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SPLITTING OF LINEAR SYSTEMS WITH IMPULSES

RAÚL NAULIN AND MANUEL PINTO

ABSTRACT. In this paper we study some dichotomic properties of the impulsive system $y' = [A(t) + B(t)]y$, $\Delta y(t_k) = [C_k + D_k]y(t_k)$. We prove that if the nonperturbed system $x' = A(t)x$, $\Delta x(t_k) = C_k x(t_k)$ has an exponential dichotomy with projection P and $PA(t) = A(t)P$, $PC_k = C_k P$, it is satisfied for all values of t and k, then there exists a change of variables $y(t) = S(t)z(t)$, reducing the perturbed system to the form $z' = [A(t) + \tilde{B}(t)]z$, $\Delta z(t_k) = [C_k + \tilde{D}_k]z(t_k)$, with the properties $P\tilde{B}(t) = \tilde{B}(t)P$, $P\tilde{D}_k = \tilde{D}_k P$. From this result follows a theorem of roughness for exponential dichotomies of impulsive systems.

1. Introduction. In the following J will denote the interval $[t_0, \infty)$; V^n will stand for the space R^n or C^n; for a vector $x \in V^n$, $|x|$ will be some fixed norm in V^n; for an $n \times n$ matrix A, $|A|$ will denote the corresponding matrix norm. In this paper the symbol $\{t_k\}$ identifies a strictly increasing sequence, contained in (t_0, ∞), with the property $\lim_{k\to\infty} t_k = \infty$. The interval $(t_{k-1}, t_k]$ will be denoted by J_k. Finally, let us denote $N = \{1, 2, 3, \dots\}$.

The theory of equations with impulsive effect is a recent branch of the theory of differential equations. Beginning with the work of Mil'man and Myshkis [**11**], this theory has been developed by the contribution of many researchers: Halanay and Wexler [**8**], Bainov et al. [**2**], Lakshmikantham et al. [**9**], etc. The theoretical questions which arise in this area attract the attention of analysts and applied mathematicians because of an increasing number of their applications to semiconductor theory, quantum mechanics, ecology, biomathematics and control theory [**2, 8, 9**].

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In this article we investigate the conditions under which the linear impulsive system

$$\left.\begin{aligned} y'(t) &= (A(t)+B(t))y, \quad t \neq t_k, \\ \Delta y(t_k) &= (C_k + D_k)y(t_k), \quad k \in N,\ t_k > 0 \end{aligned}\right\}, \tag{1}$$

where

$$\Delta y(t_k) = y(t_k^+) - y(t_k), y(t_k^+) = \lim_{t\to t_k^+} y(t),$$

can be decoupled, by means of a linear change of variables $y(t) = S(t)z(t)$ into two systems of lower dimensions. This is an important task in the theory of differential equations [**6, 14, 7, 4, 12, 13**]. Although we will touch only theoretical questions of this problem, the practical importance of this study has been recognized in the applied analysis, for example in the numerical treatment of boundary value problems [**1**]. To define the statement of our problem, let us consider P to be a given projection matrix (in all of the paper, we keep the letter P to indicate a constant matrix with the property $P^2 = P$). We seek an impulsive system,

$$\left.\begin{aligned} z'(t) &= (\tilde{A}(t)+\tilde{B}(t))z(t), \quad t \neq t_k, \\ \Delta z(t_k) &= (\tilde{C}_k + \tilde{D}_k)z(t_k), \quad k \in N \end{aligned}\right\}, \tag{2}$$

kinematically similar to system (1) [**4, 5**], such that its coefficients commute with projection P. In our research on impulsive equations, we have found that this problem has not been solved completely.

We will assume hereafter that system

$$\left.\begin{aligned} x'(t) &= A(t)x, \quad t \neq t_k, \\ \Delta x(t_k) &= C_k x(t_k), \quad k \in N \end{aligned}\right\}, \tag{3}$$

where all the matrices $I + C_k$ are invertible, satisfies the following condition (see Section 6 for commentaries of this hypothesis)

$$A(t)P = PA(t), \qquad C_kP = PC_k, \quad \forall\, t \in J,\ \forall\, k \in N. \tag{4}$$

We will investigate further under which conditions there exists a function $S : J \to V^{n\times n}$ with the following properties

S1. S is a continuously differentiable function on each interval J_k,

S2. For each impulsive time t_k, there exists the righthand side limit

$$S(t_k^+) = \lim_{t \to t_k^+} S(t).$$

S3. $S(t)$ is invertible for each $t \in J_k$ and $S(t_k^+)$ are invertible for all k.

