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GENERALIZED HÖLDER-LIKE INEQUALITIES

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ABSTRACT. Let $n \geq 2$ be a fixed integer, and let M be a one-to-one function. For a real number α, we define

$$R_{\alpha,M} = \Big\{ \mathbf{x} = (x_1, x_2, \dots, x_n) : x_1 > 0, \\ (x_i/x_1) \in \text{Domain}\,(M), i = 2, \dots, n \text{ and} \\ \Big[\alpha - \sum_{i=2}^n M(x_i/x_1)\Big] \in \text{Range}\,(M) \Big\}.$$

For $\mathbf{x} \in R_{\alpha,M}$ we define $\Phi_{\alpha,M}(\mathbf{x}) = \mathbf{x}_1 \mathbf{M}^{-1} \big[\alpha - \sum_{i=2}^n \mathbf{M}(\mathbf{x}_i/\mathbf{x}_1)\big]$. Several inequalities are presented for $\Phi_{\alpha,M}$. As special cases, these inequalities recover many known "Hölder-like" inequalities.

1. Introduction. Let $n \geq 2$ be a fixed integer, and let $\mathbf{R}$ denote the set of all real numbers. Let $a_i, b_i \in \mathbf{R}$, $i = 1, 2, \dots, n$, be such that $a_1^2 - \sum_{i=2}^n a_i^2 \geq 0$ and $b_1^2 - \sum_{i=2}^n b_i^2 \geq 0$. Then in [**1**] it was shown that

$$(1.1) \qquad \left(a_1^2 - \sum_{i=2}^n a_i^2\right)^{1/2} \left(b_1^2 - \sum_{i=2}^n b_i^2\right)^{1/2} \leq a_1 b_1 - \sum_{i=2}^n a_i b_i.$$

Inequality (1.1) was generalized by Popoviciu in [**8**] and by Bellman in [**3**] as follows. Let $p > 1$, $(1/p) + (1/q) = 1$, $a_i, b_i \geq 0$, $i = 1, 2, \dots, n$, with $a_1^p - \sum_{i=2}^n a_i^p \geq 0$, and $b_1^q - \sum_{i=2}^n b_i^q \geq 0$. Then

$$(1.2) \qquad \left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{1/q} \leq a_1 b_1 - \sum_{i=2}^n a_i b_i.$$

This is the "Hölder-like" generalization of (1.1). In [**9**] there is a very simple proof of (1.2) for $p > 1$ and the inverse inequality for $p < 1$ is given. Also, Chapter 5 in [**7**] contains generalizations of (1.2).

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In [**4**], Bjelica showed that if $a_1^p - \sum_{i=2}^n a_i^p \geq 0$ and $b_1^p - \sum_{i=2}^n b_i^p \geq 0$, then, for $0 < p \leq 2$,

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{1/p} \leq a_1 b_1 - \sum_{i=2}^n a_i b_i. \tag{1.3}$$

For a fixed integer $n \geq 2$, the authors in [**6**] introduced the following definition. For a nonzero number $p \in \mathbf{R}$, let

$$\phi_p(\mathbf{x}) = \left(x_1^p - \sum_{i=2}^n x_i^p\right)^{1/p}, \quad \mathbf{x} \in R_p,$$

where

$$R_p = \left\{\mathbf{x} = (x_1, x_2, \dots, x_n) : x_i \geq 0 (i = 1, 2, \dots, n), x_1^p \geq \sum_{i=2}^n x_i^p\right\} \quad \text{if } p > 0$$

and

$$R_p = \left\{\mathbf{x} = (x_1, x_2, \dots, x_n) : x_i > 0 (i = 1, 2, \dots, n), x_1^p > \sum_{i=2}^n x_i^p\right\} \quad \text{if } p < 0.$$

There they presented three inequalities for ϕ_p from which they deduced, among other things, the inequalities (1.1), (1.2) and (1.3).

