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QUOTIENT MAPS WITH STRUCTURE PRESERVING INVERSES

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ABSTRACT. It is proved that certain quotient maps $q : (\sum_n U_n)_{l_1} \to Y$, where U_n are finite dimensional spaces, have the following property: If E is a subspace of Y with a "good" structure of uniformly complemented finite dimensional subspaces, so is the subspace $q^{-1}(E)$ of $(\sum_n U_n)_{l_1}$. In particular, any quotient map $q : l_1 \to L_1$ has this property.

1. Introduction. Let $q : U \to Y$ be a quotient map. In general, very little is known about the connection between a subspace E of Y and the subspace $q^{-1}(E)$ of U. In this note we discuss a quotient map q, the inverse of which preserves the π property and the finite dimensional decomposition property. Recall that a space E is said to be a π_λ space, $\lambda \geq 1$, if there exist a sequence $\{E_n\}_{n=1}^\infty$ of finite dimensional subspaces of E, with $E_1 \subset E_2 \subset \cdots$ and $\overline{\cup_{n=1}^\infty E_n} = E$, and a sequence of projections $\{P_n\}_{n=1}^\infty$ of E onto E_n with $\sup_n \|P_n\| = \lambda < \infty$. E is said to be a π space (or, to have the π property) if it is a π_λ space for some $\lambda \geq 1$. The pair of sequences $(\{E_n\}_{n=1}^\infty, \{P_n\}_{n=1}^\infty)$ will be called a π *structure of* E. If E has a π structure $(\{E_n\}_{n=1}^\infty, \{P_n\}_{n=1}^\infty)$ and, for every $n, k \geq 1$, $P_n P_k = P_k P_n = P_{\min(k,n)}$, then the sequence $\{(P_n - P_{n-1})(E)\}_{n=1}^\infty$ is called a finite dimensional decomposition of E, f.d.d. for short, and E is said to have the f.d.d. property.

Our main result is the following

Theorem. *Let Y be a π_λ space with a π_λ structure $(\{Y_n\}_{n=1}^\infty, \{Q_n\}_{n=1}^\infty)$, and let $U = (\sum_{n=1}^\infty Y_n)_{l_1}$. For each $n \geq 1$, let U_n denote the subspace $\{(\underbrace{0, \ldots, 0}_{n-1}, y, 0, \ldots) \in U : y \in Y_n\}$, and denote by τ_n the*

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natural isometry of U_n onto Y_n. Let $q : U \to Y$ be the quotient map determined by the relations $q(u) = \tau_n(u)$ if $u \in U_n$ for every $n \geq 1$. Then for every subspace E of Y with a π structure (*with an* f.d.d.), *$q^{-1}(E)$ has a π structure* (*an* f.d.d., *respectively*).

We will prove the theorem in Section 2. Let us now discuss the following two examples.

Example 1. Lindenstrauss investigated in [**1**] the properties of the following quotient map $q : l_1 \to L_1[0,1]$. Let $\{u_i\}_{i=0}^{\infty}$ denote the unit vector basis of l_1 and $\chi(A)$ the indicator function of the subset $A \subset [0,1]$. For each $n \geq 0$ and $1 \leq i \leq 2^n$, put $u_i^n = u_{2^n - 1 + i}$ and define $q : l_1 \to L_1$ by $q(u_i^n) = \chi([((i-1)/2^n), (i/2^n)])$. It is clear that q satisfies the assumption of the theorem and therefore q^{-1} preserves the π and f.d.d. properties. Moreover, because l_1 is quotient homogeneous, (i.e., if $q_1 : l_1 \to L_1$ is another quotient map, then there is an automorphism T on l_1 for which $q_1 = qT$, see [**2**]) any quotient map $q_1 : l_1 \to L_1$ has the same property.

Remark. Note that the same holds for every quotient map $q : l_1 \to Y$ if Y is an $\mathcal{L}_1$-space. Indeed, in this case Y has a π_λ structure $(\{Y_n\}_{n=1}^{\infty}, \{Q_n\}_{n=1}^{\infty})$ where each Y_n has a basis $\{y_i^n\}_{i=1}^{d(n)}$, $d(n) = \dim Y_n$, satisfying the inequality $\lambda^{-1} \sum_{i=1}^{d(n)} |a_i| \leq \| \sum_{i=1}^{d(n)} a_i y_i^n \| \leq \sum_{i=1}^{d(n)} |a_i|$ for every sequence of scalars $\{a_i\}_{i=1}^{d(n)}$. Clearly $U = (\sum_n Y_n)_{l_1}$ is isomorphic to l_1 and therefore the argument presented in Example 1 proves our claim.

