$, and $x_1,x_2\in K_{q,\overline{A}}(s)$ with $K_q(s,x_1)$, $K_q(s,x_2)\leq 1$ and $K_q(x,T(x_1)-T(x_2))\geq\varepsilon$.

For every $s\in S$ we put $T_s:K_{q,\overline{A}}(s)\to K_{q,\overline{B}}(s)$, with $T_s(x)=T(x)$ for $x\in K_{q,\overline{A}}(s)$. From the above we have $\inf_{s\in S}\delta_{T_s}(\varepsilon)\geq\delta(\varepsilon)>0$ for every $\varepsilon_T<\varepsilon\leq 2m_T$, and thus $\varepsilon_{T_S}\leq\varepsilon_T$ where $T_S=(T_s)_{s\in S}$. Hence, since S is countable, from the discrete case of Lemma 1.2 we obtain $\hat{\varepsilon}(T_K)\leq\varepsilon_T$.

(ii) From [**2**] we have $\|T_J\|\leq\|M\|_\infty$. From Lemma 1.2 we obtain that, for every $\varepsilon_T<\varepsilon\leq 2m_T$, there exists $\delta(\varepsilon)>0$ such that

$J_q(s, u_1 + u_2) \leq 2(1 - \delta(\varepsilon))$, for every $s \in S$, and $u_1, u_2 \in \mathcal{A}$, with $J_q(s, u_1) \leq 1$, $J_q(s, u_2) \leq 1$ and $J_q(s, T(u_1) - T(u_2)) \geq \varepsilon$.

Let $\varepsilon_T < \varepsilon \leq 2m_T$ and $x_1, x_2 \in (A)^S_{\theta,p,q}$ with $\|x_1\|_{(A)^S_{\theta,p,q}}$, $\|x_2\|_{(A)^S_{\theta,p,q}} < 1$ and $|T_J(x_1) - T_J(x_2)|_{(B)^S_{\theta,p,q}} \geq \varepsilon$.

Then there exist $(u_1(s))_{s\in S}$ and $(u_2(s))_{s\in S}$ in $\mathcal{A}$ such that $x_1 = \sum_{s\in S} u_1(s)$, $x_2 = \sum_{s\in S} u_2(s)$ and

$$\sum_{s\in S}\left(\frac{J_q(s, u_1(s))}{s(\theta)}\right)^p < 1, \quad \sum_{s\in S}\big(\frac{J_q(s, u_1(s))}{s(\theta)}\Big)^p < 1.$$

Since $|T_J(x_1) - T_J(x_2)|_{(B)^S_{\theta,p,q}} \geq \varepsilon$, we obtain

$$\left(\sum_{s\in S}\left(\frac{J_q(s, T(u_1(s)) - T(u_2(s)))}{s(\theta)}\right)^p\right)^{1/p} \geq \varepsilon.$$

So, since S is countable, from the discrete case of Lemma 1.2 and the above we have

$$\left(\sum_{s\in S}\left(\frac{J_q s, (u_1(s) + u_2(s))}{s(\theta)}\right)^p\right)^{1/p} \leq 2(1 - \delta(\varepsilon)),$$

and thus $\|x_1 + x_2\|_{(A)^S_{\theta,p,q}} \leq 2(1 - \delta(\varepsilon))$.

The next corollary follows from Theorem 1.3.

Corollary 1.4. *Let $\overline{A} = (A_\gamma)_{\gamma\in\Gamma}$, $\overline{B} = (B_\gamma)_{\gamma\in\Gamma}$ be two i.f. and $T : \overline{A} \to \overline{B}$ an interpolation operator, with $m_T > 0$ and $\|T_\gamma\| \leq M(\gamma)$, $M \in \mathcal{L}$ and $\|M\|_\infty < +\infty$. If $\operatorname{ess\,inf}_{\gamma\in\Gamma}\delta_{T_\gamma}(\varepsilon) > 0$ for every $0 < \varepsilon \leq 2m_T$, respectively, $\operatorname{ess\,sup}_{\gamma\in\Gamma}\hat{\varepsilon}(T_\gamma) < 2m_T$, then $T_{K^s_{\theta,p,q}}$ and $T_{J^S_{\theta,p,q}}$ are uniformly convex operators, respectively uniformly nonsquare operators, for every $1 < p$, $q < +\infty$, $S \subset \mathcal{L}$ and $\theta \in D$.*

From Corollary 1.4 we obtain the next corollary.