Throughout this paper we use the following notations. Let M be a one-to-one function whose domain is a subset of the set of real numbers $\mathbf{R}$. For $\alpha \in \mathbf{R}$, we define

$$R_{\alpha,M} = \begin{cases} \mathbf{x} = (x_1, x_2, \dots, x_n) : x_1 > 0, (x_i/x_1) \in \text{Domain}\,(M) \\ (i = 2, \dots, n) \\ \text{and } [\alpha - \sum_{i=2}^n M(x_i/x_1)] \in \text{Range}\,(M) \end{cases}$$

and, for $\mathbf{x} \in R_{\alpha,M}$, we define

$$\Phi_{\alpha,M}(\mathbf{x}) = x_1 M^{-1}\left[\alpha - \sum_{i=2}^n M\left(\frac{x_i}{x_1}\right)\right].$$

In Section 2 we present an inequality involving Orlicz functions, Theorem 1, from which we deduce Popoviciu's theorem [**8**]. In Section 3, we give a general inequality, Theorem 3, which can be used to obtain several known inequalities in [**6**]. In Section 4 we present a comparison theorem that generalizes Theorem 1 in [**6**], and we also give a generalization of Bjelica's result in [**4**].

2. Orlicz functions. The introduction of Orlicz functions has been inspired by the obvious role played by the functions t^p in the definition of the l_p spaces.

An Orlicz function M is a continuous strictly increasing and convex function defined on $[0, \infty)$ such that $\lim_{t\to 0^+}(M(t)/t) = 0$ and $\lim_{t\to\infty}(M(t)/t) = \infty$. For an Orlicz function M, the function $M^*(s) = \sup_{0<t<\infty}\{st - M(t)\}$, $0 \le s < \infty$, is called the function complementary to M. For example, if $M(t) = (t^p/p)$, $p > 1$, then $M^*(s) = (s^q/q)$, where $(1/p)+(1/q) = 1$. It is clear from the definition of M^* that, for any $t, s \ge 0$, we have the so-called Young's inequality, namely,

$$ts \le M(t) + M^*(s), \tag{2.1}$$

where equality holds if and only if $s = M'_+(t)$, M'_+ being the right derivative of M. Convenient references for Orlicz functions and Orlicz spaces can be found in [**2**, Chapter 8] and [**5**, Chapter 4].

Our first theorem is a generalization of the inequality (1.2).

Theorem 1. *Let M be an Orlicz function, and let α and β be positive real numbers. Then for $\mathbf{x} = (x_1, x_2, \dots, x_n) \in R_{\alpha,M}$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in R_{\beta,M^*}$ we have*

$$\Phi_{\alpha,M}(\mathbf{x})\Phi_{\beta,M^*}(\mathbf{y}) \le (\alpha+\beta)x_1y_1 - \sum_{i=2}^{n} x_iy_i.$$

Proof. By applying Young's inequality (2.1) with $t = (\Phi_{\alpha,M}(\mathbf{x})/x_1)$

and $s = (\Phi_{\beta,M^*}(\mathbf{y})/y_1)$, we get that

$$\begin{aligned}\Phi_{\alpha,M}(\mathbf{x})\Phi_{\beta,M^*}(\mathbf{y})&\\ &\le x_1y_1\left[\alpha-\sum_{i=2}^{n}M\left(\frac{x_i}{x_1}\right)\right]+x_1y_1\left[\beta-\sum_{i=2}^{n}M^*\left(\frac{y_i}{y_1}\right)\right]\\ &=(\alpha+\beta)x_1y_1-x_1y_1\sum_{i=2}^{n}\left(M\left(\frac{x_i}{x_1}\right)+M^*\left(\frac{y_i}{y_1}\right)\right).\end{aligned}$$

Since again, by Young's inequality (2.1) we have

$$M\left(\frac{x_i}{x_1}\right)+M^*\left(\frac{y_i}{y_1}\right)\ge\frac{x_iy_i}{x_1y_1},$$

then we obtain

$$\Phi_{\alpha,M}(\mathbf{x})\Phi_{\beta,M^*}(\mathbf{y})\le(\alpha+\beta)x_1y_1-\sum_{i=2}^{n}x_iy_i,$$

as required. This completes the proof of Theorem 1. □

The following corollary is Popoviciu's theorem [**8**], which is inequality (1.2).

