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LOCAL-GLOBAL RELATIONS FOR ALMOST COMPLETELY DECOMPOSABLE GROUPS

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1. Introduction. All groups in this paper are tacitly assumed to be abelian. An almost completely decomposable group is a torsion-free group of finite rank that contains a completely decomposable group as a subgroup of finite index. Let X be such a group and A a completely decomposable subgroup of finite index. The finite group X/A has a direct decomposition into primary components, and it is natural to consider the subgroups $X^A_{\wr p}$ where

$$\frac{X^A_{\wr p}}{A} = \quad p\text{-component of} \quad \frac{X}{A}.$$

Obviously, X contains many other completely decomposable subgroups of finite index besides A, for example, the groups nA, and there is nothing canonical about the constructs $X^A_{\wr p}$. However, there are special completely decomposable subgroups of finite index, namely the *regulating subgroups* of Lady [**9**] which can be defined as the completely decomposable subgroups of minimal index $\operatorname{rgi}(X)$ in X, and more importantly their intersection, Burkhardt's [**2**] *regulator* $R(X)$ of X. The regulator is a fully invariant completely decomposable subgroup of finite index of X and clearly both $R(X)$ and $X/R(X)$ are isomorphism invariants of the almost completely decomposable group X. Thus, every almost completely decomposable group contains a canonical completely decomposable subgroup of finite index and the *p-constituents* $X_{\wr p} = X^{R(X)}_{\wr p}$ relative to this subgroup are also canonical subgroups. The purpose of this paper is to study the relationships between an almost completely decomposable group X and its primary constituents $X_{\wr p}$ in a systematic and general manner. The primes dividing the index $[X : R(X)]$ were called *relevant primes* of X by Schultz [**15**]. There is ample evidence [**5, 12, 13**] that the theory of almost completely decomposable groups is much simpler when there is

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a single relevant prime; this is what we mean by the *local* case. One of the basic results of this paper is that $R(X_{lp}) = R(X)$, Theorem 2.3, so that considering primary constituents means passing to the local case. Suppose certain questions could be settled for the local groups X_{lp}. Then the question becomes what they mean for the *global* (general) group X. This is what we mean by *local-global relations*. We consider several established and important concepts of the theory of almost completely decomposable groups and clarify their local-global relationships. This program is remarkably successful, and the connection between the global group and its local constituents is quite close. See, for example, Proposition 2.5, Theorem 3.3, Corollary 3.5, Proposition 4.1, Theorem 4.2 and Theorem 5.3.

Special local-global results have appeared in the literature, [**4**, Theorem 0.1], so that a general treatment appears timely.

The symbol $\mathbf{P}$ denotes the set of all prime numbers. The expression H_*^G denotes the purification of H in G.

We will also use the following important alternate description of regulating subgroups. The subgroups $G(\tau)$ and $G^\sharp(\tau) = (\sum_{\rho>\tau} G(\rho))_*$ denote the usual type subgroups of the torsion-free group G. A type τ is a *critical type* of G if $G(\tau)/G^\sharp(\tau) \neq 0$; $T_{\mathrm{cr}}(G)$ denotes the set of all critical types of G. If X is an almost completely decomposable group, then X has *Butler decompositions*

$$X(\tau) = A_\tau \oplus X^\sharp(\tau)$$

where A_τ is τ-homogeneous completely decomposable, and the regulating subgroups of X are exactly the groups $A = \oplus_{\rho\in T_{\mathrm{cr}}(X)} A_\rho$ obtained by selecting arbitrary *Butler complements* A_τ, the sum being automatically direct. Burkhardt's theorem on regulators says that, given any regulating subgroup $A = \oplus_{\rho\in T_{\mathrm{cr}}(X)} A_\rho$, the regulator is

$$R(X) = \bigoplus_{\rho\in T_{\mathrm{cr}}(X)} \beta_\rho^X A_\rho$$

where

$$\beta_\tau^X = \exp \frac{X^\sharp(\tau)}{R(X^\sharp(\tau))}$$

are positive integers that we will call *Burkhardt invariants*. Burkhardt's *Regulator Criterion* says that a completely decomposable subgroup U of finite index in the almost completely decomposable group X is the regulator of X if U has a decomposition into homogeneous components

$$U = \bigoplus_{\rho \in T_{\mathrm{cr}}(X)} U_\rho \quad \text{s.t.} \quad U_\tau \subset \beta_\tau X(\tau) \subset U(\tau)$$

for each critical type τ, where $\beta_\tau = \exp(X^\sharp(\tau)/U^\sharp(\tau))$.

2. Primary constituents. We start with the basic definitions.

Definition 2.1. Let X be a torsion-free finite extension of a group A, and let H be any subgroup of X. The subgroup

$$H^A_{\wr p} = \{x \in H : p^n x \in A \text{ for some } n\} = H \cap \mathbf{Z}[p^{-1}]A$$

is the *p-primary constituent* of H with respect to A. For a torsion group G, let G_p denote its p-primary component. For a positive integer n, $n_{\wr p}$ denotes its highest p-power factor so that $n = \prod_{p\in\mathbf{P}} n_{\wr p}$ is the prime factorization of n.

Some easy technicalities are listed in the next lemma.

Lemma 2.2. *Let X be a torsion-free finite extension of a group A and H some subgroup of X. Then the following hold.*

1. $H^A_{\wr p} = X^A_{\wr p} \cap H$, $H = \sum_{p\in\mathbf{P}} H^A_{\wr p}$ *and* $(H^A_{\wr p} + A) \cap (\sum_{q:q\neq p} H^A_{\wr q} + A) = A$.

2.
$$\frac{H^A_{\wr p} + A}{A} = \left(\frac{H+A}{A}\right)_p,$$
$$\left|\frac{H^A_{\wr p} + A}{A}\right| = \left|\frac{H+A}{A}\right|_{\wr p},$$
$$\exp\frac{H^A_{\wr p} + A}{A} = \left(\exp\frac{H+A}{A}\right)_{\wr p}.$$

3. *If* $X \geq H \geq A$, *then*

$$\frac{H^A_{\wr p}}{A} = \left(\frac{H}{A}\right)_p \quad \text{and} \quad \frac{H}{A} = \bigoplus_{p \in \mathbf{P}} \left(\frac{H}{A}\right)_p = \bigoplus_{p \in \mathbf{P}} \frac{H^A_{\wr p}}{A}.$$

4. *If* $X \geq H \geq A$, *then* $H^A_{\wr p} \cap \sum_{q \neq p} H^A_{\wr q} = A$.

5. $X^A_{\wr p}(\tau) = X^A_{\wr p} \cap X(\tau) = X(\tau)^A_{\wr p}$ *and* $(X^A_{\wr p})^\sharp(\tau) = X^A_{\wr p} \cap X^\sharp(\tau) = X^\sharp(\tau)^A_{\wr p}$.

Furthermore,

$$\frac{X(\tau)^A_{\wr p} + A}{A} \cong \frac{X^A_{\wr p}(\tau)}{A(\tau)} \quad \text{and} \quad \frac{X^\sharp(\tau)^A_{\wr p} + A}{A} \cong \frac{(X^A_{\wr p})^\sharp(\tau)}{A^\sharp(\tau)}.$$

Proof. Easy exercise. □

This paper deals with almost completely decomposable groups X and in this case there is the canonical choice $A = R(X)$. For this choice we will drop the reference to A and simply talk about the p-constituent of the almost completely decomposable group considered. We can now describe the local-global relationships of relevant data for the special case.