S4. The functions S and S^{-1} are bounded.

S5. The change of variables $y(t) = S(t)z(t)$ reduces System (1) to System (2) where

$$P\tilde{B}(t) = \tilde{B}(t)P, \qquad P\tilde{D}_k = \tilde{D}_k P, \quad \forall\, t,\ \forall\, k. \tag{5}$$

Sometimes we will refer to the function S with properties S1–S5 as the splitting function of System (1). In what follows we will use the following notations:

$$P_c(A) = PAP + (I-P)A(I-P), \quad P_a(A) = PA(I-P) + (I-P)AP.$$

Let us recall briefly the respective results for the differential system

$$x'(t) = A(t)x(t) \tag{6}$$

having an exponential dichotomy with projection P [**6**]:

$$\left.\begin{aligned} |\Phi(t)\Phi^{-1}(s)P| &\le Ke^{-\alpha(t-s)}, \quad t \ge s, \\ |\Phi(t)\Phi^{-1}(s)(I-P)| &\le Ke^{\alpha(t-s)}, \quad s \ge t \end{aligned}\right\}, \tag{7}$$

where K and α are positive constants and Φ is the fundamental matrix of (6).

Theorem A. *If*

$$\operatorname{Sup}\{|B(t)| : t \in J\} \le \frac{\alpha}{6K^2|P||I-P|},$$

then there exists a change of variables $y(t) = S(t)z(t)$, *transforming*

$$y'(t) = (A(t) + B(t))y(t) \tag{8}$$

into the form

$$z'(t) = (A(t) + P_c(B(t)S(t))z(t). \tag{9}$$

The splitting function S *of System (8) has the form* $S = I + H$ *where* $|H(t)| \leq 2^{-1}$ *for all* t, *and* H *satisfies the integral equation*

$$H(t) = \int_{t_0}^{t} \Gamma(t,s)P(I - H(s))B(s)(I + H(s))(I - P)\Gamma(t,s)\, ds \\ - \int_{t}^{\infty} \Gamma(s,t)(I - P)(I - H(s))B(s)(I + H(s))P\Gamma(s,t)\, ds,$$

where $\Gamma(t,s) := \Phi(t)\Phi^{-1}(s)$, *for all* $t, s \in J$.

The corresponding result, for the difference equation, was given by Papaschinopoulos [**15**], who considered the equation

$$\Delta x(n) = [C(n) + D(n)]x(n+1), \qquad \Delta x(n) = x(n+1) - x(n). \tag{10}$$

In this paper we extend Theorem A to systems with impulsive effect. We will rely on the proof of Theorem A given by Coppel in [**6**]. This way seems not only to be simple if we consider System (1) as a perturbation of System (6), but natural, since our problem is posed in a finite dimensional space.

An important feature of our results consists of the following. The transformation of splitting $y(t) = S(t)z(t)$ will be defined in Section 3 by solving a certain integral equation similar to those of the ordinary and difference equations. This is important for two reasons; first, such a result unifies the corresponding results for ordinary, difference and impulsive equations, and, second, by means of this integral equation it is possible to obtain an estimate of the size of the perturbations B and $\{D_k\}$ allowing the splitting, an estimate which is possibly not optimum, but is adequate in many applications.

Finally, we would like to emphasize the following result obtained in our paper. Let us consider the equation

$$\left.\begin{aligned} y'(t) &= (A(t)+B(t))y(t), \quad t \neq t_k, \\ \Delta y(t_k) &= C_k y(t_k), \quad k \in N \end{aligned}\right\}. \tag{11}$$

We would like to perform a change of variables on System (11) in order to obtain the split system

$$\left.\begin{aligned} z'(t) &= (A(t)+\tilde{B}(t))z(t), \quad t \neq t_k \in V^n, \\ \Delta z(t_k) &= C_k z(t_k), \quad k \in N \end{aligned}\right\}, \tag{12}$$

where the discrete equation of Systems (11) and (12) is the same, and

$$PB(t) = B(t)P, \quad \forall\, t. \tag{13}$$

We will show that, in this case, the splitting will depend only on the perturbation B of the ordinary equation (8). A similar problem will be solved for the equation

$$\left.\begin{aligned} x'(t) &= A(t)x(t), \quad t \neq t_k, \\ \Delta x(t_k) &= (C_k + D_k)x(t_k), \quad k \in N \end{aligned}\right\}. \tag{14}$$

Such problems were not treated by Coppel and Papaschinopoulos, since they dealt only with one equation. In our case we have to consider an impulse equation, where the dynamics of an ordinary equation undergoes the effect of the impulses described by a difference equation. The results obtained for equations (11) and (14) are new and cannot be considered as straightforward.