Corollary 2. *Let* $p, q \in (1,\infty)$, $(1/p)+(1/q)=1$, *and let* $x_i, y_i \ge 0$, $i = 1, 2, \dots, n$, *with* $x_1^p - \sum_{i=2}^n x_i^p \ge 0$ *and* $y_1^q - \sum_{i=2}^n y_i^q \ge 0$. *Then*

$$\left(x_1^p-\sum_{i=2}^{n}x_i^p\right)^{1/p}\left(y_1^q-\sum_{i=2}^{n}y_i^q\right)^{1/q}\le x_1y_1-\sum_{i=2}^{n}x_iy_i.$$

Proof. Let $M(t) = (t^p/p)$, $t \in [0,\infty)$. Then $M^*(t) = (t^q/q)$, $t \in [0,\infty)$. The assumptions imply that $(x_1, x_2, \dots, x_n) \in R_{(1/p),M}$ and $(y_1, y_2, \dots, y_n) \in R_{(1/q),M^*}$. Now we apply Theorem 1 with $\alpha = (1/p)$ and $\beta = (1/q)$ to get the required inequality. This completes the proof of Corollary 2. □

3. Inequalities involving monotonic functions. In this section we present general inequalities involving one-to-one functions. These inequalities can be used to recover many known inequalities.

Theorem 3. *Let $M_1, M_2, \dots, M_m$ be one-to-one real-valued functions defined in $\mathbf{R}$, and let $\sigma_1, \sigma_2, \dots, \sigma_m$ be fixed real numbers. Then we have*

(i) *If*

$$t_1 t_2 \cdots t_m \le \sigma_1 M_1(t_1) + \sigma_2 M_2(t_2) + \cdots + \sigma_m M_m(t_m) \tag{3.1}$$

for all $t_k \in \mathrm{Domain}\,(M_k)$, $k = 1, 2, \dots, m$, then

$$\prod_{k=1}^{m} \Phi_{\alpha_k, M_k}(\mathbf{x}_k) \le (\sigma_1\alpha_1 + \sigma_2\alpha_2 + \cdots + \sigma_m\alpha_m) x_{11} \cdots x_{m1} - \sum_{i=2}^{n} (x_{1i} \cdots x_{mi}) \tag{3.2}$$

for all $\alpha_k \in \mathbf{R}$ satisfying $R_{\alpha_k, M_k} \neq \varnothing$ and all $\mathbf{x}_k \in R_{\alpha_k, M_k}$, $k = 1, \dots, m$.

(ii) *If*

$$t_1 t_2 \cdots t_m \ge \sigma_1 M_1(t_1) + \sigma_2 M_2(t_2) + \cdots + \sigma_m M_m(t_m) \tag{3.3}$$

for all $t_k \in \mathrm{Domain}\,(M_k)$, $k = 1, 2, \dots, m$, then

$$\prod_{k=1}^{m} \Phi_{\alpha_k, M_k}(\mathbf{x}_k) \ge (\sigma_1\alpha_1 + \sigma_2\alpha_2 + \cdots + \sigma_m\alpha_m) x_{11} \cdots x_{m1} - \sum_{i=2}^{n} (x_{1i} \cdots x_{mi}) \tag{3.4}$$

for all $\alpha_k \in \mathbf{R}$ satisfying $R_{\alpha_k, M_k} \neq \varnothing$ and all $\mathbf{x}_k \in R_{\alpha_k, M_k}$, $k = 1, \dots, m$.

Proof. First let us prove part (i). Using (3.1), we get

$$\begin{aligned}\prod_{k=1}^{m} x_{k1} M_k^{-1}\bigg[\alpha_k - \sum_{i=2}^{n} M_k\bigg(\frac{x_{ki}}{x_{k1}}\bigg)\bigg] \\ &= \bigg(\prod_{k=1}^{m} x_{k1}\bigg)\prod_{k=1}^{m} M_k^{-1}\bigg[\alpha_k - \sum_{i=2}^{n} M_k\bigg(\frac{x_{ki}}{x_{k1}}\bigg)\bigg] \\ &\leq \bigg(\prod_{k=1}^{m} x_{k1}\bigg)\bigg\{\sum_{k=1}^{m}\sigma_k\bigg[\alpha_k - \sum_{i=2}^{n} M_k\bigg(\frac{x_{ki}}{x_{k1}}\bigg)\bigg]\bigg\} \\ &= \bigg(\prod_{k=1}^{m} x_{k1}\bigg)\bigg[\sum_{k=1}^{m}\sigma_k\alpha_k - \sum_{i=2}^{n}\sum_{k=1}^{m}\sigma_k M_k\bigg(\frac{x_{ki}}{x_{k1}}\bigg)\bigg] \\ &\leq \bigg(\prod_{k=1}^{m} x_{k1}\bigg)\bigg[\sum_{k=1}^{m}\sigma_k\alpha_k - \sum_{i=2}^{n}\prod_{k=1}^{m}\bigg(\frac{x_{ki}}{x_{k1}}\bigg)\bigg] \\ &= \bigg(\sum_{k=1}^{m}\sigma_k\alpha_k\bigg)\prod_{k=1}^{m} x_{k1} - \sum_{i=2}^{n}\bigg(\prod_{k=1}^{m} x_{ki}\bigg).\end{aligned}$$

