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THE LATTICE OF SEMILATTICE-MATRIX DECOMPOSITIONS OF A SEMIGROUP

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ABSTRACT. In this paper we investigate some general, lattice theoretical properties of semilattice-matrix decompositions of semigroups. We prove that the poset of all semilattice-matrix equivalences on an arbitrary semigroup is a complete lattice. For a fixed semilattice congruence σ on a semigroup S we prove that the set of all semilattice-matrix equivalences on S carried by σ is a complete sublattice of the lattice of equivalence relations on S, and that it is a direct product of the lattices of semilattice-left and semilattice-right equivalences on S carried by σ.

Semilattice-matrix decompositions of semigroups, including here semilattice-left and semilattice-right decompositions, form a very important type of decompositions studied in many papers. We can say that the first known example of such decompositions was the characterization of unions of groups (completely regular semigroups) as semilattices of completely simple semigroups, given by A.H. Clifford in [**7**], 1941, since by the well-known Rees-Sushkevich matrix representation theorem, completely simple semigroups can be characterized as matrices (rectangular bands) of groups, left zero bands of right groups and right zero bands of left groups. A similar property was proved by Bogdanović and Ćirić in [**5**] for left regular semigroups, which were characterized as semigroups having a semilattice-right decomposition whose components are left simple semigroups. Semilattice-right decompositions whose components are left Archimedean semigroups and nil-extensions of left groups were studied by Bogdanović and Ćirić in [**4**], 1995, and Shevrin in [**15**], 1994, respectively. Also, some other kinds of semilattice-matrix decompositions were studied by Chu, Guo and Ren in [**6**], 1989.

It is important to note that semilattice-matrix decompositions are more general than many other significant kinds of decompositions of semigroups. For example, it is evident that semilattice and matrix

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decompositions, including here left zero band and right zero band decompositions, are special cases of semilattice-matrix decompositions. More generally, all band decompositions are special cases of semilattice-matrix decompositions. This was noted by Clifford in [**8**], 1954, and it is an immediate consequence of the result of Clifford from the same paper and McLean from [**11**], 1954, by which bands are characterized as semilattices of rectangular bands. Furthermore, in many papers, semilattice-matrix decompositions have been used as the first step in band decompositions of semigroups.

In this paper we investigate some general, lattice theoretical properties of semilattice-matrix decompositions of semigroups. We prove that the poset of all semilattice-matrix equivalences on an arbitrary semigroup is a complete lattice. For a fixed semilattice congruence σ on a semigroup S we prove that the set of all semilattice-matrix equivalences on S carried by σ is a complete sublattice of the lattice of equivalence relations on S, and that it is a direct product of the lattices of semilattice-left and semilattice-right equivalences on S carried by σ.

A nonempty subset of K of a complete lattice L is called a *complete meet* (*join*) *subsemilattice* of L if it contains the meet (join) of any of its nonempty subsets. If K is both a complete meet-subsemilattice and complete joint-subsemilattice of L, then it is called a *complete sublattice* of L. For a set X, by $\mathcal{E}(X)$ we denote the lattice of all equivalence relations on X, and for a semigroup S, by $\mathrm{Con}\,(S)$ we denote the lattice of all congruence relations on S. For undefined notions and notations we refer to [**1, 2, 3, 9, 10**] and [**13**].

Let a semigroup S be a semilattice Y of semigroups S_α, $\alpha \in Y$, and for any $\alpha \in Y$, let S_α be a matrix (left zero band, right zero band) I_α of semigroups S_i, $i \in I_\alpha$. The partition of S whose components are semigroups S_i, $i \in I$, where $I = \cup_{\alpha\in Y} I_\alpha$, will be called a *semilattice matrix* (*semilattice-left, semilattice-right*) *decomposition* of S, or shortly an s-m (s-l, s-r)-decomposition of S. Let us denote by θ the corresponding equivalence relation on S, and by σ the semilattice congruence corresponding to the above considered semilattice decomposition of S. Then we will say that θ is *carried by σ as an s-m* (*s-l, s-r*)*-equivalence* and that σ is a *carrier* of θ. In other words, an equivalence relation θ on a semigroup S contained in a semilattice congruence σ on S is an s-m (s-l, s-r)-equivalence carried by σ if the restriction of θ to any σ-class T of S is a matrix (left zero band, right zero band) congruence on T.

