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MAXIMALITY OF THE HYPERCUBE GROUP

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ABSTRACT. In this paper we prove maximality of the hypercube group $\mathbf{B}_n \leqslant \mathbf{O}(n)$ for $n \geq 3$, $n \neq 4$, as a closed subgroup of $\mathbf{O}(n)$. $\mathbf{B}_4 \leqslant \mathbf{O}(4)$ is not maximal, but we are able to describe all closed supergroups of $\mathbf{B}_4$. Furthermore, we indicate how this result is used in bifurcation theory for $\mathbf{O}(n)$-equivariant equations like semilinear elliptic boundary value problems.

1. Introduction. In this paper we will discuss the symmetry group of the n-cube $[-1,1]^n \subset \mathbf{R}^n$, $n \geq 3$. We will denote this group by $\mathbf{B}_n$. The questions we are interested in are whether $\mathbf{B}_n \leqslant \mathbf{O}(n)$ is a maximal closed subgroup or, if not, which are the nontrivial closed supergroups of $\mathbf{B}_n$.

In Section 2 we prove maximality of the hypercube group $\mathbf{B}_n \leqslant \mathbf{O}(n)$ for $n \geq 3$, $n \neq 4$, in the sense that there is no nontrivial closed supergroup of $\mathbf{B}_n$ in $\mathbf{O}(n)$. $\mathbf{B}_4 \leqslant \mathbf{O}(4)$ is not maximal, but we are able to describe in Section 3 all closed supergroups of $\mathbf{B}_4$.

A first step in the proof is to show discreteness and hence finiteness of a supergroup Γ of $\mathbf{B}_n$. This follows basically from the fact that $\mathbf{B}_n$ acts irreducible on the Lie algebra of $\mathbf{O}(n)$. The finite group Γ is then set in relation to the reflection group guaranteed by reflections in Γ and their normalizer which, to the very end, determines Γ itself.

The method to determine the various normalizers is always very similar. Essentially all is based on the knowledge of a characteristic subgroup Z of the finite reflection group, say $G \leqslant \mathbf{O}(n)$. Denoting by $\mathcal{R}$ the set of roots of G we have Z acting on $\mathbf{R}\mathcal{R}$ (or a certain subset) in the natural way. Therefore, $\mathbf{R}\mathcal{R}$ decomposes in Z-orbits and elements of the normalizer of G now act on these orbits by permutation. This already enables computation of the normalizer, at least in our examples.

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These questions are of relevance in equivariant bifurcation theory (cf., e.g., [**3**]). They might be applied for instance in connection with the Equivariant Branching Lemma (see [**3**, Chap. XIII, Theor. 3.3] or [**2**]), where a maximal isotropy subgroup Σ of a closed supergroup Γ is assumed to prove bifurcating solutions of a Γ-equivariant problem which has isotropy subgroup Σ. An application in that spirit to semi-linear Neumann problems on the ball in $\mathbf{R}^n$ is given in Maier-Paape [**6**, Chap. 3]. Here solutions with isotropy subgroup $\mathbf{B}_n$ for $n \geq 3$, $n \neq 4$, are obtained. We note that a maximal closed subgroup $\Sigma \neq \mathbf{SO}\,(n)$ of $\Gamma = \mathbf{SO}\,(n)$ or $\mathbf{O}(n)$ is not a maximal isotropy subgroup of all representations of the group Γ, but by a result of Lauterbach and Maier [**5**, Theor. 6.5], Σ is a maximal isotropy subgroup for infinitely many spherical representations.

Another application of our result is given in Maier-Paape, Schmitt and Wang [**7**, Sect. 5]. Here also semi-linear Neumann problems are discussed; however, now with a homogeneous nonlinearity with critical exponent. In other words, we search for positive solutions of $-\Delta u + \lambda u = u^p$ in $\Omega \subset \mathbf{R}^n$ subject to Neumann boundary conditions. Here $\lambda \in \mathbf{R}$ is a parameter and $p = ((n+2)/(n-2))$ is the critical exponent for $\mathbf{R}^n$, $n \geq 3$.