This completes the proof of part (i). Similarly we obtain the proof of part (ii). □

Before we give some consequences of Theorem 1, we first mention the following lemma. The referee brought to our attention that this lemma recently appeared in [**10**].

Lemma 4. *Let* $p_1, p_2, \dots, p_m$ *be real numbers such that* $(1/p_1) + (1/p_2) + \cdots + (1/p_m) = 1$, $m \geq 2$. *Then*

$$(3.5)\quad t_1 t_2 \cdots t_m \geq \frac{t_1^{p_1}}{p_1} + \frac{t_2^{p_2}}{p_2} + \cdots + \frac{t_m^{p_m}}{p_m} \quad \text{for all } t_i > 0, \quad i = 1, \dots, m,$$

if and only if all p_i*'s are negative except for exactly one of them.*

Proof. Suppose that all p_i's are negative except for exactly one of them, say $p_m > 0$. Let $z_i = (1/t_i^{p_m})$, $i = 1, 2, \dots, m-1$, and let $z_m = t_1^{p_m} t_2^{p_m} \cdots t_m^{p_m}$. Also, let $q_i = |p_i|/p_m$, $i = 1, 2, \dots, m-1$ and

$q_m = (1/p_m)$. Then all the q_i's are positive and

$$\begin{aligned}\sum_{i=1}^{m}\frac{1}{q_i} &= -\frac{p_m}{p_1} - \cdots - \frac{p_m}{p_{m-1}} + p_m\\ &= p_m\left(1 - \frac{1}{p_1} - \cdots - \frac{1}{p_m}\right)\\ &= p_m\left(\frac{1}{p_m}\right) = 1.\end{aligned}$$

Hence we may apply the well-known inequality

$$x_1 x_2 \cdots x_m \le \frac{x_1^{q_1}}{q_1} + \cdots + \frac{x_m^{q_m}}{q_m}, \quad x_i \ge 0, \quad i = 1, 2, \dots, m, \tag{3.6}$$

to get that $z_1 z_2 \cdots z_m \le (z_1^{q_1}/q_1) + \cdots + (z_m^{q_m}/q_m)$. This gives

$$t_m^{p_m} \le -\frac{p_m}{p_1} t_1^{p_1} - \cdots - \frac{p_m}{p_{m-1}} t_{m-1}^{p_{m-1}} + p_m t_1 t_2 \cdots t_m.$$

Therefore after rearranging and dividing by p_m, we get that

$$t_1 t_2 \cdots t_m \ge \frac{t_1^{p_1}}{p_1} + \cdots + \frac{t_m^{p_m}}{p_m}.$$

Conversely, suppose that (3.5) holds. Then by (3.6) at least one of the p_i's must be negative. Since $\sum_{i=1}^{m}(1/p_i) = 1$, then clearly one of the p_i's is positive. Suppose that there exist at least two positive p_i's and at least one negative p_i, say $p_1, p_2, \dots, p_k > 0$ and $p_{k+1}, \dots, p_m < 0$, where $2 \le k < m$. Let $A(t_1, \dots, t_m) = t_1 t_2 \cdots t_m$ and $B(t_1, \dots, t_m) = (t_1^{p_1}/p_1) + \cdots + (t_m^{p_m}/p_m)$. For $r > 0$, we have

$$A\left(r^{k-1}, \frac{1}{r}, \dots, \frac{1}{r}, \overbrace{\quad 1 \quad}^{(k+1)st\ \text{term}}, \dots, 1\right) = 1$$