Corollary 1.5. *Let $\overline{A} = (A_\gamma)_{\gamma\in\Gamma}$ be an i.f., $1 < p$, $q < +\infty$, $S \subset \mathcal{L}$ and $\theta \in D$. If $\operatorname{ess\,inf}_{\gamma\in\Gamma}\delta_{\mathcal{A}_\gamma}(\varepsilon) > 0$ for every $0 < \varepsilon \leq 2$, respectively*

$\operatorname{ess\,sup}_{\gamma\in\Gamma}\hat{\varepsilon}(A_\gamma) < 2$, *then the interpolation spaces* $[A]^S_{\theta,p,q}$ *and* $(A)^S_{\theta,p,q}$ *are uniformly convex, respectively uniformly nonsquare.*

Corollary 1.5 for the case of the interpolation space $[A]^S_{\theta,p,q}$ was proved in [**9**].

2. Smoothness.

Lemma 2.1. *If* $(X_\gamma)_{\gamma\in\Gamma}$ *is a family of Banach spaces with* $\lim_{\tau\to 0}\operatorname{ess\,sup}_{\gamma\in\Gamma}(\rho_{X_\gamma}(\tau)/\tau) = 0$, *then* $\operatorname{ess\,inf}_{\gamma\in\Gamma}\delta_{X^*_\gamma}(\varepsilon) > 0$ *for every* $0 < \varepsilon \le 2$.

Proof. Since $\lim_{\tau\to 0}\operatorname{ess\,sup}_{\gamma\in\Gamma}(\rho_{x_\gamma}(\tau)/\tau) = 0$, we obtain that X_γ is uniformly smooth and so reflexive almost everywhere on Γ. thus, we have

$$\rho_{X_\gamma}(\tau) = \sup\left\{\frac{\tau\varepsilon}{2} - \delta_{X^*_\gamma}(\varepsilon) : 0 < \varepsilon \le 2\right\}$$

almost everywhere on Γ [**10**].

Suppose there exists $0 < \varepsilon_0 \le 2$ such that $\operatorname{ess\,inf}_{\gamma\in\Gamma}\delta_{X^*_\gamma}(\varepsilon_0) = 0$. Then

$$\begin{aligned}\operatorname*{ess\,sup}_{\gamma\in\Gamma}\rho_{X_\gamma}(\tau) &= \operatorname{ess\,sup}\left\{\sup\left\{\frac{\tau\varepsilon}{2} - \delta_{X^*_\gamma}(\varepsilon) : 0 < \varepsilon \le 2\right\} : \gamma\in\Gamma\right\}\\ &\ge \operatorname{ess\,sup}\left\{\frac{\tau\varepsilon_0}{2} - \delta_{X^*_\gamma}(\varepsilon_0) : \gamma\in\Gamma\right\}\\ &= \frac{\tau\varepsilon_0}{2} - \operatorname*{ess\,inf}_{\gamma\in\Gamma}\delta_{X^*_\gamma}(\varepsilon_0) = \frac{\tau\varepsilon_0}{2},\end{aligned}$$

which is a contradiction.

Lemma 2.2. *Let* $1 < q < +\infty$ *and* $(X_\gamma, \|\cdot\|_\gamma)$ *be a family of normed spaces with* $\lim_{\tau\to 0}\operatorname{ess\,sup}_{\gamma\in\Gamma}(\rho_X(\tau)/\tau) = 0$. *If* $\tau > 0$ *and* f, g *are functions on* Γ, *with* $f(\gamma), g(\gamma) \in X_\gamma$ *almost everywhere on* Γ, *such that* $(\int_\Gamma \|f(\gamma)\|^q_\gamma\, d\gamma)^{1/q} \le 1$ *and* $(\int_\Gamma \|g(\gamma)\|^q_\gamma\, d\gamma)^{1/q} \le \tau$, *then*

$$\left(\int_\Gamma \|f(\gamma)+g(\gamma)\|^q_\gamma\, d\gamma\right)^{1/q} + \left(\int_\Gamma \|f(\gamma)-g(\gamma)\|^q_\gamma\, d\gamma\right)^{1/q} \le \sup\{2(1-\delta_{I,p}(\varepsilon)) + \tau\varepsilon : 0 \le \varepsilon \le 2\},$$

where $(1/p)+(1/q)=1$ *and* $I=(I_\gamma)_{\gamma\in\Gamma}$, *where* $I_\gamma : X_\gamma^* \to X_\gamma^*$ *is the identity operator for* $\gamma\in\Gamma$.