2. Preliminaries. We will assume that functions $A, B : J \to V^{n\times n}$ are uniformly continuous on each interval J_k. The solutions of the involved impulsive systems are functions uniformly continuous on each interval J_k. We will denote by $C(J, \{t_k\})$ the space of such functions. $BC(J, \{t_k\})$ will denote the subspace of $C(J, \{t_k\})$ which consists of bounded functions. For a function f defined on J and a sequence of matrices $\{D_k\}$, we define

$$|f|_\infty = \text{Sup}\,\{|f(t)| : t \in J\}, \quad |\{D_k\}|_\infty = \text{Sup}\,\{|D_k| : k \in N\}.$$

It is easy to verify that the pair $(BC(J,\{t_k\}), |\ |_\infty)$ defines a Banach space.

Let us consider System (3) where we assume that all matrices $I + C_k$ are invertible. The fundamental matrix of this system is defined by

$$U(t) = \Phi(t) \prod_{[t_0,t)} \Phi^{-1}(t_i)(I + C_i)\Phi(t_i)\Phi^{-1}(t_0), \quad t \geq t_0, \tag{15}$$

where

$$\prod_{[t_0,t)} \Phi^{-1}(t_i)(I + C_i)\Phi(t_i),$$

denotes the ordered product of all matrices $\Phi^{-1}(t_i)(I + C_i)\Phi(t_i)$ such that $t_i \in (t_0, t)$, $t \geq t_0$, and we define

$$\prod_{[t_0,t_0)} \Phi^{-1}(t_i)(I + C_i)\Phi(t_i) = I.$$

$W(t, s)$ will denote the Cauchy matrix of the impulsive System (3) [**2, 9**]:

$$W(t, s) = U(t)U^{-1}(s).$$

The following properties of $U(t)$ will frequently be used.

P1. The function $U(t)$ is uniformly continuous on each J_k.

P2. $\Delta U(t_k) = (I + C_k)^{-1}C_k U(t_k^+)$.

P3. $\Delta U^{-1}(t_k) = -U(t_k^+)(I + C_k)^{-1}C_k$.

Definition 1 [**4, 5, 10**]. We shall say that System (3) has an exponential dichotomy with projection matrix P if and only if there exist positive constants $K_i \geq 1$ and α_i, $i = 1, 2$, such that

$$\left.\begin{aligned} |U(t)PU^{-1}(s)| &\leq K_1 e^{-\alpha_1(t-s)}, \quad t \geq s \geq t_0, \\ |U(t)(I - P)U^{-1}(s)| &\leq K_2 e^{\alpha_2(t-s)}, \quad s \geq t \geq t_0 \end{aligned}\right\} . \tag{16}$$

From this definition, necessarily, it follows

$$|I + C_k| \leq K_1, \quad |(I + C_k)^{-1}| \leq K_2, \quad \forall\, k. \tag{17}$$

The following conditions will be used in the writing of our theorems.

C1. System (3) has the exponential dichotomy (16).

C2. Condition (4) is satisfied.

C3. There exist a positive number l and a positive integer p such that each interval of J, of length l, contains no more than p points of the sequence $\{t_k\}$.

Condition C3 is frequently used in the theory of systems with impulsive effect [**2, 5, 9**]. It implies the following estimate

$$\sum_{(t_0,\infty)} e^{-(\alpha_1+\alpha_2)|t-t_k|} \leq \frac{2(1+p)}{1-r}, \quad r = e^{-(\alpha_1+\alpha_2)l} < 1. \tag{18}$$

3. Splitting of the perturbed system. We will modify the form of System (1) by writing it as a system which difference equation is a linear equation with advance. We do this in order to use the discrete Ricatti equation obtained in paper [**15**].