Example 2. Let $Y = l_2$, let $\{y_i\}_{i=1}^{\infty}$ be any orthonormal basis of Y and put $Y_n = [y_i]_{i=1}^{n}$. Put $U = (\sum_n Y_n)_{l_1}$, $U_n = \{(\underbrace{0, \dots, 0}_{n-1}, y, 0, \dots) : y \in Y_n\}$, and let $\tau_n : U_n \to Y_n$ be the natural isometry. Define $q : U \to Y$ by the relations $q(u) = \tau_n(u)$ if $u \in U_n$, $n = 1, 2, \dots$. Then q satisfies the assumptions of the theorem; hence, q^{-1} preserves the π and f.d.d. properties.

Let $u_i^n = (\underbrace{0, \dots, 0}_{n-1}, y_i, 0, \dots)$ for every $n \geq 1$ and $1 \leq i \leq n$. Then, clearly, kernel $(q) = [u_i^n - u_i^{n+1}]_{i=1,n=1}^{n\ \ \infty}$ and the theorem implies that every subspace V of U which contains kernel (q) has an f.d.d. In fact, one can show that every such subspace has a basis because kernel (q) has a natural basis, $q(V)$ has a basis and, as is easily seen, the projections V_n on V constructed in the proof of the theorem can be chosen so that the spaces $(V_n - V_{n-1})(V)$ have bases with uniformly bounded constants.

2. Proof of the theorem. Let us begin by taking a close look at the structure of $K = \text{kernel}\,(q)$. Let $K_n = \{u \in \sum_{i=1}^n \oplus U_i : q(u) = 0\}$.

Claim 2a.

$$K = \overline{\bigcup_{n=1}^{\infty} K_n}.$$

Indeed, if $u = \sum_{i=1}^{\infty} u_i \in K$, where $u_i \in U_i$ for $i \geq 1$, and if $\varepsilon > 0$, let N be so large that $\sum_{i=N+1}^{\infty} \|u_i\| < \varepsilon$ and let q_N denote the restriction of q to $\sum_{i=1}^N \oplus U_i$. Put $v = \sum_{i=1}^N u_i$; then $\|q_N(v)\| = \|q(u - \sum_{i=N+1}^{\infty} u_i)\| = \|q(\sum_{i=N+1}^{\infty} u_i)\| < \varepsilon$. But kernel $(q_N) = K_N$ and, by the definition of q, q_N is a quotient map of $\sum_{i=1}^N \oplus U_i$ onto Y_N. Hence, there is a $w \in K_N$ with $\|v - w\| < \varepsilon$. It follows that $\|u - w\| \leq \|u - v\| + \|v - w\| < 2\varepsilon$, proving Claim 2a. Next, note that, for every $1 \leq i \leq n$ and $u \in U_i$, $u - \tau_n^{-1}\tau_i u \in K_n$; hence we have

$$\sum_{i=1}^{n} \oplus U_i = K_n \oplus U_n \quad \text{for every} \quad n \geq 1. \tag{2.1}$$

Moreover, if $p_n : \sum_{i=1}^n \oplus U_i \to U_n$ denotes the projection onto U_n along K_n, then, because $\tau_n = q_n|_{U_n} = q|_{U_n}$ is an isometry, we have that for every $w \in K_n$ and $u \in U_n$, $\|w + u\| \geq \|q(w + u)\| = \|q(u)\| = \|u\|$ and hence $\|p_n\| = 1$. Now consider the mapping $q_{m-1} \oplus q_m = q|_{(U_{m-1} \oplus U_m)}$ which maps $U_{m-1} \oplus U_m$ onto Y_m. Put $H_{m-1} = \text{kernel}\,(q_{m-1} \oplus q_m)$; then $\dim\,(H_{m-1}) = d(m-1) = \dim Y_{m-1}$.