Proof. Let $\tau \geq 0$ and f, g be as above. We put $\sigma(\gamma)=\|f(\gamma)+g(\gamma)\|_\gamma$ and $\varphi(\gamma)=\|f(\gamma)-g(\gamma)\|_\gamma$ almost everywhere on Γ. Then $\sigma,\varphi\in L^q(\Gamma)$ such that $\|\sigma\|_{L^q(\Gamma)}\leq 1$ and $\|\varphi\|_{L^q(\Gamma)}\leq\tau$. So there exist $F,G\in L^p(\Gamma)$ for $(1/p)+(1/q)=1$ such that $\|F\|_{L^p(\Gamma)}\leq 1$, $\|G\|_{L^p(\Gamma)}\leq 1$, $\int_\Gamma F(\gamma)\cdot\sigma(\gamma)\,d\gamma=\|\sigma\|_{L^q(\Gamma)}$ and $\int_\Gamma G(\gamma)\cdot\varphi(\gamma)\,d\gamma=\|\varphi\|_{L^q(\Gamma)}$. Also, there exist $x_\gamma^*, y_\gamma^*\in B_{X_\gamma^*}$ such that $x_\gamma^*((f(\gamma)+g(\gamma))=\sigma(\gamma)$ and $y_\gamma^*((f(\gamma)-g(\gamma))=\varphi(\gamma)$ almost everywhere on Γ.

From the above and the Holder inequality, we obtain

$$\begin{aligned}
\|\sigma\|_{L^q(\Gamma)}+\|\varphi\|_{L^q(\Gamma)} &= \int_\Gamma F(\gamma)\cdot\sigma(\gamma)\,d\,dg+\int_\Gamma G(\gamma)\cdot\varphi(\gamma)\,d\gamma\\
&= \int_\Gamma F(\gamma)\cdot x_\gamma^*((f(\gamma)+g(\gamma))\,d\gamma\\
&\quad+\int_\Gamma G(\gamma)\cdot y_\gamma^*((f(\gamma)-g(\gamma))\,d\gamma\\
&= \int_\Gamma (F(\gamma)\cdot x_\gamma^*+G(\gamma)\cdot y_\gamma^*)(f(\gamma))\,d\gamma\\
&\quad+\int_\Gamma (F(\gamma)\cdot x_\gamma^*-G(\gamma)\cdot y_\gamma^*)(g(\gamma))\,d\gamma\\
&\leq \int_\Gamma \|F(\gamma)\cdot x_\gamma^*+G(\gamma)\cdot y_\gamma^*\|_\gamma\cdot\|f(\gamma)\|_\gamma\,d\gamma\\
&\quad+\int_\Gamma \|F(\gamma)\cdot x_\gamma^*-G(\gamma)\cdot y_\gamma^*\|_\gamma\cdot\|g(\gamma)\|_\gamma\,d\gamma\\
&\leq \bigg(\int_\Gamma \|F(\gamma)\cdot x_\gamma^*+G(\gamma)\cdot y_\gamma^*\|_\gamma^p\,d\gamma\bigg)^{1/p}\\
&\quad+\tau\bigg(\int_\Gamma \|F(\gamma)\cdot x_\gamma^*-G(\gamma)\cdot y_\gamma^*\|_\gamma^p\,d\gamma\bigg)^{1/p}.
\end{aligned}\tag{1}$$

From Lemma 1.2, we have

$$\varepsilon_I=\inf\{0\leq\varepsilon\leq 2:\operatorname*{ess\,inf}_{\gamma\in\Gamma}\delta_{X_\gamma^*}(\varepsilon)>0\}=0.$$

We put

$$\varepsilon_0=\bigg(\int_\Gamma \|F(\gamma)x_\gamma^*-G(\gamma)y_\gamma^*\|_\gamma^p\,d\gamma\bigg)^{1/p}.$$

From Lemma 1.2 and Remark 2 after the lemma, we get

$$\left(\int_\Gamma \|F(\gamma)x_\gamma^* + G(\gamma)y_\gamma^*\|_\gamma^p \, d\gamma\right)^{1/p} \le 2(1-\delta_{I,p}(\varepsilon_0)). \tag{2}$$

So, from (1) and (2), we obtain

$$\left(\int_\Gamma \|f(\gamma)+g(\gamma)\|_\gamma^p \, d\gamma\right)^{1/p} + \left(\int_\gamma \|f(\gamma)-g(\gamma)\|_\gamma^p \, d\gamma\right)^{1/p} \\ \le \sup\{2(1-\delta_{I,p}(\varepsilon)) + \tau\varepsilon : 0 \le \varepsilon \le 2\}.$$

Remark. It is clear that Lemma 2.2 is true not only for the space Γ but for any measure space.