The methods used in [**7**] are both variational and group theoretical. One discusses domains Ω, which are invariant under a closed subgroup Γ of $\mathbf{O}(n)$. Essentially, it is possible to construct solutions in the fixed-point space $\mathrm{Fix}\,(\Sigma)$, Σ a subgroup of Γ, which are peaked (i.e., attain their global maximum) at a finite number of well-located points on the boundary $\partial\Omega$. This information, together with the characterization of all closed supergroups of $\mathbf{B}_n$ in this paper, is enough to prove that these solutions indeed have isotropy subgroup $\Sigma = \mathbf{B}_n$, $n \geq 3$ (now $n = 4$ included). These solutions are peaked at the $2n$ intersection points of the Cartesian axes with a sphere in $\mathbf{R}^n$.

We next introduce some well-known properties of the groups $\mathbf{B}_n$ (confer Humphreys [**4**, Chap. I, Sect. 2.10] for details). $\mathbf{B}_n$ is a finite group generated by reflections at hyperplanes in $\mathbf{R}^n$, or, to use Humphrey's notation, $\mathbf{B}_n$ is the Weyl group of type B_n. Two important subgroups are S_n (permuting the canonical basis in $\mathbf{R}^n$, which we from now on call e_i, $i = 1, \ldots, n$) and $\mathcal{H}_n = (\mathbf{Z}_2)^n$ (acting by sign changes on the e_i). We have the semi-direct product $\mathbf{B}_n = \mathcal{H}_n \rtimes S_n$ and therefore $\mathcal{H}_n$ is normal in $\mathbf{B}_n$.

The results of this paper are taken in parts from the Habilitationsschrift of Maier-Paape [**6**].

2. Maximality of $\mathbf{B}_n$. In this section we will prove maximality of $\mathbf{B}_n$ as a closed subgroup of $\mathbf{O}(n)$ for $n = 3$ or $n \geq 5$. The case $n = 4$ is different and will be handled in Section 3. Before we can give the theorem on the maximality, we need a couple of auxiliary lemmas. The first one deals with the adjoint representation of $\mathbf{O}(n)$ on its Lie algebra $\mathbf{o}(n) = \{A \in \mathbf{R}^{n\times n} \mid A^T = -A\}$. For the orthogonal groups this means acting by conjugation (cf. Bröcker and Tom Dieck [**1**, Chap. I, (2.10)])

$$\begin{aligned} \mathbf{O}(n) \times \mathbf{o}(n) &\longrightarrow \mathbf{o}(n) \\ (A, B) &\longmapsto ABA^{-1}. \end{aligned}$$

Of course, with $\mathbf{O}(n)$, any subgroup of $\mathbf{O}(n)$ is acting on $\mathbf{o}(n)$ as well, by restriction of the above representation. For some subgroups this action turns out to be irreducible. Note that we always consider $n \geq 3$.

Lemma 2.1. *The adjoint action of $\mathbf{B}_n$ on $\mathbf{o}(n)$, i.e., $\mathbf{B}_n \times \mathbf{o}(n) \to \mathbf{o}(n)$, $(A, B) \mapsto ABA^{-1}$, is irreducible.*

Proof. We will show that for any $B \in \mathbf{o}(n)\backslash\{0\}$ fixed, one obtains $\operatorname{Span}\{\mathbf{B}_n B\} = \mathbf{o}(n)$. Then, clearly, there are no nontrivial $\mathbf{B}_n$-invariant subspaces of $\mathbf{o}(n)$.

We set $B_{ij} \in \mathbf{R}^{n\times n}$, $1 \leq i < j \leq n$, the matrix with (i,j)th and (j,i)th entry $\beta_{ij} = -\beta_{ji} = 1$. All other entries are zero. Then the set $\{B_{ij} \mid 1 \leq i < j \leq n\} \subset \mathbf{R}^{n\times n}$ forms the standard basis of $\mathbf{o}(n)$.