Lemma 1. *Under condition* C3, *if*

$$|D_k| \leq \sigma, \quad 0 < \sigma < \frac{1}{12K_1^2K_2^2}, \tag{19}$$

then System (1) *can be written in the form*

$$\left.\begin{aligned} y'(t) &= (A(t)+B(t))y(t), \quad t \neq t_k, \\ \Delta y(t_k) &= ((I+C_k)^{-1}C_k + \tilde{D}_k)y(t_k^+), \quad k \in N \end{aligned}\right\}, \tag{20}$$

where $\tilde{D}_k = (I+C_k+D_k)^{-1}D_k$. *Moreover,*

$$|\{\tilde{D}_k\}|_\infty \leq 2|\{D_k\}| \leq 2\sigma K_2. \tag{21}$$

Proof. From (17) we have

$$|(I+C_k)^{-1}D_k| \leq |(I+C_k)^{-1}||D_k| \leq K_2|D_k| \leq \sigma K_2 < 1.$$

Therefore, if $|D_k| \le \sigma$, all matrices $I+C_k+D_k$ are invertible. Moreover,

$$|(I+C_k+D_k)^{-1}D_k| \le \frac{\sigma K_2}{1-\sigma K_2}|D_k| \le 2K_2|\{D_k\}| \le 2\sigma K_2. \qquad \square$$

We emphasize the equivalence of System (3) and system

$$\left.\begin{array}{c} x'(t) = A(t)x(t), \quad t \ne t_k, \\ \Delta x(t_k) = (I+C_k)^{-1}C_k x(t_k^+), \quad k \in N \end{array}\right\}. \tag{22}$$

Each solution of System (3) is a solution of System (22) and, conversely, the solutions of (22) are solutions of (3). Therefore, these systems have the same fundamental matrix U.

In the following construction we will use the notations

$$H_k^+ = H(t_k^+), \quad H_k = H(t_k).$$

Regarding equation (20), let us consider the operators

$$\begin{aligned} \mathcal{O}(H)(t) &= \int_0^t W(t,s)P(I-H(s))B(s)(I+H(s))(I-P)W(s,t)\,ds \\ &\quad - \int_t^\infty W(t,s)(I-P)(I-H(s))B(s)(I+H(s))PW(s,t)\,ds, \\ \mathcal{D}(H)(t) &= \sum_{[t_0,t)} W(t,t_k)P(I-H_k)\tilde{D}_k(I+H_k^+)(I-P)W(t_k^+,t) \\ &\quad - \sum_{[t,\infty)} W(t,t_k)(I-P)(I-H_k)\tilde{D}_k(I+H_k^+)PW(t_k^+,t), \end{aligned}$$

and

$$\mathcal{T} = \mathcal{O} + \mathcal{D}. \tag{23}$$

We will call $\mathcal{O}$ the operator of ordinary splitting; $\mathcal{D}$ the operator of discrete splitting; and $\mathcal{T}$ the splitting operator of impulsive System (20). From the boundedness of the function B and the sequence $\{\tilde{D}_k\}$, the estimate (16) and the property P1, we obtain

$$\mathcal{T} : BC(J,\{t_k\}) \longrightarrow BC(J,\{t_k\}).$$

Lemma 2. *Let us assume that System* (3) *satisfies conditions* C1–C3. *Then, under conditions* (19) *and*

$$6K_1K_2|P|\,|I-P|\left(\frac{|B|_\infty}{\alpha_1+\alpha_2}+\frac{(1+p)}{1-r}|\{\tilde{D}_k\}|_\infty\right)\le 1, \tag{24}$$

there exists a unique, fixed point of $\mathcal{T}$ in the ball $|H|_\infty \le 2^{-1}$ of the space $BC(J,\{t_k\})$.

Proof. For $|H|_\infty \le 2^{-1}$, the estimates in (16) yield

$$|\mathcal{O}(H)|_\infty \le \frac{3K_1K_2}{2(\alpha_1+\alpha_2)}|P|\,|I-P|\,|B|_\infty. \tag{25}$$

The estimates (18) and (16) give

$$|\mathcal{D}(H)|_\infty \le \frac{3(1+p)}{2(1-r)}K_1K_2|P|\,|I-P|\,|\{D_k\}|_\infty. \tag{26}$$

From (24)–(26) we obtain $|\mathcal{T}(H)|_\infty \le 2^{-1}$ if $|H|_\infty \le 2^{-1}$. On the other hand, the identity

$$\begin{aligned}(I-G)B(I+G)-(I-H)B(I+H) &= (H-G)B-B(H-G)\\ &\quad+(H-G)BH+GB(H-G)\end{aligned}$$

implies

$$\begin{aligned}|\mathcal{O}(H)-\mathcal{O}(G)|_\infty &\le \frac{3K_1K_2}{\alpha_1+\alpha_2}|P|\,|I-P|\,|B|_\infty|H-G|_\infty;\\ |\mathcal{D}(H)-\mathcal{D}(G)|_\infty &\le \frac{6(1+p)K_1K_2}{1-r}|P|\,|I-P|\,|D_k|_\infty|H-G|_\infty.\end{aligned}$$