Theorem 2.3. *Let* X *be an almost completely decomposable group,* p *a prime, and* $A = \oplus_{\rho \in T_{\mathrm{cr}}(X)} A_\rho$ *a regulating subgroup of* X. *Then the following hold.*

1. $R(X) = R(X_{\wr p})$.
2. $\beta^{X_{\wr p}}_\tau = (\beta^X_\tau)_{\wr p}$.
3. $\mathrm{rgi}\, X_{\wr p} = (\mathrm{rgi}\, X)_{\wr p}$.
4. $(A_\tau)_{\wr p} = (\prod_{q: q \neq p} (\beta^X_\tau)_{\wr q}) A_\tau$, $A_{\wr p} = \oplus_{\rho \in T_{\mathrm{cr}}(X)} (A_\rho)_{\wr p}$ *completely decomposable, and* $A_{\wr p}$ *is a regulating subgroup of* $X_{\wr p}$.

Proof. 1. and 2. Set $U = R(X)$ and recall that $\beta^X_\tau = \exp(X^\sharp(\tau)/U^\sharp(\tau))$. By Burkhardt's Regulator Criterion, there is a decomposition

$U = \oplus_{\rho \in T_{\mathrm{cr}}(X)} U_\rho$ such that

$$U_\tau \subset \beta_\tau^X X(\tau) \subset U(\tau).$$

Note that

$$\exp \frac{(X_{\wr p})^\sharp(\tau)}{U^\sharp(\tau)} = \exp \frac{X^\sharp(\tau)_{\wr p}}{U^\sharp(\tau)} = (\beta_\tau^X)_{\wr p}$$

and

$$(\beta_\tau^X)_{\wr p} X_{\wr p}(\tau) = (\beta_\tau^X)_{\wr p} X(\tau)_{\wr p} \subset U(\tau).$$

Now

$$\begin{aligned} U_\tau \subset \beta_\tau^X X(\tau) &= (\beta_\tau^X)_{\wr p} \Bigg(\prod_{q:q\neq p} (\beta_\tau^X)_{\wr p} \Bigg) X(\tau) \\ &\subset (\beta_\tau^X)_{\wr p} X_{\wr p}(\tau) \subset U(\tau). \end{aligned}$$

By Burkhardt's Regulator Criterion, $U = R(X_{\wr p})$ and $\beta_\tau^{X_{\wr p}} = (\beta_\tau^X)_{\wr p}$.

3. We have with $r_\tau = \mathrm{rk}\,(X(\tau)/X^\sharp(\tau))$, that

$$[X_{\wr p} : R(X)] = (\mathrm{rgi}\, X_{\wr p}) \cdot \prod_{\rho \in T_{\mathrm{cr}}(X_{\wr p})} (\beta_\rho^{X_{\wr p}})^{r_\rho}$$

and

$$[X_{\wr p} : R(X)] = [X : R(X)]_{\wr p} = (\mathrm{rgi}\, X)_{\wr p} \cdot \prod_{\rho \in T_{\mathrm{cr}}(X)} (\beta_\rho^X)_{\wr p}^{r_\rho}.$$

It follows that $\mathrm{rgi}\,(X_{\wr p}) = (\mathrm{rgi}\, X)_{\wr p}$.

4. By Burkhardt's theorem $R(X) = \oplus_{\rho \in T_{\mathrm{cr}}(X)} \beta_\rho^X A_\rho \leq A_{\wr p}$. Obviously $(A_\tau)_{\wr p} \supset (\prod_{q:q\neq p} (\beta_\tau^X)_{\wr q}) A_\tau$. Conversely, suppose $x \in (A_\tau)_{\wr p}$ so that $p^n x = \beta_\tau^X y$ for some $y \in A_\tau$. Then $x \in (A_\tau)_*^X = A_\tau$ and $p^n x = (\beta_\tau^X)_{\wr p} (\prod_{q:q\neq p} (\beta_\tau^X)_{\wr q}) y$. It follows that $(\prod_{q:q\neq p} (\beta_\tau^X)_{\wr q})$ divides x in A_τ, i.e., $x \in (\prod_{q:q\neq p} (\beta_\tau^X)_{\wr q}) A_\tau$. Hence $A_{\wr p} = \oplus_{\rho \in T_{\mathrm{cr}}(X)} (\prod_{q:q\neq p} (\beta_\tau^X)_{\wr q}) A_\rho$ is completely decomposable and $[X_{\wr p} : A_{\wr p}] = \mathrm{rgi}\, X_{\wr p}$, so $A_{\wr p}$ is regulating in $X_{\wr p}$. $\square$

Corollary 2.4. *The almost completely decomposable group X has a regulating regulator if and only if each local constituent $X_{\wr p}$ has a regulating regulator.*

It is easy to compare the endomorphism rings of the groups $X_{\wr p}$ and the group X. This has consequences for decomposability of X. If X is an almost completely decomposable group with regulator $R = R(X)$, then, without loss of generality, $R \leq X_{\wr p} \leq X \leq \mathbf{Q}R$ where $\mathbf{Q}R$ is some divisible hull of R. Hence, every endomorphism of R, X, $X_{\wr p}$ may, and will, be considered a linear transformation of the $\mathbf{Q}$-space $\mathbf{Q}R$. Let $L = L(\mathbf{Q}R)$ denote the ring of linear transformations of $\mathbf{Q}R$. Then our point of view means that

$$\begin{aligned} \operatorname{End} R &= \{\phi \in L : R\phi \subset R\}, \\ \operatorname{End} X &= \{\phi \in L : X\phi \subset X\}, \quad \text{and} \\ \operatorname{End} X_{\wr p} &= \{\phi \in L : X_{\wr p}\phi \subset X_{\wr p}\}. \end{aligned}$$

With these identifications we have the following result.

Proposition 2.5. *Let X be an almost completely decomposable group with regulator $R = R(X)$. Then R and $X_{\wr p}$ are fully invariant in X, R is fully invariant in $X_{\wr p}$ and we have the containments*

$$\operatorname{End} X \subset \operatorname{End} X_{\wr p} \subset \operatorname{End} R.$$

Furthermore,

$$\operatorname{End} X = \bigcap_{p \in \mathbf{P}} \operatorname{End} X_{\wr p}.$$

If any one of the primary constituents is directly indecomposable, then so is X.

Proof. It is well known that regulators are fully invariant, and since $X_{\wr p}$ consists exactly of those elements in X that have p-power order modulo R, the latter is also fully invariant in X. The inclusions now follow from the descriptions of the endomorphism rings given above. Recall that $X = \sum_{p \in \mathbf{P}} X_{\wr p}$. If $\phi \in \cap_{p \in \mathbf{P}} \operatorname{End} X_{\wr p}$, then

$$X\phi \subset \sum_{p \in \mathbf{P}} X_{\wr p}\phi \subset \sum_{p \in \mathbf{P}} X_{\wr p} = X,$$

hence $\phi \in \operatorname{End} X$. $\square$

Recall that H_*^G denotes the purification of H in G.

Corollary 2.6. *Let X be an almost completely decomposable group with regulator R. A decomposition $R = A \oplus B$ induces a decomposition $X = A_*^X \oplus B_*^X$ if and only if, for each prime p, there is the induced decomposition $X_{\natural p} = A_*^{X_{\natural p}} \oplus B_*^{X_{\natural p}}$.*

3. Localization of the Regulator Criterion. The purpose of this section is to show that the Regulator Criterion of Burkhardt can be localized, i.e., if the subgroup R of the almost completely decomposable group X is the regulator of all p-primary constituents $X_{\natural p}$, then R is the regulator of X.