For a semigroup S, $\mathcal{SM}(S)$, $\mathcal{SL}(S)$ and $\mathcal{SR}(S)$ will denote the sets of all semilattice-matrix, semilattice-left and semilattice-right equivalences on S, respectively, and $\mathcal{S}(S)$ will denote the set of all semilattice congruences on S. For a $\sigma \in \mathcal{S}(S)$, $\mathcal{SM}(\sigma)$, $\mathcal{SL}(\sigma)$ and $\mathcal{SR}(\sigma)$ will denote the sets of all semilattice-matrix, semilattice-left and semilattice-right equivalences on S carried by σ, respectively.

First we prove the following:

Theorem 1. *For any semigroup S, the posets $\mathcal{SM}(S)$, $\mathcal{SL}(S)$ and $\mathcal{SR}(S)$ are complete lattices.*

Proof. The theorem will be proved for $\mathcal{SM}(S)$.

Assume an arbitrary family $\{\mu_i\}_{i\in I}$ of semilattice-matrix equivalences on S. For $i \in I$, let σ_i be a carrier of μ_i, and let

$$\mu = \bigcap_{i\in I} \mu_i \quad \text{and} \quad \sigma = \bigcap_{i\in I} \sigma_i.$$

Let us prove that μ is a semilattice-matrix congruence on S carried by σ.

Clearly, $\sigma \in \mathcal{S}(S)$, $\mu \in \mathcal{E}(S)$ and $\mu \subseteq \sigma$. Let A be an arbitrary σ-class of S. Then

$$A = \bigcap_{i\in I} A_i,$$

where, for any $i \in I$, A_i is a σ_i-class of S. By the hypothesis, for any $i \in I$, the restriction of μ_i on A_i is a matrix congruence on A_i, so the restriction of μ_i on A is also a matrix congruence on A. By this it follows that the restriction of μ on A is an intersection of matrix congruences on A, so it is also a matrix congruence on A. Therefore, we proved that $\mu \in \mathcal{SM}(S)$. This means that $\mathcal{SM}(S)$ is a complete meet-subsemilattice of $\mathcal{E}(S)$, and since it is evident that $\mathcal{SM}(S)$ contains the unity of $\mathcal{E}(S)$, then $\mathcal{SM}(S)$ is a complete lattice. $\square$

In terms of decompositions, the previous theorem is formulated in the following way:

Corollary 1. *For any semigroup S, the posets of semilattice-matrix, semilattice-left and semilattice-right decompositions of S are complete lattices.*

By the proof of Theorem 1 we also obtain the following.

Corollary 2. *For any semilattice congruence σ on a semigroup S, the posets $\mathcal{SM}(\sigma)$, $\mathcal{SL}(\sigma)$ and $\mathcal{SR}(\sigma)$ are complete lattices.*

By Theorem 1 we proved that $\mathcal{SM}(S)$, $\mathcal{SL}(S)$ and $\mathcal{SR}(S)$ are complete lattices, but we do not know whether these lattices are complete sublattices of the lattice $\mathcal{E}(S)$. This property will be proved only for the lattices $\mathcal{SM}(\sigma)$, $\mathcal{SL}(\sigma)$ and $\mathcal{SR}(\sigma)$.

First we prove the following lemma:

Lemma 1. *Let σ be a semilattice congruence on a semigroup S, and let θ be an equivalence relation on S. Then $\theta \in \mathcal{SL}(\sigma)$ if and only if $\theta \subseteq \sigma$ and for any $a, b \in S$, $a\sigma b$ implies $ab\theta a$.*

Proof. This follows by the definition of a semilattice-left equivalence and the fact that an equivalence relation τ on a semigroup T is a left zero band congruence if and only if $xy\tau x$ for all $x, y \in T$. □

For $\mathcal{SL}(\sigma)$, and dually for $\mathcal{SR}(\sigma)$, we obtain the following:

Theorem 2. *For any semilattice congruence σ on a semiring S, $\mathcal{SL}(\sigma)$ is a closed interval of $\mathcal{E}(S)$.*

Proof. Let τ denote the intersection of all elements from $\mathcal{SL}(\sigma)$. Clearly, $\tau \subseteq \sigma$. Assume $a, b \in S$ such that $a\sigma b$. Then $ab\theta a$, for any $\theta \in \mathcal{SL}(\sigma)$, whence $ab\tau a$, so $\tau \in \mathcal{SL}(\sigma)$. Therefore, τ is the smallest element of $\mathcal{SL}(\sigma)$.

Assume an arbitrary $\theta \in \mathcal{E}(S)$. If $\theta \in \mathcal{SL}(\sigma)$, then clearly $\tau \subseteq \theta \subseteq \sigma$. On the other hand, if $\tau \subseteq \theta \subseteq \sigma$, then $(a, b) \in \sigma$ implies $(ab, a) \in \tau \subseteq \theta$, so $\theta \in \mathcal{SL}(\sigma)$. Therefore, $\mathcal{SL}(\sigma)$ equals the closed interval $[\tau, \sigma]$

of $\mathcal{E}(S)$. $\square$

For an equivalence relation θ on a semigroup S, let $S(\theta)$ denote the set of all semilattice congruences on S containing θ. Since $\mathcal{S}(S)$ is a principal dual ideal of $\mathrm{Con}\,(S)$, then $\mathcal{S}(\theta)$ is a principal dual ideal both of $\mathcal{S}(S)$ and $\mathrm{Con}\,(S)$. If $\lambda \in \mathcal{SL}(S)$, then $\mathcal{C}^{\mathrm{sl}}(\lambda)$ will denote the set of semilattice congruences on S carrying λ as a semilattice-left equivalence. The place of $\mathcal{C}^{\sigma}(\lambda)$ inside $\mathcal{S}(\lambda)$ and $\mathcal{S}(S)$ is explained by the following theorem:

Theorem 3. *Let λ be a semilattice-left equivalence on a semigroup S. Then*

(a) $\mathcal{C}^{\mathrm{sl}}(\lambda)$ *is an order ideal of* $\mathcal{S}(\lambda)$;

(b) $\mathcal{C}^{\mathrm{sl}}(\lambda)$ *is a convex subset of* $\mathcal{S}(S)$ *having a smallest element.*

Proof. (a) Assume $\sigma \in \mathcal{C}^{\mathrm{sl}}(\lambda)$ and $\pi \in \mathcal{S}(\lambda)$ such that $\pi \subseteq \sigma$. If $(a,b) \in \pi$, then clearly $(a,b) \in \sigma$, and thus $(ab,a) \in \lambda$. Hence, $\pi \in \mathcal{C}^{\mathrm{sl}}(\lambda)$, which proves (a).

(b) This follows by (a) and the fact that $\mathcal{S}(\lambda)$ is a principal dual ideal of $\mathcal{S}(S)$. $\square$

By the next theorem we establish a connection between semilattice-matrix, semilattice-left and semilattice-right equivalences.

Theorem 4. *Given a semigroup S. If $\sigma_1, \sigma_2 \in \mathcal{S}(S)$, $\lambda \in \mathcal{SL}(\sigma_1)$ and $\varrho \in \mathcal{SR}(\sigma_2)$, then $\lambda \cap \varrho \in \mathcal{SM}(\sigma_1 \cap \sigma_2)$.*

Conversely, if $\sigma \in \mathcal{S}(S)$ and $\mu \in \mathcal{SM}(\sigma)$, then there exist unique $\lambda \in \mathcal{SL}(\sigma)$ and $\varrho \in \mathcal{SR}(\sigma)$ such that $\mu = \lambda \cap \varrho$.

Proof. The first statement of the theorem follows by the proof of Theorem 1 and the fact that $\mathcal{SL}(S) \cup \mathcal{SR}(S) \subseteq \mathcal{SM}(S)$.

To prove the opposite statement, assume an arbitrary $\sigma \in \mathcal{S}(S)$ and $\mu \in \mathcal{SM}(\sigma)$. Let $\{S_\alpha\}_{\alpha \in Y}$ be the set of different σ-classes of S, and for $\alpha \in Y$, let μ_α denote the restriction of μ on S_α. By the hypothesis, for any $\alpha \in Y$, μ_α is a matrix congruence on S_α, so by [**14**, Theorem

III], $\mu_\alpha = \lambda_\alpha \cap \varrho_\alpha$, where λ_α is a left zero band congruence and ϱ_α is a right zero band congruence on S_α. Let

$$\lambda = \bigcup_{\alpha\in Y} \lambda_\alpha \quad \text{and} \quad \varrho = \bigcup_{\alpha\in Y} \varrho_\alpha.$$

It is easy to verify that $\lambda \in \mathcal{SL}(\sigma)$, $\varrho \in \mathcal{SR}(\sigma)$ and $\mu = \lambda \cap \varrho$, which was to be proved.

Finally, assume another $\lambda' \in \mathcal{SL}(\sigma)$ and $\varrho' \in \mathcal{SR}(\sigma)$ such that $\lambda' \cap \varrho' = \lambda \cap \varrho = \mu$. For $\alpha \in Y$, let λ'_α and ϱ'_α denote restrictions of λ' and ϱ' on S_α, respectively. For any $\alpha \in Y$ we have that λ'_α is a left zero band congruence and ϱ'_α is a right zero band congruence on S_α, and $\lambda'_\alpha \cap \varrho'_\alpha = \mu_\alpha = \lambda_\alpha \cap \varrho_\alpha$, so by [**14**, Theorem III] we obtain that $\lambda'_\alpha = \lambda_\alpha$ and $\varrho'_\alpha = \varrho_\alpha$. Therefore,

$$\lambda' = \bigcup_{\alpha\in Y} \lambda'_\alpha = \bigcup_{\alpha\in Y} \lambda_\alpha = \lambda$$

and

$$\varrho' = \bigcup_{\alpha\in Y} \varrho'_\alpha = \bigcup_{\alpha\in Y} \varrho_\alpha = \varrho,$$

which was to be proved. □

Using the previous theorem, we also obtain the following connection between $\mathcal{SM}(\sigma)$, $\mathcal{SL}(\sigma)$ and $\mathcal{SR}(\sigma)$.

Theorem 5. *Let σ be a semilattice congruence on a semigroup S. Then the lattice $\mathcal{SM}(\sigma)$ is isomorphic to the direct product of lattices $\mathcal{SL}(\sigma)$ and $\mathcal{SR}(\sigma)$.*

Proof. Consider the mapping $\varphi : \mathcal{SL}(\sigma) \times \mathcal{SR}(\sigma) \to \mathcal{SM}(\sigma)$ defined by

$$(\lambda, \varrho)\varphi = \lambda \cap \varrho,$$

for $\lambda \in \mathcal{SL}(\sigma)$, $\varrho \in \mathcal{SR}(\sigma)$. By Theorem 4, φ is a bijection of $\mathcal{SL}(\sigma) \times \mathcal{SR}(\sigma)$ onto $\mathcal{SM}(\sigma)$. It is evident that φ is isotone. Therefore, to prove that φ is an order isomorphism, and hence a lattice isomoprhism, it remains to prove that φ^{-1} is an isotone mapping. Indeed, assume

$\lambda, \lambda' \in \mathcal{SL}(\sigma)$ and $\varrho, \varrho' \in \mathcal{SR}(\sigma)$ such that $\lambda \cap \varrho \subseteq \lambda' \cap \varrho'$. Let $(a,b) \in \lambda$. Then $(a,b) \in \sigma$. Let A denote a σ-class of S containing a and b. Since the restriction of λ on A is a congruence on A, $a, b \in A$ and $(a,b) \in \lambda$, then $(ab, b^2) \in \lambda$, and seeing that $(b^2, b) \in \lambda$, then $(ab, b) \in \lambda$. On the other hand, $(a,b) \in \sigma$ implies $(ab, b) \in \varrho$. Hence,

$$(ab, b) \in \lambda \cap \varrho \subseteq \lambda' \cap \varrho' \subseteq \lambda'.$$

By $(a,b) \in \sigma$ we also obtain $(a, ab) \in \lambda'$ which with $(ab, b) \in \lambda'$ gives $(a,b) \in \lambda'$. Therefore, $\lambda \subseteq \lambda'$. Similarly, we prove that $\varrho \subseteq \varrho'$. This completes the proof of the theorem. □

If in the previous theorem we assume that σ is the universal relation on S, then we obtain the following:

Corollary 3. *The lattice of matrix congruences on a semigroup S is isomorphic to the direct product of the lattice of left zero band congruences and the lattice of right zero band congruences on S.*

For the results concerning matrix and normal band congruences, which correspond to the above two theorems, we refer to [**12, 13**] and [**14**].

Another interesting relationship between semilattice-left and semilattice-right equivalences is given by the following theorem:

Theorem 6. *Let σ be a semilattice congruence on a semigroup S, $n \in \mathbf{Z}^+$, $\lambda, \lambda_1, \dots, \lambda_n \in \mathcal{SL}(\sigma)$ and $\varrho, \varrho_1, \dots, \varrho_n \in \mathcal{SR}(\sigma)$. Then*

(a) $\lambda\varrho = \varrho\lambda = \sigma$;

(b) $(\lambda_1 \cap \varrho_1)(\lambda_2 \cap \varrho_2) \cdots (\lambda_n \cap \varrho_n) = \lambda_1\lambda_2 \cdots \lambda_n \cap \varrho_1\varrho_2 \cdots \varrho_n$.

Proof. (a) Since $\lambda \subseteq \sigma$ and $\varrho \subseteq \sigma$ then $\lambda\varrho \subseteq \sigma$. To prove the opposite inclusion, assume an arbitrary $(a,b) \in \sigma$. Then $(a, ab) \in \lambda$ and $(ab, b) \in \varrho$, whence $(a,b) \in \lambda\varrho$; hence, $\lambda\varrho = \sigma$. Analogously, we prove that $\varrho\lambda = \sigma$.

(b) Assume $(a,b) \in (\lambda_1 \cap \varrho_1)(\lambda_2 \cap \varrho_2) \cdots (\lambda_n \cap \varrho_n)$. Then there exist $x_1, \dots, x_{n-1} \in S$ such that

$$(a, x_1) \in \lambda_1 \cap \varrho_1, (x_1, x_2) \in \lambda_2 \cap \varrho_2, \dots, (x_{n-1}, b) \in \lambda_n \cap \varrho_n,$$

and clearly $(a,b) \in \lambda_1\lambda_2\cdots\lambda_n$ and $(a,b) \in \varrho_1\varrho_2\cdots\varrho_n$. This means that

$$(\lambda_1 \cap \varrho_1)(\lambda_2 \cap \varrho_2)\cdots(\lambda_n \cap \varrho_n) \subseteq \lambda_1\lambda_2\cdots\lambda_n \cap \varrho_1\varrho_2\cdots\varrho_n.$$

To prove the opposite inclusion, assume $(a,b) \in \lambda_1\lambda_2\cdots\lambda_n \cap \varrho_1\varrho_2\cdots\varrho_n$. Then there exists $x_1,\dots,x_{n-1},y_1,\dots,y_{n-1} \in S$ such that

$$\begin{aligned} &(a,x_1) \in \lambda_1, \quad (x_1,x_2) \in \lambda_2, \cdots (x_{n-1},b) \in \lambda_n, \\ &(a,y_1) \in \varrho_1, \quad (y_1,y_2) \in \varrho_2, \cdots (y_{n-1},b) \in \varrho_n. \end{aligned} \tag{1}$$

It is clear that $a,b,x_1,\dots,x_{n-1},y_1,\dots,y_{n-1}$ belong to the same σ-class of S. By this, it follows that

$$(x_1, x_1y_1) \in \lambda_1 \quad \text{and} \quad (y_1, x_1y_1) \in \varrho_1,$$

which with (1) gives $(a, x_1y_1) \in \lambda_1 \cap \varrho_1$. Similarly we prove that $(x_{n-1}y_{n-1}, b) \in \lambda_n \cap \varrho_n$. Moreover, for an arbitrary i, $1 \le i \le n-2$, we have that

$$\begin{aligned} (x_iy_i, x_i), (x_i, x_{i+1}), (x_{i+1}, x_{i+1}y_{i+1}) \in \lambda_{i+1} \\ (x_iy_i, y_i), (y_i, y_{i+1}), (y_{i+1}, x_{i+1}y_{i+1}) \in \varrho_{i+1}, \end{aligned}$$

whence it follows that $(x_iy_i, x_{i+1}y_{i+1}) \in \lambda_{i+1} \cap \varrho_{i+1}$. Therefore

$$(a, x_1y_1) \in \lambda_1 \cap \varrho_1, (x_1y_1, x_2y_2) \in \lambda_2 \cap \varrho_2, \dots, (x_{n-1}y_{n-1}, b) \in \lambda_n \cap \varrho_n,$$

so $(a,b) \in (\lambda_1 \cap \varrho_1)(\lambda_2 \cap \varrho_2)\cdots(\lambda_n \cap \varrho_n)$. Hence,

$$\lambda_1\lambda_2\cdots\lambda_n \cap \varrho_1\varrho_2\cdots\varrho_n \subseteq (\lambda_1 \cap \varrho_1)(\lambda_2 \cap \varrho_2)\cdots(\lambda_n \cap \varrho_n),$$

which completes the proof of (b). □

As a consequence of the previously obtained result we obtain the following:

Corollary 4. *A semilattice-left equivalence λ and a semilattice-right equivalence ϱ on a semigroup S can have at most one common carrier.*

Proof. If λ and ϱ have a common carrier, then by Theorem 6 it follows that $\lambda\varrho = \varrho\lambda$, and it is the unique common carrier of λ and ϱ. □

Finally, using Theorem 6, we prove that $\mathcal{SM}(\sigma)$ is also a complete sublattice of $\mathcal{E}(S)$.

Theorem 7. *For any semilattice congruence σ on a semigroup S, $\mathcal{SM}(\sigma)$ is a complete sublattice of $\mathcal{E}(S)$.*

Proof. By the proof of Theorem 1, $\mathcal{SM}(\sigma)$ is a complete meet-subsemilattice of $\mathcal{E}(S)$, so it remains to prove that it is a complete join-subsemilattice of $\mathcal{E}(S)$.

Assume an arbitrary nonempty subset $\{\mu_i\}_{i\in I}$ of $\mathcal{SM}(\sigma)$. By Theorem 4, for any $i \in I$, there exist unique $\lambda_i \in \mathcal{SL}(\sigma)$ and $\varrho_i \in \mathcal{SR}(\sigma)$ such that $\mu_i = \lambda_i \cap \varrho_i$. Let

$$\mu = \bigvee_{i\in I} \mu_i, \quad \lambda = \bigvee_{i\in I} \lambda_i \quad \text{and} \quad \varrho = \bigvee_{i\in I} \varrho_i$$

in $\mathcal{E}(S)$. Let us prove that

$$\mu = \lambda \cap \varrho. \tag{2}$$

Assume $(a, b) \in \mu$. Then $(a, b) \in \mu_{i_1}\mu_{i_2}\cdots\mu_{i_n}$ for some $i_1, i_2, \dots, i_n \in I$, so by Theorem 6 we have that

$$\begin{aligned}(a, b) \in \mu_{i_1}\mu_{i_2}\cdots\mu_{i_n} &= (\lambda_{i_1} \cap \varrho_{i_1})(\lambda_{i_2} \cap \varrho_{i_2})\cdots(\lambda_{i_n} \cap \varrho_{i_n})\\ &= \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} \cap \varrho_{i_1}\varrho_{i_2}\cdots\varrho_{i_n} \subseteq \lambda \cap \varrho.\end{aligned}$$

Therefore, we proved that $\mu \subseteq \lambda \cap \varrho$.

To prove the opposite inclusion, assume $(a, b) \in \lambda\cap\varrho$. Then $(a, b) \in \lambda$ and $(a, b) \in \varrho$, whence

$$(a, b) \in \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} \quad \text{and} \quad (a, b) \in \varrho_{i_{n+1}}\varrho_{i_{n+2}}\cdots\varrho_{i_{n+k}},$$

for some $i_1, i_2, \dots, i_n, i_{n+1}, i_{n+2}, \dots, i_{n+k} \in I$. Since $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_n}$, $\varrho_{i_{n+1}}, \varrho_{i_{n+2}}, \dots, \varrho_{i_{n+k}}$ are reflexive relations, then

$$\begin{aligned}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} &\subseteq \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}\lambda_{i_{n+1}}\lambda_{i_{n+2}}\cdots\lambda_{i_{n+k}},\\ \varrho_{i_{n+1}}\varrho_{i_{n+2}}\cdots\varrho_{i_{n+k}} &\subseteq \varrho_{i_1}\varrho_{i_2}\cdots\varrho_{i_n}\varrho_{i_{n+1}}\varrho_{i_{n+2}}\cdots\varrho_{i_{n+k}},\end{aligned}$$

so by Theorem 6 we have that

$$\begin{aligned}(a,b) &\in \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n+k}} \cap \varrho_{i_1}\varrho_{i_2}\cdots\varrho_{i_{n+k}}\\ &= (\lambda_{i_1}\cap\varrho_{i_1})(\lambda_{i_2}\cap\varrho_{i_2})\cdots(\lambda_{i_{n+k}}\cap\varrho_{i_{n+k}})\\ &= \mu_{i_1}\mu_{i_2}\cdots\mu_{i_{n+k}} \subseteq \mu.\end{aligned}$$

Hence we proved that $\lambda\cap\varrho \subseteq \mu$, which completes the proof of (2). Since by Theorem 2 and its dual we have that $\lambda \in \mathcal{SL}(\sigma)$ and $\varrho \in \mathcal{SR}(\sigma)$, then by Theorem 4 we obtain $\mu \in \mathcal{SM}(\sigma)$, which was to be proved. □

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