and

$$\begin{aligned}&B\left(r^{k-1}, \frac{1}{r}, \dots, \frac{1}{r}, \overbrace{\quad 1 \quad}^{(k+1)st\ \text{term}}, \dots, 1\right)\\ &\qquad = \frac{r^{p_1(k-1)}}{p_1} + \frac{1}{p_2 r^{p_2}} + \cdots + \frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m} \to \infty\end{aligned}$$

as $r \to \infty$ because $\lim_{r\to\infty}(r^{p_1(k-1)}/p_1) = \infty$. Thus, for r large enough, we have

$$A\bigg(r^{k-1}, \frac{1}{r}, \cdots, \frac{1}{r}, 1, \cdots, 1\bigg) < B\bigg(r^{k-1}, \frac{1}{r}, \cdots, \frac{1}{r}, 1, \cdots, 1\bigg).$$

This contradiction with (3.5) completes the proof of the lemma.

The following result appears in Corollary 1 in [**6**].

Corollary 5. *Let $p_1, p_2, \dots, p_m$ be real numbers such that $(1/p_1) + \cdots + (1/p_m) = 1$, $m \geq 2$. Then*

(i) *If all p_i's are negative except for exactly one of them, then we have*

$$\prod_{k=1}^{m} \Phi_{1,t^{p_k}}(\mathbf{x}_k) \geq \Phi_{1,t}(\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_m)$$

for all $\mathbf{x}_k \in R_{1,t^{p_k}}$, $k = 1, 2, \dots, m$.

(ii) *If all p_i's are positive, then*

$$\prod_{k=1}^{m} \Phi_{1,t^{p_k}}(\mathbf{x}_k) \leq \Phi_{1,t}(\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_m)$$

for all $\mathbf{x}_k \in R_{1,t^{p_k}}$, $k = 1, 2, \dots, m$, where

$$\mathbf{x}_1\mathbf{x}_2\dots\mathbf{x}_m = (x_{11}\dots x_{m1}, \dots, x_{1n}\dots x_{mn}).$$

Proof. (i) Suppose that all p_i's are negative except for exactly one of them. Then inequality (3.5) holds. For each $k \in \{1, \dots, m\}$, let $\alpha_k = 1$, $\sigma_k = (1/p_k)$, and let $M_k(t) = t^{p_k}$, $t \in (0, \infty)$. Then (3.3) becomes (3.5) and consequently (3.3) holds. Now applying Theorem 3 (ii) we get

$$\prod_{k=1}^{m} \Phi_{1,t^{p_k}}(\mathbf{x}_k) \geq \Phi_{1,t}(\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_m)$$
$$\text{for all } \mathbf{x}_k \in R_{1,t^{p_k}}, \quad k = 1, 2, \dots, m.$$

(ii) Suppose that all p_i's are positive. Then (3.6) holds. For each $k \in \{1, \dots, m\}$, let $\alpha_k = 1$, $\sigma_k = (1/p_k)$, and let $M_k(t) = t^{p_k}$, $t \in [0, \infty)$. Then (3.1) becomes (3.6) and consequently (3.1) holds. Hence we may apply Theorem 3 (i) to get that $\prod_{k=1}^m \Phi_{1,t^{p_k}}(\mathbf{x}_k) \le \Phi_{1,t}(\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m)$ for all $\mathbf{x}_k \in R_{1,t^{p_k}}$, $k = 1, 2, \dots, m$.

Lemma 6. *Let M be a strictly increasing continuous function defined on $[0, \infty)$ with $M(0) = 0$ and $M(\infty) = \infty$. For each $s \in (0, \infty)$, let $\tilde{M}(s) = \max_{t \in (0,\infty)}\{M(t) - st\}$ be well defined. Then $\tilde{M}$ is continuous, strictly decreasing and* Range $(\tilde{M}) = (0, \infty)$.

Proof. We divide the proof into three steps.

Step 1. $\tilde{M}$ is strictly decreasing on $(0, \infty)$.