Lemma 3.1. *Let G be a torsion-free abelian group with subgroup U that is a Butler group. Let τ be a type and k, l relatively prime natural numbers. If*

$$U(\tau) = U^k \oplus U^\sharp(\tau) = U^l \oplus U^\sharp(\tau)$$

where $U^k \subset kG(\tau)$ and $U^l \subset lG(\tau)$, then there exists a subgroup U^{kl} of U such that

$$U(\tau) = U^{kl} \oplus U^\sharp(\tau) \quad \text{with} \quad U^{kl} \subset klG(\tau).$$

Proof. By hypothesis,

$$\begin{aligned} U(\tau) &\subset [U^\sharp(\tau) + kG(\tau)] \cap [U^\sharp(\tau) + lG(\tau)] \\ &\subset U^\sharp(\tau) + kG(\tau) \cap [U^\sharp(\tau) + lG(\tau)] \quad \text{(modular law)} \\ &\subset U^\sharp(\tau) + kG(\tau) \cap [kU^\sharp(\tau) + lU^\sharp(\tau) + lG(\tau)] \quad (ak + bl = 1) \\ &\subset U^\sharp(\tau) + kU^\sharp(\tau) + kG(\tau) \cap lG(\tau) \\ &\qquad\qquad\qquad \text{(modular law and } U^\sharp(\tau) \subset G(\tau)) \\ &\subset U^\sharp(\tau) + klG(\tau) \quad (k, l \text{ relatively prime}). \end{aligned}$$

Intersecting with $U(\tau)$, we obtain

$$U(\tau) \subset U^\sharp(\tau) + [klG(\tau) \cap U(\tau)].$$

Using Butler decomposition, it follows that $U(\tau) = U^\sharp(\tau) \oplus U^{kl}$ with $U^{kl} \subset klG(\tau)$. Indeed, the induced epimorphism $klG(\tau) \cap U(\tau) \to U(\tau)/U^\sharp(\tau)$ is splitting by Baer's lemma. $\square$

An immediate consequence of Lemma 3.1 is the following corollary.

Corollary 3.2. *Let G be a torsion-free abelian group with subgroup U that is a Butler group. Let τ be a type and h a natural number with primary decomposition $h = \prod_{p|h} h_{\wr p}$. Let $U(\tau) = U^p \oplus U^\sharp(\tau)$ where the U^p are maximal τ-homogeneous summands with $U^p \subset h_{\wr p}G(\tau)$ for all primes p dividing h. Then there is a maximal τ-homogeneous summand U^h of U, i.e., $U(\tau) = U^h \oplus U^\sharp(\tau)$ such that $U^h \subset hG(\tau)$.*

Theorem 3.3. *Let X be an almost completely decomposable group with completely decomposable subgroup R of finite index. If R is the regulator of $X_{\wr p} = \mathbf{Z}[p^{-1}]R \cap X$ for every prime p, then R is the regulator of X.*

Proof. Let τ be a fixed type. By the Regulator Criterion of Burkhardt, for each prime p we have a decomposition $R = \oplus_{\sigma \in T_{\mathrm{cr}}(R)} R_{p,\sigma}$ such that

$$R_{p,\tau} \subset \beta_\tau^{X_{\wr p}} X_{\wr p}(\tau) \subset R_{p,\tau}(\tau),$$

where

$$\beta_\tau^{X_{\wr p}} = \exp(X_{\wr p}^\sharp(\tau)/R^\sharp(\tau))$$

is the Burkhardt invariant and a power of p. Moreover, the $R_{p\tau}$ are maximal τ-homogeneous summands of R for each prime p. Since, by Theorem 2.3, $\beta_\tau^X = \prod_{p \in \mathbf{P}} \beta_\tau^{X_{\wr p}} = \exp(X^\sharp(\tau)/R^\sharp(\tau))$, we obtain by Corollary 3.2 the existence of a maximal τ-homogeneous summand R_τ of R such that $R_\tau \subset \beta_\tau^X X(\tau) \subset R(\tau)$. But the type τ was arbitrary, hence the Regulator Criterion holds for the decomposition $R = \oplus_{p \in T_{\mathrm{cr}}(R)} R_\rho$, and R is the regulator of X. $\square$

We apply the local-global principle to a fundamental question for almost completely decomposable groups.

Question 3.4. *Let R be a completely decomposable group of finite rank and G a finite abelian group. Describe the class $C(R,G)$ of almost completely decomposable groups X such that $R = R(X)$ and $X/R \cong G$. In particular, determine when $C(R,G)$ is nonvoid.*

Note that $C(R,G)$ may well be void. For example, there is no proper torsion-free extension of a p-divisible group R by a p-group G. The following corollary of Theorem 3.3 reduces the question to the local case.

Corollary 3.5. *Let R be a completely decomposable group of finite rank, S a finite set of prime numbers and G_p, $p \in s$, a set of finite p-groups. If, for each $p \in S$, there exists a group Y_p such that $R \le Y_p \le \mathbf{Q}R$, $Y_p/R \cong G_p$, Y_p is almost completely decomposable, and $R = R(Y_p)$, then $X = \sum_{p\in S} Y_p$ is an almost completely decomposable group such that $R(X) = R$, $X/R \cong \oplus_{p\in S} G_p$ and $X_{\wr p} = Y_p$.*

Proof. Let $X = \sum_{p\in S} Y_p$. Then

$$\frac{X}{R} = \bigoplus_{p\in S} \frac{Y_p}{R}$$

by consideration of orders. Hence, $X_{\wr p} = Y_p$ and $R(X) = R$ by Theorem 3.3. $\square$

We consider the local case and first derive a necessary condition for $C(R,G)$ to be nonempty. We will write $p\tau = \tau$ to say that the type τ is p-divisible, i.e., infinite at p.

Proposition 3.6. *Let X be an almost completely decomposable group. Suppose that X has a regulating regulator R such that X/R is p-primary. Set*

$$r_\tau = \operatorname{rk}\left(\frac{R(\tau)}{R^\sharp(\tau)}\right).$$

Then

$$\dim\left(\frac{X}{R}[p]\right) \le \operatorname{rk} R - \max_{\tau\in T_{\mathrm{cr}}(R)} \left\{ \sum \{r_\rho : \rho \ge \tau, \rho \ne p\rho\}\right\} - \sum_{\rho:\rho=p\rho} r_\rho.$$

Proof. Let $R = \oplus_{\rho \in T_{\mathrm{cr}}(X)} R_\rho$ be a decomposition of R into homogeneous components, and let D_p be the maximal p-divisible subgroup of X. It is easy to see that $D_p = \oplus_{\rho = p\rho} R_\rho$ is pure in X and

$$X = D_p \oplus Y \quad \text{where} \quad Y = \Big(\bigoplus_{\rho : \rho \neq p\rho} R_\rho \Big)_*^X .$$

Since R is regulating in X, we have $X(\tau) \subset R$ for all critical types τ. Note that $D_p + X(\tau) = D_p + D_p(\tau) + Y(\tau) = D_p \oplus Y(\tau)$ is pure in X. Thus

$$\frac{X}{R} \cong \frac{X/(D_p \oplus Y(\tau))}{R/(D_p \oplus Y(\tau))}$$

is a homomorphic image of the torsion-free abelian group $X/(D_p \oplus Y(\tau))$ of rank $\mathrm{rk}\,(X) - \sum_{\rho : \rho = p\rho} \mathrm{rk}\,(R_\rho) - \sum_{\rho \geq \tau, \rho \neq p\rho} \mathrm{rk}\,(R_\rho) = \mathrm{rk}\, X - \sum_{\rho : \rho = p\rho} r_\rho - \sum_{\rho \geq \tau, \rho \neq p\rho} r_\rho$. The claimed inequality follows by considering p-independent sets in X/R and $X/(D_p \oplus Y(\tau))$ [**6**, p. 144].