From (24), for $|H|_\infty \le 2^{-1}$ and $|G|_\infty \le 2^{-1}$, we obtain

$$|\mathcal{T}(H)-\mathcal{T}(G)|_\infty \le \frac{1}{2}|H-G|, \quad |H|_\infty,\ |G|_\infty \le 2^{-1}.$$

Thus, the operator $\mathcal{T}$ acting on the ball $|H|_\infty \le 2^{-1}$ into itself is a contraction. Therefore, in this ball, it has a fixed point we will denote by H. □

Lemma 3. *Under the conditions of Lemma* 2*, on each interval* J_k *the change of variables* $y(t) = S(t)z(t)$, $S = I+H$ *reduces the ordinary differential System* (8) *to the form* (9).

Proof. The identity $H = \mathcal{T}(H)$ implies that H is differentiable on each J_k (at each time t_k the derivative of H is understood as the righthand side derivative). From the definition of $\mathcal{O}$ and $\mathcal{D}$, we obtain

$$\begin{aligned} H'(t) = & A(t)\mathcal{O}(H)(t) - \mathcal{O}(H)(t)A(t) \\ & + U(t)PU^{-1}(t)(I - H(t))B(t)(I + H(t))U(t)PU^{-1}(t) \\ & + U(t)(I-P)U^{-1}(t)(I-H(t))B(t)(I+H(t))U(t)(I-P)U^{-1}(t) \\ & + A(t)\mathcal{D}(H)(t) - \mathcal{D}(H)(t)A(t). \end{aligned}$$

From condition (4) we have $U(t)PU^{-1}(t) = P$. Therefore,

$$H'(t) = A(t)H(t) - H(t)A(t) + P_a((I - H(t))B(t)(I + H(t))).$$

From the identity $H = \mathcal{T}(H)$ we obtain $PHP = 0$, $(I-P)H(I-P) = 0$, implying

$$H(t) = H(t)P + PH(t), \quad t \in J_k. \tag{27}$$

This allows us to write, on each interval J_k, the identity

$$H'(t) = A(t)H(t) - H(t)A(t) + P_a(B(t)(I+H(t))) - HP_c(B(t)(I+H(t))).$$

Since $S = I + H$, then S satisfies

$$S' = AS - SA + BS - SP_c(BS)$$

which is a necessary and sufficient condition in order that the change of variables $y(t) = S(t)z(t)$ transforms the equation (8) into the form (9). □

Now we will examine the effect of the change of variables $y(t) = S(t)z(t)$ on the second equation in System (1). Accordingly, let us define the continuous functions

$$J_1(t) = \int_0^t PU^{-1}(s)(I - H(s))B(s)(I + H(s))U(s)(I - P)\, ds$$

and

$$J_2(t) = \int_t^\infty (I - P)U(s)(I - H(s))B(s)(I + H(s))U(s)P\, ds.$$

Lemma 4. *At each time t_k, the operator of ordinary splitting satisfies*

$$\Delta\mathcal{O}(H)(t_k) = (I + C_k)^{-1} C_k \mathcal{O}(H)(t_k^+) - \mathcal{O}(H)(t_k)(I + C_k)^{-1} C_k. \tag{28}$$

Proof. Since

$$\mathcal{O}(H)(t) = U(t)J_1(t)U^{-1}(t) - U(t)J_2(t)U^{-1}(t),$$

then, using P2 and P3, we have

$$\begin{aligned}\Delta\mathcal{O}(H)(t_k) &= \Delta(U(t_k)J_1(t_k)U^{-1}(t_k)) - \Delta(U(t_k)J_2(t_k)U^{-1}(t_k)) \\ &= \Delta(U(t_k))J_1(t_k)U^{-1}(t_k^+) + U(t_k)J_1(t_k)\Delta U^{-1}(t_k)) \\ &\quad - \Delta U(t_k)J_2(t_k)U^{-1}(t_k^+)) - \Delta U(t_k)J_2(t_k)\Delta U^{-1}(t_k)).\end{aligned}$$