Claim 2b. *$\sum_{m=1}^n H_m$ is a Schauder decomposition of K_{n+1} and $\sum_{m=1}^{\infty} H_m$ is an* f.d.d. *of K.*

Indeed, let R_i denote the natural projection of U onto U_i, let $h_m \in H_m$ for $1 \leq m \leq n$, and suppose that $h_n = u+v$ where $u = R_n h_n \in U_n$ and $v = R_{n+1}h_n \in U_{n+1}$. Then $q(u+v) = q(h_n) = 0$ and, since the maps q_{n+1} and q_n restricted to U_{n+1} and U_n, respectively, are isometries, we get that $\|u\| = \|v\|$ and the maps $\widetilde{R_n} = R_n|_{H_n}$ and $\widetilde{R_{n+1}} = R_{n+1}|_{H_n}$ are isomorphisms satisfying

$$\|\widetilde{R_n}h_n\| = \frac{1}{2}\|h_n\| = \|\widetilde{R_{n+1}}h_n\|. \tag{2.2}$$

It follows that $\|\sum_{m=1}^{n-1} h_m\| \leq \|\sum_{m=1}^{n} h_m\|$ and, since $\dim H_m = \dim Y_{m-1}$, this implies that $\sum_{m=1}^{n} \oplus H_m = K_{n+1}$; hence, in view of Claim 2a, $\sum_{m+1}^{\infty} \oplus H_m$ is an f.d.d. of K. Moreover, the natural projections $W_n : K \to \sum_{m=1}^{n-1} H_m = K_n$ have norm $\|W_n\| = 1$. This proves Claim 2b. Note that $qR_n|_{(U_{n-1}\oplus U_n)}$ is a quotient map of $U_{n-1} \oplus U_n$ onto Y_n, the kernel of which is U_{n-1}; hence, U_{n-1} is isomorphic to H_{n-1} via (2.2).

Assume that E is a subspace of Y with a π_λ structure $(\{E_n\}_{n=1}^{\infty}, \{P_n\}_{n=1}^{\infty})$. A standard small perturbation argument allows us to assume without loss of generality that $\cup_{n=1}^{\infty} E_n \subset \cup_{n=1}^{\infty} Y_n$ because $\overline{\cup_{n=1}^{\infty} Y_n} = Y$, see, e.g., [**3**, Lemma 2.1]. Also, because each E_n is contained in Y_m for a sufficiently large m, allowing finite numbers of repetitions of E_ns in the sequence, we may assume that $E_n \subseteq Y_n$ for every $n \geq 1$.

Let us now construct a π structure in $q^{-1}(E)$. We start with the definition of the finite dimensional subspaces F_n of $q^{-1}(E)$ which will determine the π structure. For every $n \geq 1$, let $G_n = q_n^{-1}(E_n)$ and put $F_n = G_n + K_n$; then, (2.1) ensures that this is a direct sum and, for each $f = g + h$ with $g = q_n^{-1}(e)$, $e \in E_n$ and $h \in K_n$, we have that

$$\|f\| \geq \|g\| = \|e\| \tag{2.3}$$

because $q|_{G_n}$ is an isometry. We must show that $F_n \subset F_{n+1}$. Indeed, let $g_1 = \tau_{n+1}^{-1}\tau_n g$; then $g_1 \in U_{m+1}$ and, putting $h_0 = g_1 - g$, we have that $q(g_1) = q(g) = e$ and hence $h_0 \in H_n \subset K_{n+1}$. Consequently, $f = g + h = g_1 - h_0 + h$ where $g_1 = q_{n+1}^{-1}(e) \in q_{n+1}^{-1}(E_{n+1}) = G_{n+1}$ and $h - h_0 \in K_{n+1}$. This establishes the inclusion $F_n \subset F_{n+1}$. Since $\overline{\cup_{n=1}^{\infty} K_n} = q^{-1}(0)$ and, for each $n \geq 1$, $q|_{G_n}$ is an isometry, we get that $q^{-1}(E) = \overline{\cup_{n=1}^{\infty} F_n}$. We proceed to construct projections V_n of $q^{-1}(E)$ onto F_n which will eventually determine the π structure of $q^{-1}(E)$. Recall that W_n denotes the natural projection of K onto K_n and define the operator V_n on $\cup_{j=1}^{\infty} F_j$ as follows: if $k \geq n$ and $f = g+h$ with $g \in G_k$ and $h \in K_k$, then