We will show that the above necessary condition is also sufficient if the critical typeset is assumed to be an antichain. We first assume that R is *rigid*, meaning that $T_{\mathrm{cr}}(R)$ is an antichain and $\mathrm{rk}\,(R(\tau)/R^\sharp(\tau)) = 1$ for each critical type τ.

Lemma 3.7. *Let p be a prime. Let $R = \oplus_{i=1}^n R_i$ be a rigid p-reduced completely decomposable group with critical typeset $T = \{\mathrm{tp}\,(R_1), \dots , \mathrm{tp}\,(R_n)\}$.*

If X is an almost completely decomposable group with regulator R, then the p-rank of the regulator quotient X/R is less than or equal to $n-1$.

Conversely, given a finite p-group G of rank less than or equal to $n-1$, there is an almost completely decomposable group X such that R is the regulator of X and $X/R \cong G$.

Proof. The necessity of the inequality was shown in Proposition 3.6. For the sufficiency we construct an almost completely decomposable group X with $R \subset X \subset \mathbf{Q}R$ such that $X/R \cong \oplus_{j=2}^n \mathbf{Z}(l_j)$, where the l_j, $j = 2, \dots , n$ are powers of p, possibly 1, and R is the regulator of X. Write $R = \oplus_{i=1}^n S_i a_i$, where $1 \in S_i \subset \mathbf{Z}_p$. In particular, $\mathrm{ht}_p(a_i) = 0$

for all i. The desired group X will be

$$X = R + \sum_{j=2}^{n} \mathbf{Z} g_j, \quad \text{where} \quad g_j = (l_j)^{-1}(a_{j-1} + a_j),\ j = 2, \dots, n.$$

It is routine to check that $S_i a_i$ is pure in X.

First we determine that the quotient X/R is isomorphic to $\oplus_{j=2}^{n} \mathbf{Z}(l_j)$. It is plain that $\oplus_{j=2}^{n} \mathbf{Z}(l_j)$ maps onto X/R since the generators $\bar{g}_j = g_j + R$ of X/R have orders l_j.

Suppose that $\sum_{j=2}^{n} h_j \bar{g}_j = 0$ with $h_j \in \mathbf{Z}$. It is enough to show that $l_j \mid h_j$, in $\mathbf{Z}$, for each j. Since

$$\sum_{j=2}^{n} h_j g_j = \frac{h_2}{l_2} a_1 + \left(\frac{h_2}{l_2} + \frac{h_3}{l_3} \right) a_2 + \dots + \left(\frac{h_j}{l_j} + \frac{h_{j+1}}{l_{j+1}} \right) a_j + \cdots + \frac{h_n}{l_n} a_n \in R \subset \bigoplus_{i=1}^{n} \mathbf{Z}_p a_i,$$

we see that (h_2/l_2), $(h_n/l_n) \in \mathbf{Z}_p$; whence $(h_3/l_3) \in \mathbf{Z}_p$, and inductively $(h_j/l_j) \in \mathbf{Z}_p$, therefore $\mathbf{Z}$, for all j.

Secondly we show that R is the regulator of X. For this it is enough to prove that $X(\tau) \subset R(\tau)$ for all critical types τ. Let $a \in X(\tau)$. There are $h_j \in \mathbf{Z}$ and $s_j \in \mathbf{Q}$ such that $a = \sum_{j=2}^{n} h_j g_j + \sum_{j=1}^{n} s_j a_j \in R$. Then

$$\begin{aligned} a &= \sum_{j=2}^{n} \left(\frac{h_j}{l_j} \right) (a_{j-1} + a_j) + \sum_{j=1}^{n} s_j a_j \\ &= \left(\frac{h_2}{l_2} + s_1 \right) a_1 + \left(\frac{h_2}{l_2} + \frac{h_3}{l_3} + s_2 \right) a_2 + \dots + \left(\frac{h_n}{l_n} + s_n \right) a_n. \end{aligned}$$

Since $a \in X(\tau)$ and T is an antichain, at most one a_j can have a nonzero coefficient. A modification of the inductive argument above shows that $h_j/l_j \in \mathbf{Z}$ for all j, so that $a \in X(\tau) \cap R = R(\tau)$ as desired. □

The following block-rigid case generalizes a result of Kozhukhov [**7**, Corollary 1].

Proposition 3.8. *Let p be a prime and R a completely decomposable group of finite rank. Suppose that $T_{\mathrm{cr}}(R)$ is an antichain. Let $r_\tau = \mathrm{rk}\,(R(\tau)/R^\sharp(\tau))$.*

If X is an almost completely decomposable group with regulator R, then the regulator quotient X/R is restricted by the inequality

$$\dim\left(\frac{X}{R}[p]\right) \le \mathrm{rk}\,R - \max\{r_\tau : \tau \in T_{\mathrm{cr}}(R), \tau \ne p\tau\} - \sum_{\rho:\rho=p\rho} r_\rho.$$

Conversely, let G be a finite p-group G satisfying

$$\dim\left(G[p]\right) \le \mathrm{rk}\,R - \max\{r_\tau : \tau \in T_{\mathrm{cr}}(R), \tau \ne p\tau\} - \sum_{\rho:\rho=p\rho} r_\rho,$$

then there exists an almost completely decomposable group X such that $R(X) = R$ and $X/R \cong G$.

Proof. The necessity of the inequality was shown in Proposition 3.6. For the sufficiency we assume without loss of generality that R is p-reduced, since any p-divisible summand of R may be set aside and added on after the construction is complete.

The basic idea of the proof is to decompose R into rigid summands of largest possible ranks, to decompose G correspondingly, and to apply Lemma 3.7.

Let $\tau_1, \tau_2, \dots, \tau_t$ be a listing of the critical typeset $T_{\mathrm{cr}}(R)$ such that $r_1 \le r_2 \le \cdots \le r_t$ where $r_i = r_{\tau_i}$. Let $R = \oplus_{i=1}^t R_i$ be a homogeneous decomposition of R such that R_i is homogeneous of type τ_i, and let $R_i = \oplus_{j=1}^{r_i} S_{ij}$ be a decomposition of each R_i into rank-one summands. Groups of the form $\oplus_j S_{ij}$ are rigid summands of R that appear in the diagram below as horizontal slabs, the widest ones being obtained for $i = 1$, and r_1 such slabs can be formed. The process continues in this fashion and is formally described below.

Setting $r_0 = 0$, for $r_{k-1} < j \le r_k$, let $W_j = \oplus_{i=k}^t S_{ij}$. Note that

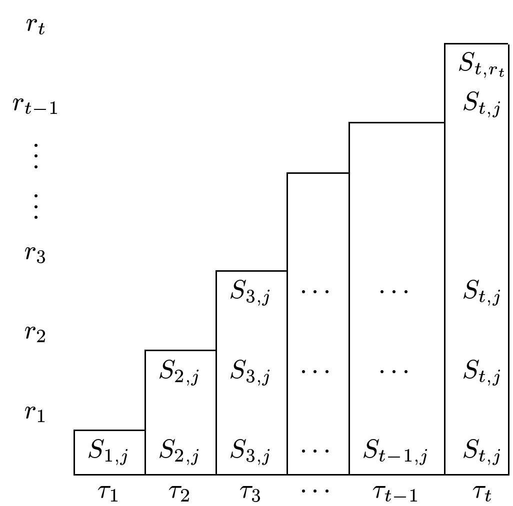
$$t = \mathrm{rk}\,W_1 = \cdots = \mathrm{rk}\,W_{r_1} > \mathrm{rk}\,W_{r_1+1} = \cdots = \mathrm{rk}\,W_{r_2} > \cdots.$$