By the definition of $\tilde{M}$, we have, for each $s \in (0, \infty)$, $\tilde{M}(s) = M(t_s) - st_s$ for some $t_s \in (0, \infty)$. Now let $s_1, s_0 \in (0, \infty)$. Now using the definition of $\tilde{M}$, we get

$$\begin{aligned}\tilde{M}(s_1) + (s_1 - s_0)t_{s_1} &= M(t_{s_1}) - s_1 t_{s_1} + (s_1 - s_0)t_{s_1} \\ &= M(t_{s_1}) - s_0 t_{s_1} \\ &\le \tilde{M}(s_0) = M(t_{s_0}) - s_0 t_{s_0} \\ &= M(t_{s_0}) - s_1 t_{s_0} + (s_1 - s_0)t_{s_0} \\ &\le \tilde{M}(s_1) + (s_1 - s_0)t_{s_0}.\end{aligned} \tag{3.7}$$

If $0 < s_0 < s_1 < \infty$, then the first inequality in (3.7) gives $\tilde{M}(s_1) < \tilde{M}(s_0)$, which completes the proof of Step 1.

Step 2. $\tilde{M}$ is continuous on $(0, \infty)$.

Let $s_0 \in (0, \infty)$ be fixed. From (3.7) we get, for each $s \in (0, \infty)$,

$$|\tilde{M}(s) - \tilde{M}(s_0)| \le \max(t_s, t_{s_0})|s - s_0|. \tag{3.8}$$

Let $s_1 \in (0, s_0)$ be a fixed number. By comparing the first and last term in (3.7) we obtain that $t_{s_0} \le t_{s_1}$ and, in a similar way, that $t_s \le t_{s_1}$ for all $s > s_1$. This with (3.8) implies that $|\tilde{M}(s) - \tilde{M}(s_0)| \le t_{s_1}|s - s_0|$ if s is close enough to s_0. Consequently, $\tilde{M}$ is continuous at s_0, and hence this completes the proof of Step 2.

Step 3. Range $(\tilde{M}) = (0, \infty)$.

First let us show that $\lim_{s\to 0^+}\tilde{M}(s)=\infty$. For each $t\in(0,\infty)$, we have $\tilde{M}(s)\geq M(t)-st$. Therefore, $\liminf_{s\to 0^+}\tilde{M}(s)\geq \lim_{s\to 0^+}[M(t)-st]=M(t)$. Since $\sup_{t\in(0,\infty)}M(t)=\infty$, then $\lim_{s\to 0^+}\tilde{M}(s)=\infty$.

Next let us show that $\lim_{s\to\infty}\tilde{M}(s)=0$. Since $\tilde{M}$ is strictly decreasing on $(0,\infty)$ by Step 1, and $\tilde{M}(s)\geq 0$ for all $s\in(0,\infty)$, then there exists $l\geq 0$ such that $\lim_{s\to\infty}\tilde{M}(s)=l$. From the first inequality in (3.7) we obtain that, for all $s\in(0,\infty)$, $\tilde{M}(s+1)+((s+1)-s)t_{s+1}\leq \tilde{M}(s)$. Hence $0<t_{s+1}\leq \tilde{M}(s)-\tilde{M}(s+1)$. Consequently, we get that $\lim_{s\to\infty}t_{s+1}=0$. But we have $0\leq\tilde{M}(s)=M(t_s)-st_s\leq M(t_s)$ for all $s>0$. Therefore, letting $s\to\infty$, we get that $0\leq\limsup_{s\to\infty}\tilde{M}(s)\leq \lim_{s\to\infty}M(t_s)=M(0)=0$. Hence, $\lim_{s\to\infty}\tilde{M}(s)=0$. This proves that Range $(\tilde{M})=(0,\infty)$, as required, and that ends the proof of the lemma. □

Corollary 7. *Let M be a strictly increasing continuous function defined on $[0,\infty)$ with $M(0)=0$ and $M(\infty)=\infty$. For each $s\in(0,\infty)$, let $\tilde{M}(s)=\max_{t\in(0,\infty)}\{M(t)-st\}$ be well defined. If $\alpha,\beta\in(0,\infty)$, then $R_{\alpha,M}\neq\varnothing$ and $R_{\beta,\tilde{M}}\neq\varnothing$. Moreover, if $\mathbf{x}\in R_{\alpha,M}$ and $\mathbf{y}\in R_{\beta,\tilde{M}}$, then we have*

$$\Phi_{\alpha,M}(\mathbf{x})\Phi_{\beta,\tilde{M}}(\mathbf{y})\geq(\alpha-\beta)x_1y_1-\sum_{i=2}^{n}x_iy_i. \tag{3.9}$$

Proof. By the definition of $\tilde{M}$, we have $t_1t_2\geq M(t_1)-\tilde{M}(t_2)$, $t_1\in[0,\infty)$ and $t_2\in(0,\infty)$. By Lemma 6 we obtain that $R_{\beta,\tilde{M}}\neq\varnothing$. Now applying Theorem 3 (ii) with $M_1=M$, $M_2=\tilde{M}$, $\alpha_1=\alpha$, $\beta_1=\beta$, $\sigma_1=1$ and $\sigma_2=-1$, we get inequality (3.9) as required.