The discrete variant of Lemma 2.2 is the following.

Let $(X_n)_{n\in /N}$ be a sequence of normed spaces with $\lim_{\tau\to 0}(\sup_n \rho_{X_n}(\tau)/\tau) = 0$. If $\tau > 0$ and $x_n, y_n \in X_n$ for $n \in N$ such that $(\sum_{n=1}^\infty \|x_n\|^q)^{1/q} \le 1$ and $(\sum_{n=1}^\infty \|y_n\|^q)^{1/q} \le \tau$, then

$$\left(\sum_{n=1}^\infty \|x_n+y_n\|^q\right)^{1/q} + \left(\sum_{n=1}^\infty \|x_n-y_n\|^q\right)^{1/q} \\ \le \sup\{2(1-\delta_{I,p}(\varepsilon)) + \tau\varepsilon : 0 \le \varepsilon \le 2\},$$

where $(1/p)+(1/q)=1$ and $I=(I_n)_{n\in /N}$, where $I_n : X_n^* \to X_n^*$ is the identity operator for $n \in /N$.

Theorem 2.3. *Let* $\overline{A} = (A_\gamma)_{\gamma\in\Gamma}$ *be an i.f., such that* $\lim_{\tau\to 0} \operatorname{ess\,sup}_{\gamma\in\Gamma}(\rho_{A_\gamma}(\tau)/\tau) = 0$. *Then the interpolation space* $[A]_{\theta,p,q}^S$ *is uniformly smooth for every* $1 < p$, $q < +\infty$, $\theta \in D$ *and* $S \subset \mathcal{L}$.

Proof. Let $1 < p$, $q < +\infty$, $\theta \in D$ and $S \subset \mathcal{L}$. We denote by $I_\gamma : A_\gamma^* \to A_\gamma^*$ the identity operator for $\gamma \in \Gamma$ and $I = (I_\gamma)_{\gamma\in\Gamma}$.

From Lemma 2.2 we have that

$$K_{q,\overline{A}}(s, x+y) + K_{q,\overline{A}}(s, x-y) \le \sup\{2(1-\delta_{I,p}(\varepsilon)) + \tau\varepsilon : 0 \le \varepsilon \le 2\}$$

for every $\tau > 0$, $s \in S$, $s, y \in K_{q,\overline{A}}(s)$, with $K_{q,\overline{A}}(s,x) < 1$ and $K_{q,\overline{A}}(s,y) < \tau$.

From the above, we get

$$\sup_{s\in S} \rho_{K_q(s)}(\tau) \le \sup \left\{ \frac{\tau\varepsilon}{2} - \delta_{I,p}(\varepsilon) : 0 \le \varepsilon \le 2 \right\},$$

and, thus, it is easy to see that $\lim_{\tau\to 0}(\sup_{s\in S} \rho_{K_q(s)}(\tau)/\tau) = 0$. Let $\tau > 0$ and $x_1, x_2 \in [A]^S_{\theta,p,q}$, with $\|x_1\|_{[A]^S_{\theta,p,q}} < 1$ and $\|x_2\|_{[A]^S_{\theta,p,q}} < \tau$. Since S is countable, from the above and the discrete case of Lemma 2.2, we obtain

$$\|x_1+x_2\|_{[A]^S_{\theta,p,q}} + \|x_1-x_2\|_{[A]^S_{\theta,p,q}} \le \sup\{2(1-\delta_{I_K,q}(\varepsilon))+\tau\varepsilon : 0 \le \varepsilon \le 2\},$$

where $I_K = (I_{K_q(s)})_{s\in S}$ and $I_{K_q(s)} : K_q^*(s) \to K_q^*(s)$ is the identity operator for $s \in S$. So

$$\rho_{[A]^S_{\theta,p,q}}(\tau) \le \sup \left\{ \frac{\tau\varepsilon}{2} - \delta_{I_K,q}(\varepsilon) : 0 \le \varepsilon \le 2 \right\}.$$

From this we obtain $\lim_{\tau\to 0}(\rho_{[A]^S_{\theta,p,q}}/\tau) = 0$.