Furthermore, if $W = \oplus_{j=r_{t-1}+1}^{r_t} S_{tj}$

$$R = \left(\bigoplus_{j=1}^{r_1} W_j\right) \oplus \left(\bigoplus_{j=r_1+1}^{r_2} W_j\right) \oplus \cdots \oplus \left(\bigoplus_{j=r_{t-2}+1}^{r_{t-1}} W_j\right) \oplus W.$$

Note that some of these sums may be empty, namely if $r_i = r_{i+1}$. Each W_j is a rigid completely decomposable group. Let us recall that there are r_1 groups $W_1 \cong \cdots \cong W_{r_1}$ of rank t. There are $r_2 - r_1$ groups W_j of rank $t-1$, etc., and there are $r_{t-1} - r_{t-2}$ groups of rank 2. Observe that the ranks $\operatorname{rk}(W_j) - 1$ add precisely to $\operatorname{rk} R - r_t$. Let G be a finite p-group of rank less than or equal to $\operatorname{rk} R - r_t$. Hence there is a decomposition $G = \oplus_j G_j$ of G such that $\operatorname{rk} G_j \leq \operatorname{rk} W_j - 1$ for all j. By Lemma 3.7, there is an almost completely decomposable group X_j with regulator W_j and regulator quotient $X_j/W_j \cong G_j$ for each j. Let $X = W \oplus \bigoplus_j X_j$. Then $X/R \cong G$ and, since $X(\tau) = W(\tau) \oplus (\bigoplus_j X_j(\tau)) = W(\tau) \oplus (\bigoplus_j W_j(\tau)) = R(\tau)$, we proved that R is the regulating regulator of X. $\square$

We will not formulate the obvious extension to the global case.



4. Regulating subgroups of primary constituents. By Theorem 2.3 every regulating subgroup A of X produces a family $\{A_{\wr p} : p \in \mathbf{P}\}$ of regulating subgroups of the p-constituents $X_{\wr p}$, and $A = \sum_{p \in \mathbf{P}} A_{\wr p}$. Theorem 4.2 below asserts that the correspondence $A \mapsto \{A_{\wr p} : p \in \mathbf{P}\}$ is bijective. For the proof we will use a formula counting the number of different regulating subgroups in an almost completely decomposable group. This result is due to Mutzbauer [**14**, Theorem 5.2]. We first apply it to an almost completely decomposable group and its primary constituents. The symbol $\operatorname{Regg} X$ denotes the set of all regulating subgroups of X.

Proposition 4.1. *Let X be an almost completely decomposable group. Then*

$$|\operatorname{Regg} X_{\wr p}| = \prod_{\rho \in T_{\mathrm{cr}}(X)} (\operatorname{rgi} X^\sharp(\rho))_{\wr p}^{r_\rho(X)}$$

where $r_\rho = r_\rho(X) = \operatorname{rk}(X(\rho)/X^\sharp(\rho))$. Furthermore,

$$|\operatorname{Regg} X| = \prod_{p \in \mathbf{P}} |\operatorname{Regg} X_{\wr p}|.$$

Proof. The Mutzbauer formula applied to $X_{\wr p}$ is

$$|\operatorname{Regg} X_{\wr p}| = \prod_{\rho \in T_{\mathrm{cr}}(X_{\wr p})} (\operatorname{rgi}(X_{\wr p}^\sharp(\rho)))^{r_\rho(X_{\wr p})}.$$

Now use that $T_{\mathrm{cr}}(X_{\wr p}) = T_{\mathrm{cr}}(X)$, $r_\rho(X_{\wr p}) = r_\rho(X)$, and $\operatorname{rgi}(X_{\wr p}^\sharp(\rho)) = \operatorname{rgi}(X^\sharp(\rho))_{\wr p}$ by Theorem 2.3. The second formula follows from the first since every integer equals the product of its primary constituents. $\square$

We are now ready for the main result of this section.

Theorem 4.2. *Let X be an almost completely decomposable group. If $A \in \operatorname{Regg} X$, then $A_{\wr p}$ is a regulating subgroup of the constituent $X_{\wr p}$. Conversely, if for each prime p the group $A(p)$ is a regulating subgroup of $X_{\wr p}$, then $A = \sum_{p \in \mathbf{P}} A(p)$ is a regulating subgroup of X and $A_{\wr p} = A(p)$.*

Proof. As mentioned above, we have the assignment

$$\operatorname{Regg} X \longrightarrow \prod_{p \in \mathbf{P}} \operatorname{Regg}(X_{\wr p}) : A \longmapsto [\dots, A_{\wr p}, \dots]_{p \in \mathbf{P}}$$

which is injective since $A = \sum_{p\in\mathbf{P}} A_{\wr p}$ by Lemma 2.2. By Proposition 4.1 the sets $\operatorname{Regg} X$ and $\prod_{p\in\mathbf{P}} \operatorname{Regg}(X_{\wr p})$ have the same finite cardinalities, and hence the assignment is bijective. $\square$

These results enable us to generalize local results on regulating quotients to the global case. A *regulating quotient* of an almost completely decomposable group X is any group isomorphic to a quotient X/A where A is some regulating subgroup of X. All regulating quotients must have the same order $\operatorname{rgi} X$, but Lady [**9**] showed that nonisomorphic regulating quotients are possible. Burkhardt [**2**, Corollary 2], proved the striking result that a *p-local almost completely decomposable group X that possesses a cyclic regulating quotient and a p-elementary regulating quotient has every group of order* $\operatorname{rgi} X$ *as a regulating quotient.*

We will extend this result to the global case.

Proposition 4.3. *Let X be an almost completely decomposable group that contains regulating subgroups A_1 and A_2 such that X/A_1 is cyclic and X/A_2 is elementary. Then, given any group G of order* $\operatorname{rgi} X$*, there is a regulating subgroup A of X such that $X/A \cong G$.*

Proof. Since

$$\frac{X_{\wr p}}{(A_i)_{\wr p}} = \frac{X_{\wr p}}{A_i \cap X_{\wr p}} \cong \frac{X_{\wr p} + A_i}{A_i} \leq \frac{X}{A_i}$$

for $i = 1, 2$, it follows that each primary constituent $X_{\wr p}$ has a cyclic regulating quotient, modulo $(A_1)_{\wr p}$, and a p-elementary regulating quotient, modulo $(A_2)\wr p$. Let G be any group of order $\operatorname{rgi} X$, and let $G = \bigoplus_{p\in\mathbf{P}} G_p$ be its primary decomposition. Then $|G_p| = \operatorname{rgi}(X)_{\wr p}$. By Burkhardt's theorem there exist regulating subgroups B_p of $X_{\wr p}$ such that $X_{\wr p}/B_p \cong G_p$. By Theorem 4.2, $A = \sum_{p\in\mathbf{P}} B_p$ is a regulating subgroup of X such that $B_p = A_{\wr p}$. We now have

$$\frac{X}{R} = \bigoplus_{p\in\mathbf{P}} \frac{X_{\wr p}}{R}, \quad \frac{A}{R} = \bigoplus_{p\in\mathbf{P}} \frac{A_{\wr p}}{R}, \quad B_p = A_{\wr p} \leq X_{\wr p}.$$

Hence,

$$\frac{X}{A} \cong \frac{X/R}{A/R} \cong \bigoplus_{p\in\mathbf{P}} \frac{X_{\wr p}/R}{A_{\wr p}/R} \cong \bigoplus_{p\in\mathbf{P}} \frac{X_{\wr p}}{A_{\wr p}} \cong \bigoplus_{p\in\mathbf{P}} G_p = G. \qquad \square$$

In [**12**] the possible regulating quotients of a p-local group with a cyclic regulating quotient were completely determined, extending Burkhardt's results. It is clear that the description of the possible regulating quotients can now also be extended to the global case. We will not do so here, since the local results are already fairly lengthy to formulate, and the global results would be more so without giving new insights.