From this decomposition we obtain (28). □

In order to prove the forthcoming lemma, we introduce the notations:

$$\Lambda_1(t) = \sum_{[t_0,t)} P(I - H_k)\tilde{D}_k(I + H_k^+)(I - P),$$

and

$$\Lambda_2(t) = \sum_{[t,\infty)} (I - P)(I - H_k)\tilde{D}_k(I + H_k^+)P.$$

Consequently, we can write the operator $\mathcal{D}$ in the form

$$\mathcal{D}(H)(t) = U(t)\Lambda_1(t)U^{-1}(t) - U(t)\Lambda_2(t)U^{-1}(t).$$

Lemma 5. *At each time t_k, the operator of discrete splitting satisfies*

$$\begin{aligned}\Delta\mathcal{D}(H)(t_k) &= (I+C_k)^{-1}C_k\mathcal{D}(H)(t_k^+) - \mathcal{D}(H)(t_k)(I+C_k)^{-1}C_k \\ &\quad + P_a((I-H_k)\tilde{D}_k(I+H_k^+).\end{aligned} \tag{29}$$

Proof. Using P2 and P3, we have

$$\begin{aligned}\Delta\mathcal{D}(H)(t_k) &= \Delta U(t_k)\Lambda_1(t_k)U^{-1}(t_k^+) + U(t_k)\Lambda_1(t_k)\Delta U^{-1}(t_k) \\ &\quad - \Delta U(t_k)\Lambda_2(t_k)U^{-1}(t_k^+) + U(t_k)\Lambda_2(t_k)\Delta U^{-1}(t_k) \\ &\quad + U(t_k)PU^{-1}(t_k)(I-H_k)\tilde{D}_k(I+H_k^+)(I-P) \\ &\quad + U(t_k^+)(I-P)U^{-1}(t_k)(I-H_k)\tilde{D}_k(I+H_k^+)P \\ &= C_kU(t_k^+)(\Lambda_1(t_k^+)-\Lambda_2(t_k^+))U^{-1}(t_k^+) \\ &\quad - U(t_k^+)(\Lambda_1(t_k^+)-\Lambda_2(t_k))U^{-1}(t_k)C_k \\ &\quad + P(I-H_k)\tilde{D}_k(I+H_k^+)(I-P) \\ &\quad + (I-P)(I-H_k)\tilde{D}_k(I+H_k^+)P.\end{aligned}$$

This identity implies (29). □

Lemma 6. *Let us assume that System* (3) *satisfies conditions* C1–C3*. Moreover, let us assume that conditions* (19) *and* (24) *hold. Then there exists a splitting function of System* (20), $S = I + H$, $|H|_\infty \le 2^{-1}$, *reducing System* (20) *to the form*

$$\left.\begin{aligned} z'(t) &= (A(t) + P_c(B(t)S(t)))z(t), \quad t \neq t_k, \\ \Delta z(t_k) &= ((I+C_k)^{-1}C_k + P_c(\tilde{D}_kS(t_k^+)))z(t_k^+), \quad k \in N \end{aligned}\right\}. \tag{30}$$

Proof. Lemma 3 establishes that the change of variables $y(t) = S(t)z(t)$ reduces the equation (8) into the form (9). Lemmas 4 and 5 imply that

$$\begin{aligned}\Delta H(t_k) &= (I+C_k)^{-1}C_kH(t_k^+) - H(t_k)(I+C_k)^{-1}C_k \\ &\quad + P_a((I-H_k)\tilde{D}_k(I+H_k^+)).\end{aligned} \tag{31}$$

From the property (27) we obtain

$$H_k^+ P + P H_k^+ = H_k^+.$$

Plugging this last identity into (31) we obtain

$$\begin{aligned}\Delta H(t_k) &= (I + C_k)^{-1} C_k H(t_k^+) - H(t_k)(I + C_k)^{-1} C_k \\ &\quad + P_a(\tilde{D}_k(I + H_k^+)) - H_k P_c(\tilde{D}_k(I + H_k^+)),\end{aligned}$$

which is a necessary and sufficient condition in order to reduce the difference equation of System (20) to the form

$$\Delta y(t_k) = ((I + C_k)^{-1} C_k + P_c(\tilde{D}_k(S(t_k^+)))y(t_k^+). \quad \square$$