It is not difficult to see that, for any fixed $B \in \mathbf{o}(n)\backslash\{0\}$ there is at least one index pair (i,j) such that $B_{ij} \in \operatorname{Span}\{\mathbf{B}_n B\}$ (add matrices of the form CBC^{-1}, $C \in \mathcal{H}_n$ to B in order to obtain more and more zero columns and rows). Hence, it suffices to show $\operatorname{Span}\{\mathbf{B}_n B_{ij}\} = \mathbf{o}(n)$ for any $1 \leq i < j \leq n$. Using the permutation matrices of S_n applied to B_{ij}, this is obvious. □

Our next goal is to calculate the normalizers $N_{\mathbf{O}(n)}(\mathbf{B}_n) = \{\gamma \in \mathbf{O}(n) \mid \gamma \mathbf{B}_n \gamma^{-1} = \mathbf{B}_n\}$. However, before we get there, we need an auxiliary lemma on some normal subgroup of $\mathbf{B}_n$.

Lemma 2.2. *For $n = 3$ or $n \geq 5$, $\mathcal{H}_n$ is the only normal subgroup of $\mathbf{B}_n$ with order 2^n.*

Proof. In order to find a contradiction, assume there were a normal subgroup $X \lhd \mathbf{B}_n$, $X \neq \mathcal{H}_n$ and $|X| = 2^n$. Then, of course, $X\mathcal{H}_n \gneqq \mathcal{H}_n$ is a normal 2-subgroup of $\mathbf{B}_n$ as well: $\mathcal{H}_n \neq X\mathcal{H}_n \lhd \mathbf{B}_n$. Furthermore, $X\mathcal{H}_n \cap S_n$ is normal in S_n and therefore $\{\mathbf{1}\} \neq X\mathcal{H}_n \cap S_n \lhd S_n$. This follows from the fact that for any $\gamma \in \mathbf{B}_n \backslash \mathcal{H}_n$, and in particular for any element $\gamma \in X \backslash \mathcal{H}_n$, there exists an element $\xi \in \mathcal{H}_n$ such that $\gamma\xi \in S_n \backslash \{1\}$. The normal subgroups of S_n, however, are very well known.

Since $n = 3$ or $n \geq 5$, the normal subgroups of S_n are S_n, A_n (the alternating group) or $\{\mathbf{1}\}$. Hence either S_n or A_n must be a subgroup of the 2-group $X\mathcal{H}_n$. But this is impossible since the order of A_n or S_n is divisible by three for $n \geq 3$. $\square$

We note that for $n = 4$ there is an additional normal subgroup of S_n with four elements (Klein's 4-group V_4), making this kind of argument impossible. With this lemma, we can now calculate the normalizer of $\mathbf{B}_n$.

Theorem 2.3. *For $n = 3$ or $n \geq 5$, we have $N_{\mathbf{O}(n)}(\mathbf{B}_n) = \mathbf{B}_n$.*

Proof. We just have to show $N := N_{\mathbf{O}(n)}(\mathbf{B}_n) \leqslant \mathbf{B}_n$. Since $\mathcal{H}_n \lhd \mathbf{B}_n$, for any given $\gamma \in N$ also $\gamma\mathcal{H}_n\gamma^{-1} \lhd \mathbf{B}_n$. The order of $\gamma\mathcal{H}_n\gamma^{-1}$ is again 2^n, so due to uniqueness of the normal subgroups with this order, we derive

$$\gamma\mathcal{H}_n\gamma^{-1} = \mathcal{H}_n \quad \text{for any} \quad \gamma \in N. \tag{2.1}$$

The set of one-dimensional coordinate subspaces $\mathcal{U} := \{U \mid \dim U = 1, U = \operatorname{Span}\{e_i\}\}$ will now turn out to be crucial. Note that $\mathcal{H}_n$ lets all $U \in \mathcal{U}$ be invariant and no other one-dimensional subspaces. In other words we have:

(i) $\xi U = U$ for all $\xi \in \mathcal{H}_n$ and $U \in \mathcal{U}$.

(ii) If $\xi V = V$ for all $\xi \in \mathcal{H}_n$ and $\dim V = 1$, then $V \in \mathcal{U}$.