5. Localizations in anti-representations. The results in this section have appeared in similar form in [**11**]. We rephrase and reprove them in a form that is closer to standard approaches and to our preceding discussions.

Let X be an almost completely decomposable group and A a completely decomposable subgroup of finite index in X. Let e be a positive integer such that $eX \subset A$. Let

$$- : A \longrightarrow A/eA$$

denote the natural epimorphism. So, in particular, $\bar{A} = A/eA$ and $\bar{a} = a + eA$. By the *anti-representation* of X with respect to A, we mean

$$\operatorname{Rep}(X, A, e) = \{\bar{A}; \overline{A(\tau)}, \tau \in T_{\mathrm{cr}}(A), \overline{eX}\}.$$

This object may be viewed as a finite abelian group $\bar{A}$ with distinguished subgroups $\overline{A(\tau)}$, $\tau \in T_{\mathrm{cr}}(A)$ and $\overline{eX}$. It may also be viewed as an anti-representation of the finite poset $T_{\mathrm{cr}}(A) \cup \{\infty\}$ over the ring $\mathbf{Z}/e\mathbf{Z}$ with $\tau \mapsto \overline{A(\tau)}$ for $\tau \in T_{\mathrm{cr}}(A)$ and $\infty \mapsto \overline{eX}$. This is an anti-representation since $\overline{A(\tau)} \subset \overline{A(\sigma)}$ if $\sigma \leq \tau$. The element ∞ is assumed to be incomparable with any of the types τ, so that nothing is stipulated about the position of $\overline{eX}$ relative to the "type subgroups" $\overline{A(\tau)}$.

The anti-representation is particularly interesting when $A = R(X)$. Given $\operatorname{Rep}(X, A, e)$, it can be seen within the anti-representation itself whether $A = R(X)$. In fact, the following translation of Burkhardt's Regulator Criterion appears in [**11**] and in [**8**].

Lemma 5.1. *Let X be an almost completely decomposable group, A a completely decomposable subgroup of finite index and e an integer such that $eX \subset A$. Then $A = R(X)$ if and only if, for each critical type τ,*

(i) $\beta_\tau = \exp(\overline{A(\tau)} \cap \overline{eX}) = \exp(\overline{A^\sharp(\tau)} \cap \overline{eX})$, *and*

(ii) *there exist decompositions* $\overline{A(\tau)} = U_\tau \oplus \overline{A^\sharp(\tau)}$ *such that* $(e/\beta_\tau)U_\tau \subset \overline{eX}$.

Suppose that two almost completely decomposable groups X and Y are given having the common regulator $A = R(X) = R(Y)$. Assume further that e is an integer such that $eX, eY \subset A$. There is a natural equivalence relation between such groups X, Y, called *type-isomorphism*, namely, representation isomorphism of the anti-representations $\operatorname{Rep}(X, A, e)$ and $\operatorname{Rep}(Y, A, e)$. More precisely, let

$$\operatorname{TypAut}(\bar{A}) = \{\xi \in \operatorname{Aut}\bar{A} : \forall\, \tau \in T_{\mathrm{cr}}(A), \overline{A(\tau)}\xi \subset \overline{A(\tau)}\}.$$

The groups X and Y are *type-isomorphic* if there exists $\xi \in \operatorname{TypAut}\bar{A}$ such that $\overline{eX}\xi = \overline{eY}$. It was shown in [**11**] that the groups X and Y are type-isomorphic if and only if they are near-isomorphic. Near-isomorphism is the weakening of isomorphism due to Lady [**9, 10**]. Our goal is to show, again, that the Regulator Criterion holds globally if and only if it holds locally for each prime, and that two almost completely decomposable groups are globally nearly isomorphic if and only if they are locally nearly isomorphic for each prime.

We begin by examining the anti-representation of the primary constituents and their relationship to the anti-representation of the original group.

Lemma 5.2. *Let X be an almost completely decomposable group and suppose that* $\operatorname{Rep}(X, A, e) = \{\bar{A}; \overline{A(\tau)}, \tau \in T_{\mathrm{cr}}(X), \overline{eX}\}$ *is an anti-representation of X. Let X_{lp} be the p-constituent of X with respect to A. Then*

$$\operatorname{Rep}(X_{lp}, A, e) = \{\bar{A}; \overline{A(\tau)}, \tau \in T_{\mathrm{cr}}(X), (\overline{eX})_p\}.$$

Proof. It is routine to verify that $\overline{eX_{lp}} = (\overline{eX})_p$. □

We are now prepared for the reduction theorem. It says that the local-global relations work perfectly with respect to near-isomorphism.

Theorem 5.3 (Primary Reduction Theorem). (i) *Let X be an almost completely decomposable group, A a completely decomposable subgroup and e a positive integer such that $eX \subset A$. Then $A = R(X)$ if and only if $A = R(X_{\wr p})$ for all primes p.*

(ii) *If X and Y are almost completely decomposable groups, then X is nearly isomorphic to Y if and only if $X_{\wr p}$ is nearly isomorphic to $Y_{\wr p}$ for all primes p.*

Proof. (i) For the first part of the Regulator Criterion (Lemma 5.1), observe that

$$\exp(\overline{A(\tau)} \cap \overline{eX}) = \exp(\overline{A^\sharp(\tau)} \cap \overline{eX}) \Longleftrightarrow$$
$$\forall\, p \in \mathbf{P},\ \exp(\overline{A(\tau)} \cap (\overline{eX})_p) = \exp(\overline{A^\sharp(\tau)} \cap (\overline{eX})_p).$$

This says that the first condition of the Regulator Criterion is satisfied globally if and only if it is satisfied locally.

Now suppose that $A = R(X)$. We then have the first condition of the Regulator Criterion both globally and locally, and in addition

$$\beta_\tau = \prod_{p \in \mathbf{P}} \beta_{\tau,p} \tag{5.4}$$

where

$$\beta_\tau = \exp(\overline{A(\tau)} \cap \overline{eX}) = \exp(\overline{A^\sharp(\tau)} \cap \overline{eX})$$

and

$$\beta_{\tau,p} = \exp(\overline{A(\tau)} \cap \overline{eX_{\wr p}}) = \exp(\overline{A^\sharp(\tau)} \cap (\overline{eX})_p).$$

By the second part of the Regulator Criterion, there exist decompositions $\overline{A(\tau)} = K_\tau \oplus \overline{A^\sharp(\tau)}$ such that $(e/\beta_\tau)K_\tau \subset \overline{eX}$. Then also $(e/\beta_{\tau,p})K_\tau \subset \overline{eX}$, so that the complements K_τ work for all primes p simultaneously.