Theorem 2.4. *Let $\overline{A} = (A_\gamma)_{\gamma\in\Gamma}$ be an i.f., such that $\lim_{\tau\to 0} \operatorname{ess\,sup}_{\gamma\in\Gamma}(\rho_{A_\gamma}(\tau)/\tau) = 0$. Then the interpolation space $(A)^S_{\theta,p,q}$ is uniformly smooth for every $1 < p$, $q < +\infty$, $\theta \in D$ and $S \subset \mathcal{L}$.*

Proof. Let $1 < p$, $q < +\infty$, $\theta \in D$ and $S \subset \mathcal{L}$. From Lemma 2.2, we obtain

$$J_q(s, x+y) + J_q(s, x-y) \le \sup\{2(1-\delta_{I,p}(\varepsilon)) + \tau\varepsilon : 0 \le \varepsilon \le 2\},$$

for every $\tau > 0$, $s \in S$, $x, y \in J_q(s)$ with $J_q(s,y) \le 1$ and $J_q(s,y) \le \tau$, where $I = (I_\gamma)_{\gamma\in\Gamma}$ and $I_\gamma : A^*_\gamma \to A^*_\gamma$ is the identity operator for $\gamma \in \Gamma$. So, for every $\tau > 0$, we have

$$\sup_{s\in S} \rho_{J_q(s)}(\tau) \le \sup \left\{ \frac{\tau\varepsilon}{2} - \delta_{I,p}(\varepsilon) : 0 \le \varepsilon \le 2 \right\},$$

and thus it is easy to see that

$$\lim_{\tau\to 0} \frac{\sup_{s\in S} \rho_{J_q}(\tau)}{\tau} = 0. \tag{1}$$

Let $\tau > 0$ and $x_1, x_2 \in (A)^S_{\theta,p,q}$ with $\|x_1\|_{(A)^S_{\theta,p,q}} < 1$ and $\|x_2\|_{(A)^S_{\theta,p,q}} < \tau$. Then there exist $(u_1(s))_{s\in S}$ and $(u_2(s))_{s\in S}$ with $u_1(s), u_2(s) \in J_q(s)$,

$$\bigg(\sum_{s\in S}\bigg(\frac{J_q(s,u_1(s))}{s(\theta)}\bigg)^p\bigg)^{1/p} < 1$$

and

$$\bigg(\sum_{s\in S}\bigg(\frac{J_q(s,u_2(s))}{s(\theta)}\bigg)^p\bigg)^{1/p} < \tau.$$

Since S is countable, from (1) and the discrete case of Lemma 2.2, we obtain

$$\begin{aligned}\bigg(\sum_{s\in S}\bigg(\frac{J_q(s,u_1(s)+u_2(s))}{s(\theta)}\bigg)^p\bigg)^{1/p} & \\ + \bigg(\sum_{s\in S}\bigg(\frac{J_q(s,u_1(s)-u_2(s))}{s(\theta)}\bigg)^p\bigg)^{1/p} & \\ \le \sup\{2(1-\delta_{I,q}(\varepsilon)) + \tau\varepsilon : 0 \le \varepsilon \le 2\}, &\end{aligned}$$

where $I_J = (I_{J_q(s)})_{s\in S}$ and $I_{J_q(s)} : J_q(s) \to J_q(s)$ is the identity operator for $s \in S$. Thus,

$$\|x_1+x_2\|_{(A)^S_{\theta,p,q}} + \|x_1-x_2\|_{(A)^S_{\theta,p,q}} \le \sup\{2(1-\delta_{1,q}(\varepsilon))+\tau\varepsilon : 0 \le \varepsilon \le 2\}.$$

So

$$\sup_{s\in S} \rho_{(A)^S_{\theta,p,q}} \le \sup\bigg\{\frac{\tau\varepsilon}{2} - \delta_{I_J,q}(\varepsilon) : 0 \le \varepsilon \le 2\bigg\}.$$

From the above inequality, we obtain $\lim_{\tau\to 0}(\rho_{(A)^S_{\theta,p,q}}/\tau) = 0$.

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Department of Mathematics, University of Sofia, Boul. James Bouchier 5, Sofia 1126, Bulgaria
E-mail address: `ludmilan@fmi.uni-sofia.bg`

Department of Mathematics, University of Athens, Panepistemiopolis, 15784 Athens, Greece
E-mail address: `Tzaharia@atlas.uoa.gr`