The following result is part (ii) of Corollary 2 in [**6**].

Corollary 8. *Let $0<p<1$, $(1/p)+(1/q)=1$. Then for $x_i\geq 0$, $x_1^p\geq\sum_{i=2}^n x_i^p$, $y_i>0$ and $y_1^q>\sum_{i=2}^n y_i^q$, we have*

$$\left(x_1^p-\sum_{i=2}^{n}x_i^p\right)^{1/p}\left(y_1^q-\sum_{i=2}^{n}y_i^q\right)^{1/q}\geq x_1y_1-\sum_{i=2}^{n}x_iy_i. \tag{3.10}$$

Proof. Let $M(t)=(t^p/p)$, $t \in [0,\infty)$. Then $\tilde{M}(s)=\sup_{0<t<\infty}\{M(t)-st\} = s^q/|q|$, $0 < s < \infty$. The conditions on x_i and y_i imply that $(x_1, x_2, \dots, x_n) \in R_{(1/p),M}$ and $(y_1, y_2, \dots, y_n) \in R_{(1/|q|),\tilde{M}}$. Now applying Corollary 7 with $\alpha = (1/p)$ and $\beta = 1/|q|$, we get exactly inequality (3.10), as required. □

4. A comparison theorem and further inequalities. In this section we present a theorem that will provide a generalization of Theorem 1 in [**6**]. Moreover, we combine this with Theorem 3 to deduce Bejelica's result [**4**].

Theorem 9. *Let M_1 and M_2 be two strictly increasing functions from the interval I onto I, where I is either $[0,1]$ or $(0,1)$. Suppose that $M_1(t) < M_2(t)$ for all $t \in (0,1)$. Then we have*:

If $\mathbf{x} \in R_{1,M_2}$ then $\mathbf{x} \in R_{1,M_1}$ and

(i) *If $I = [0,1]$, then $\Phi_{1,M_2}(\mathbf{x}) \le \Phi_{1,M_1}(\mathbf{x})$. Equality holds if and only if either $x_i = 0$ for all $i \ge 2$ or $x_{i_0} = x_1$ for some $i_0 \ge 2$ and $x_i = 0$ for all $i \ge 2$, $i \ne i_0$.*

(ii) *If $I = (0,1)$, then $\Phi_{1,M_2}(\mathbf{x}) < \Phi_{1,M_1}(\mathbf{x})$.*

Proof. From the conditions on M_1 and M_2 we obtain that if $I = [0,1]$, then

$$M_1(0) = M_2(0) = 0 \quad \text{and} \quad M_1(1) = M_2(1) = 1. \tag{4.1}$$

Now let $I = [0,1]$ and let $\mathbf{x} \in R_{1,M_2}$. By (4.1) and since $M_1(t) < M_2(t)$ for all $t \in (0,1)$, we have

$$0 \le \left[1 - \sum_{i=2}^{n} M_2\left(\frac{x_i}{x_1}\right)\right] \le \left[1 - \sum_{i=2}^{n} M_1\left(\frac{x_i}{x_1}\right)\right] \le 1. \tag{4.2}$$

Note that $(x_i/x_1) \in \text{Domain}\,(M_2) = \text{Domain}\,(M_1)$. From (4.2) we obtain that $\mathbf{x} \in R_{1,M_1}$ and, since $M_2^{-1}(t) < M_1^{-1}(t)$ for all $t \in (0,1)$, $M_2^{-1}(0) = M_1^{-1}(0) = 0$, $M_2^{-1}(1) = M_1^{-1}(1) = 1$ and M_j^{-1}, $j = 1,2$, is strictly increasing in I, that

$$M_2^{-1}\left[1 - \sum_{i=2}^{n} M_2\left(\frac{x_i}{x_1}\right)\right] \le M_1^{-1}\left[1 - \sum_{i=2}^{n} M_1\left(\frac{x_i}{x_1}\right)\right].$$