Now fix some $\gamma \in N$ and $U \in \mathcal{U}$. Then, due to (2.1), γU is invariant under $\mathcal{H}_n$ as well. Hence $\gamma U \in \mathcal{U}$. But $U \in \mathcal{U}$ was arbitrary and therefore γ is just permuting the coordinate subspaces. This already determines γ fully and we conclude $\gamma \in \mathbf{B}_n$. $\square$

We need one more auxiliary lemma concerning finite reflection groups, i.e., finite subgroups of $\mathbf{O}(n)$ which are generated by finitely many reflections at hyperplanes in $\mathbf{R}^n$.

Lemma 2.4. $\mathbf{B}_n \leqslant \mathbf{O}(n)$ *is for* $n = 3$ *or* $n \geq 5$ *a maximal finite reflection group.*

Proof. Since the action of the finite reflection group $\mathbf{B}_n$ on $\mathbf{R}^n$ is irreducible, the action of any supergroup of $\mathbf{B}_n$ is irreducible as well. But all (irreducible) finite reflection groups are very well known and characterized (cf., e.g., Humphreys [**4**, Chap. I, Sect. 2]). Now all possible irreducible finite reflection groups are excluded to be supergroups of $\mathbf{B}_n$ by order considerations. A list of their orders is given again in [**4**, Chap. I, Subsect. 2.11].

For any subgroup $\Sigma \leqslant \mathbf{O}(n)$ we introduce the set

$$\mathcal{T}_\Sigma := \{\gamma \in \Sigma \mid \gamma \text{ is a reflection at a hyperplane}\}$$

and denote by $\langle \mathcal{T}_\Sigma \rangle$ the subgroup of Σ which is generated by the reflections in Σ. Assume $r \in \mathcal{T}_\Sigma$ is a reflection at a hyperplane $M \subset \mathbf{R}^n$. Then for arbitrary $\gamma \in \Sigma$ also $\gamma r \gamma^{-1} \in \Sigma$ is a reflection (now at the hyperplane γM). Hence, $\gamma r \gamma^{-1} \in \mathcal{T}_\Sigma$. Iterating this argument a finite number of times gives:

Lemma 2.5. *If* $\Sigma \leqslant \mathbf{O}(n)$ *is finite, then* $\langle \mathcal{T}_\Sigma \rangle \triangleleft \Sigma$.

Now we have all preliminaries at hand to show the main theorem of this section.

Theorem 2.6. *For* $n = 3$ *or* $n \geq 5$, $\mathbf{B}_n$ *is a maximal closed subgroup of* $\mathbf{O}(n)$.

Proof. Assume some closed subgroup Γ of $\mathbf{O}(n)$ with

$$\mathbf{B}_n \leqslant \Gamma \leqslant \mathbf{O}(n) \quad \text{but} \quad \Gamma \neq \mathbf{O}(n), \tag{2.2}$$

is given. Our goal is to show $\Gamma = \mathbf{B}_n$. Of course, Γ is a Lie group, since it is a closed subgroup of the Lie group $\mathbf{O}(n)$. Therefore Γ has a Lie algebra $\mathfrak{g} \subset \mathbf{o}(n)$ and Γ acts on $\mathfrak{g}$ through the adjoint representation (cf. Bröcker and Tom Dieck [**1**, Chap. I])

$$\begin{aligned} \Gamma \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (A, B) &\longmapsto ABA^{-1}. \end{aligned}$$

Since $\mathbf{B}_n \leqslant \Gamma$, also $\mathbf{B}_n$ acts on $\mathfrak{g}$ by restriction. Therefore, $\mathfrak{g} \subset \mathbf{o}(n)$ is a $\mathbf{B}_n$-invariant subspace of $\mathbf{o}(n)$. But, by Lemma 2.1, $\mathbf{B}_n$ acts irreducible on $\mathbf{o}(n)$ yielding that $\mathfrak{g}$ is a trivial subspace of $\mathbf{o}(n)$, i.e., $\mathfrak{g} = \{0\}$ or $\mathfrak{g} = \mathbf{o}(n)$. The latter is only possible for $\Gamma = \mathbf{SO}\,(n)$ or $\Gamma = \mathbf{O}(n)$ which is excluded due to (2.2). Hence $\mathfrak{g} = \{0\}$ and the compactness of Γ gives even that Γ is finite.