$$V_n f = q_n^{-1} P_n q_k(g) + W_n(h). \tag{2.4}$$

This definition obviously depends on the representation $f = g+h$ in F_k. However, suppose that $g = q_k^{-1}(e)$ and $f = g_1 + h_1$ where $g_1 \in G_{k+1}$ and $h_1 \in K_{k+1}$; then the above argument for the inclusion $F_n \subset F_{n+1}$ shows that there is an $h_0 \in H_n$ for which $g_1 = g + h_0$ and $h_1 = h - h_0$. Hence, $q_{k+1}(g_1) = q(g_1) = q(g) = e$ and $W_n(h_0) = 0$. Therefore,

$$\begin{aligned} q_n^{-1} P_n q_{k+1}(g_1) + W_n(h_1) &= q_n^{-1} P_n e + W_n(h - h_0) \\ &= q_n^{-1} P_n q_k(g) + W_n(h) \\ &= V_n(f). \end{aligned}$$

This shows that the definition of V_n does not depend on the choice of k and V_n is well defined. Let us show that $V_n^2 = V_n$. If $f = g + h$ with $g \in G_k$, $h \in K_k$ and $k \geq n$, then the representation of $V_n f$ in F_n is clearly $V_n f = q_n^{-1} P_n q_k(g) + W_n(h)$. Hence, by (2.4), $V_n^2 f = q_n^{-1} P_n q_n q_n^{-1} P_n q_k(g) + W_n^2(h) = q_n^{-1} P_n q_k(g) + W_n(h) = V_n f$. Suppose that the given sequence of projections mutually commute and let $m < n$. Because $W_m W_n = W_m$, we get that $V_m V_n f = q_m^{-1} P_m q_n q_n^{-1} P_n q_k(g) + W_m W_n(h) = q_m^{-1} P_m q_k(g) + W_m(h) = V_m f$, and hence

$$V_m V_n = V_m = V_n V_m. \tag{2.5}$$

Before proceeding to estimate the norm of V_n, let us prove the following

Lemma. *Let X be a Banach space, let $\lambda \geq 1$ and let g, h, u and v be elements of X satisfying the following four inequalities: $\|g\| \leq \|g+h\|$, $\|u\| \leq \|u+v\|$, $\|u\| \leq \lambda\|g\|$ and $\|v\| \leq \lambda\|h\|$. Then $\|u+v\| \leq 3\lambda\|g+h\|$.*

Proof. If $\|v\| \leq 2\|u\|$, then $\|u+v\| \leq \|u\| + \|v\| \leq 3\|u\| \leq 3\lambda\|g\| \leq 3\lambda\|g+h\|$. Suppose that $\|v\| > 2\|u\|$, then since $\|h\| \leq$

$\|g\| + \|g+h\| \leq 2\|g+h\|$ and $\|h\| \geq \lambda^{-1}\|v\| > 2\lambda^{-1}\|u\|$, we get that $\|u+v\| \leq \|u\| + \|v\| \leq (3/2)\|v\| \leq (3/2)\lambda\|h\| \leq 3\lambda\|g+h\|$. □

Let us complete the proof of the theorem by showing that $\|V_n\| \leq 3\lambda$. For any $k \geq n$ and $f = g + h \in G_k \oplus K_k$, put $u = q_n^{-1}P_n q_k(g)$ and $v = W_n(h)$. Then, by (2.3), $\|g\| \leq \|g+h\|$ and $\|u\| \leq \|u+v\|$. Moreover, $\|u\| \leq \|P_n\|\|g\| \leq \lambda\|g\|$ and $\|v\| \leq \|W_n\|\|h\| \leq \|h\| \leq \lambda\|h\|$. It follows from the lemma that $\|V_n f\| = \|q_n^{-1}P_n q_k g + W_n(h)\| = \|u+v\| \leq 3\lambda\|g+h\| = 3\lambda\|f\|$. Extending V_n to all of $q_n^{-1}(E)$ by continuity, we complete the construction of the π structure of $q^{-1}(E)$. Equality (2.5) takes care of the f.d.d. case.

Remark. Our proof shows that, under the assumptions of the theorem, q^{-1} preserves Grothendieck's bounded approximation property and the commuting bounded approximation property.

REFERENCES

1. J. Lindenstrauss, *On a certain subspace of* l_1, Bull. Acad. Polon. Sci. **12** (1964), 539–542.

2. J. Lindenstrauss and H.P. Rosenthal, *Automorphisms in* c_0*,* l_1*, and* m, Israel J. Math. **7** (1969), 227–239.

3. M. Zippin, *Applications of Michael's continuous selection theorem to operator extension problems*, Proc. Amer. Math. Soc., to appear.

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