Theorem 1. *Under conditions* C1–C3, (19) *and* (24), *the change of variables* $y(t) = S(t)z(t)$ *defined in Lemma* 6, *splits System* (1) *into the form*

$$\begin{aligned} z' &= (A(t) + P_c(B(t)S(t)))z, \quad t \neq t_k, \\ \Delta z(t_k) &= (C_k + \hat{D}_k) z(t_k^+), \quad k \in N, \end{aligned} \tag{32}$$

where

$$\hat{D}_k = (I - P_c(\tilde{D}_k S(t_k^+)))^{-1}(I + C_k) P_c(\tilde{D}_k S(t_k^+))(I + C_k)$$

satisfies the estimate

$$|\hat{D}_k| \leq 12\sigma K_1^2 K_2,$$

and all the matrices $I + C_k + \hat{D}_k$ *are invertible.*

Proof. The splitting of System (1), accomplished by function S, was proven by Lemma 6. Condition (19) implies the estimate (21). Consequently,

$$|P_c(\tilde{D}_k S(t_k^+))| \leq 2|S|_\infty |\tilde{D}_k| \leq 6\sigma K_2 \leq 2^{-1},$$

then all matrices $I - P_c(\tilde{D}_k S(t_k^{-1}))$ are invertible. From this estimate it follows

$$|\hat{D}_k| \leq 12\sigma K_2 |I + C_k|^2 \leq 12\sigma K_1^2 K_2.$$

From

$$|(I + C_k)^{-1}\hat{D}_k| \le 12\sigma K_1^2 K_2^2 < 1,$$

we infer the invertibility of matrices $I + C_k + \hat{D}_k$. $\square$

4. Invariance of impulsive equations. By the definition of operator $\mathcal{D}$ we observe that $\mathcal{D} = 0$ if $D_k = 0$, for all k. In this case H will be a solution of the integral equation

$$H = \mathcal{O}(H).$$

Theorem 2. *Under conditions* C1–C2, *for any continuous function* B *satisfying*

$$6\frac{K_1K_2}{\alpha_1 + \alpha_2}|P|\,|I - P|\,|B|_\infty \le 1,$$

there exists a function $S : J \to V^{n\times n}$ *satisfying* S1–S4 *such that* $y(t) = S(t)z(t)$ *transforms equation* (11) *to equation* (12) *with the property* (13).

Regarding equation (14), we observe that $B = 0$. In this case H is a solution of the equation

$$H = \mathcal{D}(H).$$

Theorem 3. *Under conditions* C1, C2, C3 *and* (19), *for any bounded sequence* $\{D_k\}$ *satisfying*

$$\frac{12(1+p)}{1-r}K_1K_2|P|\,|I-P|\,|\{D_k\}|_\infty \le 1$$

there exists a function $S : J \to V^{n\times n}$ *satisfying* S1–S4 *such that the change of variables* $x(t) = S(t)y(t)$ *converts equation* (14) *to the form*

$$z'(t) = A(t)z(t),$$
$$\Delta z(t_k) = (C_k + \tilde{D}_k)z(t_k^+),$$

where, for each integer k, $\tilde{D}_k$ *commutes with the projection matrix* P.

5. Roughness of exponential dichotomies. From Theorem 1, a proof of the roughness of the exponential dichotomy (16) follows. Let us consider System (3). Denoting

$$A(t) = \operatorname{diag}\{A_1(t), A_2(t)\},$$
$$\tilde{P}_c(B(t)S(t)) = \operatorname{diag}\{\hat{B}_1(t), \hat{B}_2(t)\},$$

and

$$C_k = \operatorname{diag}\{C_k^1(t), C_k^2(t)\},$$
$$\hat{D}_k = \operatorname{diag}\{\hat{D}_k^1(t), \hat{D}_k^2(t)\},$$

we may decompose System (30) into

$$\left.\begin{aligned} z_1'(t) &= (A_1(t) + \hat{B}_1(t))z_1(t), \quad t \neq t_k, \\ \Delta z_1(t_k) &= (C_k^1 + \hat{D}_k^1)z_1(t_k), \quad k \in N \end{aligned}\right\}, \tag{33}$$

$$\left.\begin{aligned} z_2'(t) &= (A_2(t) + \hat{B}_2(t))z_2(t), \quad t \neq t_k, \\ \Delta z_2(t_k) &= (C_k^2 + \hat{D}_k^2)z_2(t_k), \quad k \in N \end{aligned}\right\}. \tag{34}$$

We will denote by U_i, $i = 1, 2$, the fundamental matrices of the linear systems

$$\left.\begin{aligned} x_i'(t) &= A_i(t)x_i(t), \quad t \neq t_k, \\ \Delta x_i(t_k) &= C_k^i x_i(t_k^+), \quad k \in N \end{aligned}\right\}.$$