Using the notation introduced above we have $\langle \mathcal{T}_{\mathbf{B}_n} \rangle = \mathbf{B}_n$, since $\mathbf{B}_n$ is a finite reflection group. Finiteness of Γ makes $\langle \mathcal{T}_\Gamma \rangle$ a finite reflection group. But since $\mathcal{T}_{\mathbf{B}_n} \subset \mathcal{T}_\Gamma$ we find $\mathbf{B}_n = \langle \mathcal{T}_{\mathbf{B}_n} \rangle \leqslant \langle \mathcal{T}_\Gamma \rangle$. Using the maximality of $\mathbf{B}_n$ as a finite reflection group, by Lemma 2.4 for $n = 3$ or $n \geq 5$ this is only possible in case $\mathbf{B}_n = \langle \mathcal{T}_\Gamma \rangle$.

Using Lemma 2.5 for Γ we conclude $\Gamma = N_\Gamma(\langle \mathcal{T}_\Gamma \rangle) = N_\Gamma(\mathbf{B}_n) \leqslant N_{\mathbf{O}(n)}(\mathbf{B}_n)$. On the other hand, due to Theorem 2.3 $N_{\mathbf{O}(n)}(\mathbf{B}_n) = \mathbf{B}_n$ yielding $\Gamma \leqslant \mathbf{B}_n$ and therefore by assumption (2.2) even $\Gamma = \mathbf{B}_n$. □

3. Dimension four. Dimension four is different, $\mathbf{B}_4$ cannot be a maximal closed subgroup of $\mathbf{O}(4)$, since it has even a finite reflection supergroup $\mathbf{F}_4$. For a definition of the finite reflection group $\mathbf{F}_4$ (the Weyl group of type $\mathbf{F}_4$) confer again Humphreys [**4**, Chap. I, Sec. 2.10]. Other possible irreducible finite reflection groups are again excluded to be supergroups of $\mathbf{B}_4$ by order consideration. We note that $|\mathbf{F}_4| : |\mathbf{B}_4| = 3$ and we may generate $\mathbf{F}_4 = \langle \mathbf{B}_4, \gamma_f \rangle$ with

$$\gamma_f := \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Before we can make similar arguments as we did in the proof of Theorem 2.6, we have to determine $N_{\mathbf{O}(4)}(\mathbf{B}_4)$ and $N_{\mathbf{O}(4)}(\mathbf{F}_4)$. To do that, we need information on the subgroups of $\mathbf{B}_4$ and $\mathbf{F}_4$. We denote $\mathcal{H}_4^e := \{\gamma \in \mathcal{H}_4 \mid \det \gamma = 1\}$. Hence, $\gamma \in \mathcal{H}_4^e$ are diagonal matrices with an even amount of -1's on the diagonal. Define furthermore $Z_4 := \mathcal{H}_4^e \rtimes V_4$, where again $V_4 = \{\mathbf{1}, (12)(34), (13)(24), (14)(23)\} \leqslant S_4 \leqslant \mathbf{B}_4$ is Klein's 4-group.

According to Humphreys [**4**, pp. 42–45], the Weyl group $\mathbf{D}_4$ of type D_4 in $\mathbf{B}_4$ satisfies

$$\mathbf{D}_4 = \mathcal{H}_4^e \rtimes S_4 \quad \text{and therefore} \quad Z_4 \triangleleft \mathbf{D}_4.$$

We also find

$$\mathbf{F}_4 = \mathbf{D}_4 \rtimes \tilde{S}_3 \quad \text{in particular} \quad \mathbf{D}_4 \triangleleft \mathbf{F}_4,$$

for some permutation group $\tilde{S}_3$ which is *not* a subgroup of $\mathbf{B}_4$.

Lemma 3.1. *We have $Z_4 \triangleleft \mathbf{F}_4$ and it is the only normal subgroup of $\mathbf{F}_4$ with 32 elements.*

Proof. Firstly, Z_4 is a characteristic subgroup of $\mathbf{D}_4$, since it is the unique normal subgroup of $\mathbf{D}_4$ with order $|Z_4|$. The last statement is a consequence of the fact that $\mathbf{D}_4/Z_4 \cong S_3$ which has no power two normal subgroup, yielding that Z_4 is the largest normal 2-subgroup in $\mathbf{D}_4$.

Now $Z_4 \triangleleft \mathbf{F}_4$ follows immediately from $\mathbf{D}_4 \triangleleft \mathbf{F}_4$. Uniqueness of Z_4 as a normal subgroup of $\mathbf{F}_4$ with this order follows from the argument just given above once we have established $\mathbf{F}_4/Z_4 \cong S_3 \times S_3$, since this again has no normal 2-subgroup.

Letting $Y := \mathbf{F}_4/Z_4$ and $U := \mathbf{D}_4/Z_4$, we have

$$U \triangleleft Y, \quad U \cong S_3 \quad \text{and} \quad Y/U \cong \mathbf{F}_4/\mathbf{D}_4 \cong S_3.$$

Introducing $C_Y(U) := \{c \in Y \mid cu = uc \text{ for all } u \in U\}$, one easily shows that $Y = U \cdot C_Y(U)$. But $U \triangleleft Y$ implies $C_Y(U) \triangleleft Y$. Since also $U \cap C_Y(U) = \{\mathbf{1}\}$ we find $Y = U \times C_Y(U)$ and the claim is established using $U \cong S_3$ and $C_Y(U) \cong Y/U \cong S_3$. □

We introduce the roots of $\mathbf{F}_4$ (cf. again [**4**, Chap. I, Sect. 2.10]). In the case of $\mathbf{F}_4$ there are 24 roots of length 1 and 24 of length $\sqrt{2}$:

$$\pm e_i, \quad 1 \le i \le 4, \quad (\pm e_1 \pm e_2 \pm e_3 \pm e_4)/2,$$
$$\pm e_i \pm e_j, \quad 1 \le i < j \le 4.$$

Let $\mathcal{E}_4$ denote the set of those 48 root vectors. We define

$$\mathcal{U}_4 := \{U \subset \mathbf{R}^4 \mid \dim U = 1, U = \operatorname{Span}\{v\} \text{ for some } v \in \mathcal{E}_4\} = \mathbf{R}\mathcal{E}_4,$$

which is a set of finitely many one-dimensional subspaces of $\mathbf{R}^4$. Note that the elements in $\mathbf{F}_4$ permute these one-dimensional subspaces of $\mathcal{U}_4$, in fact all elements in the normalizer $N_{\mathbf{O}(4)}(\mathbf{F}_4)$ act as a permutation on $\mathcal{U}_4$, since for any finite reflection group G with root system $\mathcal{R}$ the normalizer of G in $\mathbf{O}(n)$ is equal to $\{\gamma \in \mathbf{O}(n) \mid \gamma\mathcal{R} \subset \mathbf{R}\mathcal{R}\}$. Consider now

$$\gamma_x := \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix},$$

and observe $\gamma_x \notin \mathbf{F}_4$, but γ_x preserves the roots of $\mathbf{F}_4$ up to scalars and $\gamma_x^2 \in \mathbf{F}_4$. Therefore,

$$X_4 := \langle \mathbf{F}_4, \gamma_x \rangle \leqslant N_{\mathbf{O}(4)}(\mathbf{F}_4)$$

yields a supergroup of $\mathbf{F}_4$ with $|X_4| = 2 \cdot |\mathbf{F}_4|$. In particular, even $\mathbf{F}_4$ cannot be a maximal closed subgroup of $\mathbf{O}(n)$.

Theorem 3.2. $N_{\mathbf{O}(4)}(\mathbf{F}_4) = X_4$.

Proof. With $\mathbf{F}_4$ also Z_4 acts on the set $\mathcal{U}_4$. Consider Z_4-orbits $\mathcal{O}_{Z_4}(U)$, with $U \in \mathcal{U}_4$. For instance, $\mathcal{O}_{Z_4}(\operatorname{Span}\{e_1\})$ contains $\operatorname{Span}\{e_i\}$, $i = 1, \dots, 4$. Similarly, we find that the sets $\mathcal{O}_{Z_4}(\operatorname{Span}\{e_1 + e_2\})$, $\mathcal{O}_{Z_4}(\operatorname{Span}\{e_1 + e_3\})$, $\mathcal{O}_{Z_4}(\operatorname{Span}\{e_1 + e_4\})$, $\mathcal{O}_{Z_4}(\operatorname{Span}\{(-e_1 + e_2 + e_3 + e_4)/2\})$ and $\mathcal{O}_{Z_4}(\operatorname{Span}\{(e_1 + e_2 + e_3 + e_4)/2\})$ always contain four one-dimensional subspaces and are pairwise disjoint.

For simplicity we write $\mathcal{O}_i$, $i = 1, \dots, 6$, for the above occurring orbits (in that order). Consider now $\gamma \in N := N_{\mathbf{O}(4)}(\mathbf{F}_4)$. Then $\gamma Z_4 \gamma^{-1}$ is like Z_4 normal in $\mathbf{F}_4$. But due to Lemma 3.1 this implies

$$\gamma Z_4 \gamma^{-1} = Z_4 \quad \text{for all} \quad \gamma \in N. \tag{3.1}$$

We next claim that $\gamma\mathcal{O}_1 \in \{\mathcal{O}_i, i = 1, \dots, 6\}$ for all $\gamma \in N$, where $\mathcal{O}_1$ was the set of coordinate subspaces. But, obviously from (3.1), we have that $\gamma\mathcal{O}_1 = \gamma Z_4(\mathrm{Span}\,\{e_1\}) = Z_4(\mathrm{Span}\,\{\gamma e_1\}) = Z_4(\mathrm{Span}\,\{r\})$ for some $r = r_\gamma \in \mathcal{E}_4$. Hence $\gamma\mathcal{O}_1 \in \{\mathcal{O}_i, i = 1, \dots, 6\}$.

Observe now that for any $\mathcal{O}_i$, $i = 1, \dots, 6$, there is some $\xi_i \in X_4$, not unique, such that $\mathcal{O}_i = \xi_i\mathcal{O}_1$. Now for fixed $\gamma \in N$ there is some $i_0 \in \{1, \dots, 6\}$ with $\gamma\mathcal{O}_1 = \mathcal{O}_{i_0} = \xi_{i_0}\mathcal{O}_1$ and therefore $(\xi_{i_0}^{-1}\gamma)\mathcal{O}_1 = \mathcal{O}_1$. This means $\xi_{i_0}^{-1}\gamma$ permutes coordinate subspaces which is only possible for $\xi_{i_0}^{-1}\gamma \in \mathbf{B}_4$. Hence $\gamma \in \xi_{i_0}\mathbf{B}_4 \subset X_4$. This proves $N \subset X_4$ and therefore indeed $N = N_{\mathbf{O}(4)}(\mathbf{F}_4) = X_4$. □

The other normalizer of interest is the one of $\mathbf{B}_4$. Our approach will be quite similar. The relevant power two normal subgroup is $\tilde{Z}_4 := \mathcal{H}_4 \rtimes V_4$.

Lemma 3.3. *We have $\tilde{Z}_4 \triangleleft \mathbf{B}_4$, and it is the unique normal subgroup of $\mathbf{B}_4$ with 2^6 elements.*

Proof. Note that $\mathbf{B}_4/\tilde{Z}_4 \cong S_3$ which has no power two normal subgroup. Again $\tilde{Z}_4$ is the largest power two normal subgroup of $\mathbf{B}_4$ and therefore unique. □

We also need the root system of $\mathbf{B}_4$ (cf. [**4**, Chap. I, Sect. 2.10]). These roots are

$$\pm e_i, \quad 1 \le i \le 4 \quad \text{and} \quad \pm e_i \pm e_j, \quad 1 \le i < j \le 4.$$

The set of these 32 vectors is similarly denoted by $\tilde{\mathcal{E}}_4$ and

$$\tilde{\mathcal{U}}_4 := \{U \subset \mathbf{R}^4 \mid \dim U = 1, U = \mathrm{Span}\,\{v\} \text{ for some } v \in \tilde{\mathcal{E}}_4\} = \mathbf{R}\tilde{\mathcal{E}}_4$$

is now the relevant set of finitely many one-dimensional subspaces of $\mathbf{R}^4$.

Theorem 3.4. $N_{\mathbf{O}(4)}(\mathbf{B}_4) = \mathbf{B}_4$.

Proof. Our proceeding is very similar to the proof of Theorem 3.2. We therefore skip arguments whenever possible. For any $U \in \tilde{\mathcal{U}}_4$ consider the $\tilde{Z}_4$-orbits $\mathcal{O}_{\tilde{Z}_4}(U)$. $\tilde{\mathcal{U}}_4$ decomposes into four $\tilde{Z}_4$-orbits given by $\mathcal{O}_i$, $i = 1, \dots, 4$, introduced in the proof of Theorem 3.2. Using Lemma 3.3 we obtain similarly as in (3.1)

$$\gamma \tilde{Z}_4 \gamma^{-1} = \tilde{Z}_4 \quad \text{for all} \quad \gamma \in N_{\mathbf{O}(4)}(\mathbf{B}_4)$$

giving $\gamma \mathcal{O}_1 \in \{\mathcal{O}_i \mid i = 1, \dots, 4\}$ for all $\gamma \in N_{\mathbf{O}(4)}(\mathbf{B}_4)$ with the same arguments. But the rest of the proof works unchanged giving $N_{\mathbf{O}(4)}(\mathbf{B}_4) \leqslant X_4$. One can now easily check that neither X_4 nor $\mathbf{F}_4$ normalize $\mathbf{B}_4$ and the only possibility left is $N_{\mathbf{O}(4)}(\mathbf{B}_4) = \mathbf{B}_4$. $\square$

We are now ready to state the main theorem on maximality in dimension four.

Theorem 3.5. *The groups $\mathbf{F}_4$ and X_4 are the only nontrivial closed supergroups of $\mathbf{B}_4$ in $\mathbf{O}(4)$. In particular, X_4 is a maximal closed subgroup of $\mathbf{O}(4)$.*

Proof. Our proof is along the lines of the proof of Theorem 2.6. Assume some closed subgroup Γ of $\mathbf{O}(4)$ with

$$\mathbf{B}_4 \leqslant \Gamma \leqslant \mathbf{O}(4) \quad \text{but} \quad \Gamma \neq \mathbf{F}_4, X_4, \mathbf{O}(4),$$

is given. Again our goal is to show $\Gamma = \mathbf{B}_4$. With the same arguments as before we conclude that Γ is finite. Thus $\langle \mathcal{T}_\Gamma \rangle \geqslant \mathbf{B}_4$ is again a finite reflection group with only two possibilities, either

$$\langle \mathcal{T}_\Gamma \rangle = \mathbf{B}_4 \quad \text{or} \quad \langle \mathcal{T}_\Gamma \rangle = \mathbf{F}_4.$$

Again $\langle \mathcal{T}_\Gamma \rangle$ is normal in Γ by Lemma 2.5. Together with Theorems 3.2 and 3.4 we therefore conclude

$$\Gamma = N_\Gamma(\langle \mathcal{T}_\Gamma \rangle) = \begin{cases} N_\Gamma(\mathbf{B}_4) \leqslant N_{\mathbf{O}(4)}(\mathbf{B}_4) = \mathbf{B}_4 \\ N_\Gamma(\mathbf{F}_4) \leqslant N_{\mathbf{O}(4)}(\mathbf{F}_4) = X_4. \end{cases}$$

In any case $\mathbf{B}_4 \leqslant \Gamma \leqslant X_4$, but since $\Gamma \neq \mathbf{F}_4, X_4$, we conclude $\Gamma = \mathbf{B}_4$. $\square$

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