Conversely, suppose that $A = R(X_{\wr p})$ for each prime p. Then the first condition of the Regulator Criterion is again satisfied locally and globally and, in addition, (5.4) holds. Since the second condition of the Regulator Criterion is satisfied locally, we have groups $K(\tau,p)$ such that $\overline{A(\tau)} = K(\tau,p) \oplus \overline{A^\sharp(\tau)}$ and $(e/\beta_{\tau,p})K(\tau,p) \subset \overline{eX}$. We now write the groups $K(\tau,p)$ as direct sums of their primary components

$$K(\tau,p) = \bigoplus_{q\in\mathbf{P}} K(\tau,p)_q.$$

Let $K_\tau = \bigoplus_{p\in\mathbf{P}} K(\tau,p)_p$. Then $\overline{A(\tau)} = K_\tau \oplus \overline{A^\sharp(\tau)}$ since this is true for each primary component of the finite abelian groups. In order to show that $(e/\beta_\tau)K_\tau \subset \overline{eX}$, it suffices to show this for each primary component $K(\tau,p)_p$. The latter is the case since

$$\begin{aligned}\frac{e}{\beta_\tau}K(\tau,p)_p &= \frac{e}{\beta_\tau}\Bigg(\prod_{q\neq p}\beta_{\tau,q}\Bigg)K(\tau,p)_p \\ &= \frac{e}{\beta_{\tau,p}}K(\tau,p)_p \subset \frac{e}{\beta_{\tau,p}}K(\tau,p) \subset \overline{eX}.\end{aligned}$$

(ii) Suppose that X and Y are nearly isomorphic. Then $R(X) \cong R(Y)$ and $X/R(X) \cong Y/R(Y)$ [**11**]. We therefore assume without loss of generality that X and Y have a common regulator A and consider the anti-representations $\operatorname{Rep}(X,A,e)$ and $\operatorname{Rep}(Y,A,e)$ where $e = \exp(X/A) = \exp(Y/A)$. Since X and Y are nearly isomorphic, there is a $\xi \in \operatorname{TypAut}\bar{A}$ such that $\overline{eX}\xi = \overline{eY}$. Then $(\overline{eX})_p\xi = (\overline{eY})_p$ also, and this shows that the localizations are type-isomorphic and thus nearly isomorphic.

Conversely assume that the p-localizations of X and Y are nearly isomorphic for each p. Then again we may assume that X and Y have a common regulator A and isomorphic regulator quotients X/A, Y/A, and hence they may be studied in their related anti-representations. By assumption there exist $\xi_p \in \operatorname{TypAut}\bar{A}$ such that $(\overline{eX})_p\xi_p = (\overline{eY})_p$. We must construct a type-isomorphism that works globally. To do this, factor e as $e = \prod_{p\in\mathbf{P}} e_{\wr p}$ and set $\breve{e}_{\wr p} = \prod_{q\neq p} e_{\wr q}$. The integers $\breve{e}_{\wr p}$ are relatively prime and there exist integers u_p such that

$$1 = \sum_{q\in\mathbf{P}} u_q\breve{e}_{\wr q}.$$

Let

$$\xi = \sum_{q\in\mathbf{P}} u_q\breve{e}_{\wr q}\xi_q.$$

Then $\xi \in \operatorname{End} \bar{A}$ and, $\bar{A}$ being finite, ξ is an automorphism if it is injective. To show the latter, let $x \in \operatorname{Ker} \xi$ and decompose x into primary components, $x = \sum_{q \in \mathbf{P}} x_q$, $x_q \in (\bar{A})_q$. Then $\sum_{q \in \mathbf{P}} u_q \breve{e}_{\wr q}(x\xi_q) = 0$ and by order considerations it follows that $u_p \breve{e}_{\wr p} x \xi_p = 0$ individually for each prime p. Since ξ_p is injective, we have $u_p \breve{e}_{\wr p} x = 0$, and finally $x = \sum_{q \in \mathbf{P}} u_q \breve{e}_{\wr q} x = 0$. To show that $\overline{eX}\xi \subset \overline{eY}$, it suffices to consider primary components. We find that

$$(\overline{eX})_p \xi \subset \sum_{q \in \mathbf{P}} u_q \breve{e}_{\wr q} (\overline{eX})_p \xi_q = u_p \breve{e}_{\wr p} (\overline{eX}) \xi_p \subset (\overline{eY})_p \subset \overline{eY}. \qquad \square$$

6. ACD groups with a cyclic regulating quotient. Almost completely decomposable groups X that contain a regulating subgroup A such that X/A is cyclic are said to have a *cyclic regulating quotient* and are called *crq-groups* for short. They were studied in the local case in [**12**] and in the global case in [**3**].

Suppose that $X = A + \mathbf{Z}n^{-1}a$ is a crq-group with regulating subgroup A and regulating index n. We will show first that each primary constituent $X_{\wr p}$ of X is again a crq-group and find the representation of $X_{\wr p}$ naturally associated with the representation $X = A + \mathbf{Z}n^{-1}a$. Recall that $\exp(X/A) = \exp(X/R(X))$ since X/A is cyclic, and hence $a = n(n^{-1})a \in R(X)$. Further, if $a = \sum_{\rho \in T_{\mathrm{cr}}(X)}, a_\rho$ is a representation of a with respect to some decomposition $A = \bigoplus_{\rho \in T_{\mathrm{cr}}(X)} A_\rho$, then even $a_\tau \in R(X)$.

Proposition 6.1. *Let $X = A + \mathbf{Z}n^{-1}a$ be a crq-group with completely decomposable subgroup $A = \bigoplus_{\rho \in T_{\mathrm{cr}}(A)} A_\rho$. Write $a = \sum_{\rho \in T_{\mathrm{cr}}(A)} a_\rho$ with $a_\rho \in A_\rho$. Then $X_{\wr p}$ is a crq-group with the representation $X_{\wr p} = A_{\wr p} + \mathbf{Z}n_{\wr p}^{-1}a$, $A_{\wr p} = \bigoplus_{\rho \in T_{\mathrm{cr}}(X)} (A_\rho)_{\wr p}$ where $(A_\rho)_{\wr p} = (\prod_{q \neq p} (\beta_\rho^X)_{\wr q}) A_\rho$, and $a = \sum_{\rho \in T_{\mathrm{cr}}(X)} a_\rho$ is the representation of $a \in R(X) \subset A_{\wr p}$ with respect to the decomposition $A_{\wr p} = \bigoplus_{\rho \in T_{\mathrm{cr}}(X)} (A_\rho)_{\wr p}$.*

Proof. Note that

$$\frac{X_{\wr p}}{A_{\wr p}} \cong \frac{X_{\wr p}}{A \cap X_{\wr p}} \cong \frac{A + X_{\wr p}}{A} \leq \frac{X}{A};$$

hence, $X_{\wr p}/A_{\wr p}$ is cyclic. Since $[X:(A+X_{\wr p})] = \operatorname{rgi} X/\operatorname{rgi} X_{\wr p} = n/n_{\wr p}$, it follows that $A + X_{\wr p} = A + \mathbf{Z}n_{\wr p}^{-1}a$. Thus, every $x \in X_{\wr p}$ can be written as $x = y + kn_{\wr p}^{-1}a$ for some $y \in A$ and $k \in \mathbf{Z}$. Moreover, $y = x - kn_{\wr p}^{-1}a \in A \cap X_{\wr p} = A_{\wr p}$. Hence, $X_{\wr p} = A_{\wr p} + \mathbf{Z}n_{\wr p}^{-1}a$. By Theorem 2.3 $A_{\wr p} = \bigoplus_{\rho\in T_{\mathrm{cr}}(X)}(A_\rho)_{\wr p}$ is a homogeneous decomposition of $A_{\wr p}$, and $a_\tau \in \beta_\tau^X A_\tau \subset (\prod_{q\neq p}(\beta_\tau^X)_{\wr q})A_\tau = (A_\tau)_{\wr p}$. $\square$

This result reduces questions on general crq-groups to questions on local crq-groups and the results of [**12**] might be applied. However, there is a problem. Recall that a group is *clipped* if it has no completely decomposable direct summand. The results in Mader-Vinsonhaler [**12**] mostly assume that $X_{\wr p}$ is clipped. To make this assumption for each p is rather restrictive while the assumption that X is clipped does not imply that the primary constituents $X_{\wr p}$ are clipped. We give a simple example.

Example 6.2. There exist clipped crq-groups all of whose nonzero primary constituents have rational direct summands.

Proof. Let

$$A = \mathbf{Z}[2^{-1}]a_1 \oplus \mathbf{Z}[3^{-1}]a_2 \oplus \mathbf{Z}[5^{-1}]a_3,$$

and

$$X = A + \mathbf{Z}\frac{1}{30}(a_1+a_2+a_3).$$

It is routine to check that A is the regulating regulator of X either ad hoc or by the general method provided in [**3**, Theorem 3.4]. Then

$$\begin{aligned} X_{\wr 2} = A + \mathbf{Z}(a_1+a_2+a_3)/2 &= \mathbf{Z}[2^{-1}]a_1 \\ &+ \left[(\mathbf{Z}[3^{-1}]a_2 \oplus \mathbf{Z}[5^{-1}]a_3) + \mathbf{Z}(a_2+a_3)/2\right] \end{aligned}$$

and similarly

$$X_{\wr 3} = \mathbf{Z}[3^{-1}]a_2 + \left[(\mathbf{Z}[2^{-1}]a_1 \oplus \mathbf{Z}[5^{-1}]a_3) + \mathbf{Z}(a_1+a_3)/3\right],$$

$$X_{\wr 5} = \mathbf{Z}[5^{-1}]a_3 + \left[(\mathbf{Z}[2^{-1}]a_1 \oplus \mathbf{Z}[3^{-1}]a_2) + \mathbf{Z}(a_1+a_2)/5\right]. \qquad \square$$

We would like to extract from a standard representation $X = A + \mathbf{Z}n^{-1}a$, $A = \sum_{\rho\in T_{\mathrm{cr}}(A)} A_\rho$ regulating in X, $a = \sum_{\rho\in T_{\mathrm{cr}}(A)} a_\rho$ with $a_\rho \in A_\rho$, the near-isomorphism class to which X belongs. This was achieved in the local case in [**12**, Theorem 3.2, Theorem 3.9, Corollary 3.13] as follows.

Lemma 6.3. *Let $X = A + \mathbf{Z}p^{-k}a$, A regulating in X, $a \in A$, be a p-local crq-group that is clipped and whose regulating index is p^k. Let $A = \bigoplus_{\rho\in T_{\mathrm{cr}}(A)} A_\rho$ be a homogeneous decomposition of A and write $a = \sum_{\rho\in T_{\mathrm{cr}}(A)} a_\rho$, $a_\rho \in A_\rho$. Then the greatest common divisors (computed in A) $\gcd^A(p^k, a_\tau)$ form a complete set of near-isomorphism invariants for X.*

Even if a global crq-group X is clipped, its p-constituents $X_{\wr p}$ need not be clipped as we have seen. This means that the greatest common divisors $\gcd^A(n, a_\tau)$ of Proposition 6.1 need not be near-isomorphism invariants of X. In the global case the greatest common divisors depend on the particular homogeneous decomposition of the regulating subgroup A. In [**3**] it was shown that a given decomposition of A can be changed in such a way that the values of the greatest common divisors are maximized, and when this is done we have arrived at a *top decomposition* of A and the values $\gcd^A(n, a_\tau)$ are the same for every top decomposition [**3**, Proposition 2.9]. We remark that the local equivalent of a top decomposition is the assumption that the group is decomposed into a direct sum of a completely decomposable group and a clipped group, and the regulating subgroup used is a direct sum of regulating subgroups of the two summands. The key is to work with a top decomposition.

Lemma 6.4. *Let $X = A + \mathbf{Z}n^{-1}a$ such that A is regulating in X and n is the regulating index. Suppose that $A = \bigoplus_{\rho\in T_{\mathrm{cr}}(A)} A_\rho$ is a top decomposition and $a = \sum_{\rho\in T_{\mathrm{cr}}(A)} a_\rho$, $a_\rho \in A_\rho$. Then the associated representation of the primary constituent $X_{\wr p}$ is obtained in the form*

$$\begin{aligned} X_{\wr p} &= \bigoplus\{(A_\rho)_{\wr p} : \gcd^A(n_{\wr p}, a_\rho) = n_{\wr p}\} \\ &\quad \oplus \left(\bigoplus\{(A_\rho)_{\wr p} : \gcd^A(n_{\wr p}, a_\rho) < n_{\wr p}\}\right)_*^{X_{\wr p}} \end{aligned}$$

where the first summand is completely decomposable and the second *summand is clipped. The greatest common divisors* $\mathrm{gcd}^A(n_{\wr p}, a_\tau) = \mathrm{gcd}^{A_{\wr p}}(n_{\wr p}, a_\tau)$ *are near-isomorphism invariants of* $X_{\wr p}$ *and the integers*

$$\mathrm{gcd}^A(n, a_\tau) = \prod_{p \in \mathbf{P}} \mathrm{gcd}^{A_{\wr p}}(n_{\wr p}, a_\tau)$$

are near-isomorphism invariants of X.

Proof. By Proposition 6.1 we have $X_{\wr p} = A_{\wr p} + \mathbf{Z}(n_{\wr p})^{-1}(\sum_{\rho \in T_{\mathrm{cr}}(A)} a_\rho)$ and the claimed representation of $X_{\wr p}$ is an immediate consequence. The fact that A is regulating in X and the fact that A is in top decomposition imply that $(\bigoplus\{(A_\rho)_{\wr p} : \mathrm{gcd}^A(n_{\wr p}, a_\rho) < n_{\wr p}\})_*^{X_{\wr p}}$ satisfies the conditions of [**12**, Theorem 3.2] and hence is clipped. $\square$

We now have the following classification theorem for global crq-groups.

Corollary 6.5. *Let* X *and* Y *be crq-groups given in the form* $X = A + \mathbf{Z}n^{-1}a$, $a \in A$, $Y = A + \mathbf{Z}n^{-1}b$, $b \in A$, *such that* A *is regulating in both groups* X *and* Y *and* $n = \mathrm{rgi}\,(X) = \mathrm{rgi}\,(Y)$. *Suppose that* $A = \bigoplus_{\rho \in T_{\mathrm{cr}}(A)} A_\rho$ *is a top decomposition of* A *for* X, $a = \sum_{\rho \in T_{\mathrm{cr}}(A)} a_\rho$ *with* $a_\rho \in A_\rho$, *and that* $A = \bigoplus_{\rho \in T_{\mathrm{cr}}(A)} B_\rho$ *is a top decomposition of* A *for* Y, $b = \sum_{\rho \in T_{\mathrm{cr}}(A)} b_\rho$ *with* $b_\rho \in B_\rho$. *Then* X *is nearly isomorphic with* Y *if and only if* $\mathrm{gcd}^A(n, a_\tau) = \mathrm{gcd}^A(n, b_\tau)$ *for each* $\tau \in T_{\mathrm{cr}}(X)$.

7. Direct decompositions. It was shown in [**5**] and independently in [**13**] that direct decompositions of local almost completely decomposable groups are unique up to near-isomorphism. This means that the well-known pathology of direct decompositions of almost completely decomposable groups enters in the process of globalization, and the following question shapes up as one of the most prominent in the theory of almost completely decomposable groups.

Question 7.1. *Find compatibility criteria assuring that local decompositions of an almost completely decomposable group can be lifted to a global decomposition.*

An answer to this question would explain the pathologies of direct decompositions of almost completely decomposable groups. In [**1**], the relatively simple case of block-rigid crq-groups was studied and the problem of finding the decompositions of such groups was reduced to a factorization problem of the regulating index satisfying certain constraints. The (near-)classification of block-rigid crq-groups by integral invariants was used prominently in the proof.

A more special question is the following.

Question 7.2. *Find indecomposability criteria for global almost completely decomposable groups.*

We do not have an answer in the crq case.

Question 7.3. *Find indecomposability criteria for global crq-groups.*

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