Multiplying both sides by x_1 we get that $\Phi_{1,M_2}(\mathbf{x}) \le \Phi_{1,M_1}(\mathbf{x})$. The proof of (ii) is similar. For the remainder of (i), it is clear that from (4.1), (4.2) and the fact that $M_1(t) < M_2(t)$ for all $t \in (0,1)$ that equality holds if and only if $(x_i/x_1) = 0$ or 1 for each $i \ge 2$. Since $\mathbf{x} \in R_{1,M_2}$, then $[1 - \sum_{i=2}^n M_2(x_i/x_1)] \in \text{Range}\,(M_2) = [0,1]$. Hence, we must have either $(x_i/x_1) = 0$ for all $i \ge 2$ or $(x_{i_0}/x_1) = 1$ for some $i_0 \ge 2$ and $(x_i/x_1) = 0$ for all $i (\ne i_0) \ge 2$. This completes the proof of the theorem.

Remark 1. We note that the inequality $\Phi_{1,M_2}(\mathbf{x}) \le \Phi_{1,M_1}(\mathbf{x})$ in (i) of Theorem 9 still holds if we replace the assumption $M_1(t) < M_2(t)$ for all $t \in (0,1)$ by the assumption $M_1(t) \le M_2(t)$ for all $t \in (0,1)$.

The following corollary is Theorem 1 in [**6**].

Corollary 10. *If* $0 < p < q$ *and* $\mathbf{x} \in R_p$, *then* $\mathbf{x} \in R_q$ *and* $\phi_p(\mathbf{x}) \le \phi_q(\mathbf{x})$. *Equality holds if and only if* $x_2 = \cdots = x_n = 0$ *or* $x_1 = x_s$ *for some* $s \in \{2, \dots, n\}$ *and* $x_i = 0$ *for* $i \in \{2, \dots, n\} \backslash \{s\}$.

If $p < q < 0$ *and* $\mathbf{x} \in R_q$, *then* $\mathbf{x} \in R_p$ *and* $\phi_p(\mathbf{x}) < \phi_q(\mathbf{x})$.

Proof. Let $0 < p < q$. Let $M_1(t) = t^q$ and $M_2(t) = t^p$, $t \in [0,1]$. Then $R_q = R_{1,M_1} \cup \{0\}$ and $R_p = R_{1,M_2} \cup \{0\}$. Now apply Theorem 9 (i) to get the required conclusion. Similarly if $p < q < 0$ we take $M_1(t) = t^{-p}$ and $M_2(t) = t^{-q}$, $t \in (0,1)$, and we apply Theorem 9 (ii).

Corollary 11. *Let* M *be a strictly increasing function from* $[0,1]$ *onto* $[0,1]$ *with* $M(t) \ge t^2$ *for all* $t \in (0,1)$. *If* $\mathbf{x}, \mathbf{y} \in R_{1,M}$, *then* $\mathbf{x}, \mathbf{y} \in R_{1,t^2}$ *and*

$$\Phi_{1,M}(\mathbf{x})\Phi_{1,M}(\mathbf{y}) \le \Phi_{1,t^2}(\mathbf{x})\Phi_{1,t^2}(\mathbf{y}) \le \Phi_{1,t}(\mathbf{xy}),$$

where $\mathbf{xy} = (x_1y_1, \dots, x_ny_n)$.

Proof. The first inequality follows from Theorem 9 and the remark following it by taking $M_1(t) = t^2$ and $M_2 = M$. The second inequality follows from Theorem 3 (i) by taking $m = 2$, $M_1(t) = M_2(t) = t^2$, $\alpha_1 = \alpha_2 = 1$ and $\sigma_1 = \sigma_2 = (1/2)$.

The above corollary is a generalization of Bjelica's result in [**4**].

As a direct consequence of Corollary 11, we obtain Corollary 3 in [**6**]:

Corollary 12. *If* $0 < p \leq 2$ *and* $\mathbf{x}, \mathbf{y} \in R_p$, *then*

$$\phi_p(\mathbf{x})\phi_p(\mathbf{y}) \leq \phi_2(\mathbf{x})\phi_2(\mathbf{y}) \leq \phi_1(\mathbf{xy}).$$

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