Matrices U_i satisfy

$$|U_1(t)U_1^{-1}(s)| \leq K_1 e^{-\alpha_1(t-s)}, \quad t \geq s,$$
$$|U_2(t)U_2^{-1}(s)| \leq K_2 e^{\alpha_2(t-s)}, \quad s \geq t.$$

Using the estimate of $\hat{D}_k$ given in Theorem 1, we obtain that the fundamental matrices X_1 and X_2 of Systems (33) and (34) have the following estimates

$$|X_1(t)X_1^{-1}(s)| \leq \tilde{K}_1 e^{-(\alpha_1-\gamma_1)(t-s)}, \quad t \geq s, \tag{35}$$

where

$$\gamma_1 = \frac{p}{l}\ln(1 + 12\sigma K_1^3 K_2^2) + 3K_1|B|_\infty,$$

and

$$|X_2(t)X_2^{-1}(s)| \le \tilde{K}_2 e^{(\alpha_2-\gamma_2)(t-s)}, \quad s \ge t, \tag{36}$$

where

$$\gamma_2 = \frac{p}{l}\ln(1+12\sigma K_1^2 K_2^3) + 3K_2|B|_\infty.$$

The above estimates can be obtained by standard arguments of the Gronwall inequality for piecewise continuous functions [**5, 3**]. These estimates prove the following

Theorem 4. *Under conditions* C1–C3, (19) *and* (24), *if*

$$\alpha_i > \frac{p}{l}\ln(1+12\sigma K_i K_1^2 K_2^2) + 3K_i|B|_\infty, \quad i=1,2,$$

then System (1) *has the exponential dichotomy*

$$\left.\begin{aligned} |U(t)PU^{-1}(s)| &\le \hat{K}_1 e^{-(\alpha_1-\gamma_1)(t-s)}, \quad t \ge s \ge t_0, \\ |U(t)(I-P)U^{-1}(s)| &\le \hat{K}_2 e^{(\alpha_2-\gamma_2)(t-s)}, \quad s \ge t \ge t_0 \end{aligned}\right\},$$

where $\hat{K}_i$, $i=1,2$, *are constants.*

6. Commentaries. In [**4, 5**], the notion of the Riesz integral is used to establish the following result.

Theorem B. *Let us assume that* $U(t)$, *the fundamental matrix of the linear impulsive system* (3) [**2**] *satisfies*

$$|U(t)PU^{-1}(t)| \le M < \infty, \quad \forall\, t.$$

Under these circumstances, there exists a kinematically similar system to (3)

$$\left.\begin{aligned} x'(t) &= \hat{A}(t)x(t), \quad t \ne t_k, \\ \Delta x(t_k) &= \hat{C}_k x(t_k), \quad k \in N \end{aligned}\right\}, \tag{37}$$

such that the matrices $\hat{A}(t)$ *and* $\hat{C}_k$ *commute with projection* P.

This theorem justifies the assumption (4).

The proof of the roughness of an exponential dichotomy displayed in [**5**] is complicated because it relies on techniques closer to problems in spaces of infinite dimensions. Consequently, the roughness Theorem 4, obtained by splitting System (1) is simpler. It is worth noticing that Theorem 4 gives concrete bounds for coefficients B and $\{D_k\}$ of equation (1) allowing an exponential dichotomy. Theorem 11.1 in [**5**] states the existence of an exponential dichotomy for System (1) "for sufficiently small δ_1 and δ_2 (where $|B|_\infty \le \delta_1$, $|\{D_k\}| \le \delta_2$)." The proof of Theorem 11.1 does not allow one to find concrete bounds for δ_1 and δ_2.

Another feature distinguishing Theorem 11.1 of [**5**] and Theorem 4 of our text concerns the condition of boundedness of the upper and lower general exponents of the space of solutions of equation (3) imposed in Theorem 11.1. According to Theorem 4 this condition is superfluous.

Finally, we emphasize the invariance of equations (11) and (14) under the splitting $y(t) = S(t)z(t)$ proven in our paper. It is not clear how these results can be deduced from the theory of exponential dichotomies exposed in the monography [**5**].

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Departamento de Matemáticas, Universidad de Oriente, Apartado 245, Cumaná 6101-A, Venezuela
E-mail address: rnaulin@cumana.sucre.udo.edu.ve

Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile