ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 4, Winter 2002

SEMIREGULAR, SEMIPERFECT AND PERFECT RINGS RELATIVE TO AN IDEAL

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ABSTRACT. Let I be an ideal of a ring R. Consider the following conditions on R:

1. If X is a finitely generated submodule of a finitely generated projective module P, then $X = A \oplus B$ where A is a summand of P and $B \subseteq P \cdot I$.

2. If X is a submodule of a finitely generated projective module P, then $X = A \oplus B$ where A is a summand of P and $B \subseteq P \cdot I$.

3. If X is a submodule of a projective module P, then $X = A \oplus B$ where A is a summand of P and $B \subseteq P \cdot I$.

When I is the Jacobson radical $J(R)$ of R, these conditions characterize semiregular rings, semiperfect rings and right perfect rings, respectively. In this paper we completely characterize these conditions for the cases when I is the right singular ideal, or the right socle, or the intersection of any two of the three ideals. As applications, structure theorems are obtained for right CEP-rings R with $J(R)^2 = 0$ and for QF-rings R with $J(R)^2 = 0$.

All rings R are associative and have an identity, unless otherwise specified, and modules are unitary right modules over R. For an R-module M, $J(M)$, $Z(M)$ and $\mathrm{Soc}\,(M)$ are the Jacobson radical, the singular submodule and the socle of M, respectively. We use Z_r, Z_l, S_r and S_l to indicate the right singular ideal, the left singular ideal, the right socle and the left socle of R, respectively.

1. I-Semiregular rings.

The following lemma has been observed in [**21**, Lemma 1.1] when K is a principal right ideal of R.

Lemma 1.1. *Let I be an ideal of the ring R. The following conditions are equivalent for a right ideal K of R:*

1991 AMS *Mathematics Subject Classification.* Primary 16P99, 16N99.

The research was supported by the NSERC grant OGP0194196 and a grant from the Ohio State University.
Received by the editors on July 27, 2001, and in revised form on November 7, 2001.

(1) *There exists* $e^2 = e \in K$ *with* $(1-e)K \subseteq I$.

(2) *There exists* $e^2 = e \in K$ *with* $K \cap (1-e)R \subseteq I$.

(3) $K = eR \oplus S$ *where* $e^2 = e$ *and* $S \subseteq I$.

Proof. (1) $\Rightarrow$ (2). This is obvious since $K \cap (1-e)R \subseteq (1-e)K$.

(2) $\Rightarrow$ (3). Let $S = K \cap (1-e)R$.

(3) $\Rightarrow$ (1). For $a \in K$, write $a = er + s$ where $r \in R$ and $s \in S$. Then $ea = er + es$ and $(1-e)a = a - ea = s - es \in I$. So $(1-e)K \subseteq I$. □

Following [**21**], R is called a *right I-semiregular ring* if every principal right ideal K of R satisfies the equivalent conditions of Lemma 1.1.

Clearly R is a (von Neumann) regular ring if and only if R is right (respectively left) (0)-semiregular and R is semiregular (or f-semiperfect) if and only if R is right (respectively left) $J(R)$-semiregular. The right Z_r semiregular rings, called right weakly continuous rings, are studied in [**21**]. Let δ_r be the ideal of R defined by $\delta_r/S_r = J(R/S_r)$. The right δ_r-semiregular rings are discussed in [**23**].

The next lemma is due to Baccella [**6**].

Lemma 1.2. *For a ring R, idempotents of R/S_r lift to idempotents of R.*

Proof. Let $x \in R$ with $x^2 - x \in S_r$. Write $S_r = S_1 \oplus S_2$ where S_1 is the sum of all nilpotent minimal right ideals and S_2 is the sum of all idempotent minimal right ideals. Then both S_1 and S_2 are ideals of R and $S_1^2 = 0$. Write $x^2 - x = a_1 + a_2$ where $a_1 \in S_1$ and $a_2 \in S_2$. Since a_2R is a direct sum of finitely many idempotent minimal right ideals, it is standard to show that a_2R is a direct summand of R_R. So $a_2R = fR$ for some $f^2 = f \in R$. Write $f = a_2b$ where $b \in R$ and let $c = bf$. Then $a_2 = fa_2 = a_2(bf)a_2 = a_2ca_2$ and $c \in S_2$. It follows that $x^2 - x = a_1 + a_2ca_2 = a_1 + (x^2 - x - a_1)c(x^2 - x - a_1) = (x^2 - x)c(x^2 - x) + b_1$ where $b_1 \in S_1$. Let $y = 1 - (x-1)c(x-1)$. Then $xyx = x^2 - (x^2 - x)c(x^2 - x) = x^2 - (x^2 - x - b_1) = x + b_1$ and hence $(xy)^2 = xy + b_1y$ with $b_1y \in S_1$. Since $S_1^2 = 0$, there exists $e^2 = e \in R$ such that $e - xy \in S_1$. So $e - x = (e - xy) + (xy - x) =$

$(e - xy) - x(x-1)c(x-1) \in S_1 + S_2 = S_r$. $\square$

Lemma 1.3. *For a ring R, let $\overline{R} = R/S_r$. If idempotents of $\overline{R}/J(\overline{R})$ lift to idempotents of $\overline{R}$, then idempotents of R/δ_r lift to idempotents of R.*

Proof. Let $x \in R$ with $x^2 - x \in \delta_r$. Then $\bar{x} \in R/S_r$ and $\bar{x}^2 - \bar{x} \in J(\overline{R}) = \delta_r/S_r$. By the hypothesis, there exists $\bar{a}^2 = \bar{a} \in \overline{R}$ such that $\bar{x}\bar{a} \in \delta_r/S_r$. Thus, $a^2 - a \in S_r$ and $x - a \in \delta_r$. By Lemma 1.2, there exists $e^2 = e \in R$ such that $a - e \in S_r$. So $x - e = (x-a) + (a-e) \in \delta_r$. $\square$

The right δ_r-semiregular rings were characterized in [**23**, Theorem 3.5]. A new characterization of such rings is given in the next theorem.

Theorem 1.4. *A ring R is a right δ_r-semiregular ring if and only if R/S_r is semiregular.*

Proof. By [**23**, Theorem 3.5], R is a right δ_r-semiregular ring if and only if R/δ_r is a regular ring and idempotents lift modulo δ_r. Thus the implication "$\Rightarrow$" follows immediately. Suppose that R/S_r is a semiregular ring. Then $R/\delta_r \cong \overline{R}/J(\overline{R})$ is regular and idempotents of R/δ_r lift to idempotents of R by Lemma 1.3. Thus, R is right δ_r-semiregular. $\square$

Following Ara [**2**], we say that an ideal I of a ring R is an exchange ring if, for every $x \in I$, there exists $e^2 = e \in xI$ such that $1 - e \in (1-x)R$. This extends the concept of a unital exchange ring to rings without unit.

Corollary 1.5. *Let R/S_r be a semiregular ring. Then R is an exchange ring and every finitely generated projective R-module is isomorphic to a direct sum of right ideals of the form eR, $e^2 = e$.*

Proof. Suppose that R/S_r is semiregular. Then R/S_r is an exchange ring by Warfield [**22**]. By [**6**, Lemma 1.2], S_r is an exchange ring. Since idempotents of R/S_r lift to idempotents (Lemma 1.2), a result of Ara [**2**, Theorem 2.2] asserts that R is an exchange ring. The second part follows from a well-known result of Warfield [**22**, Theorem 1]. $\square$

Theorem 1.6. *The following are equivalent for a ring R:*

(1) *R is right S_r-semiregular.*

(2) *For any $a \in R$, $aR = eR \oplus U$ where $e^2 = e$ and $U \subseteq J(R) \cap S_r$.*

(3) *R/S_r is a regular ring.*

(4) *If X is a finitely generated submodule of a (finitely generated) projected module P, then $X = A \oplus B$ where A is a summand of P and $B \subseteq \mathrm{Soc}\,(P)$.*

Proof. The implications (4) $\Rightarrow$ (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (2). Let $a \in R$. By (1), $aR = eR \oplus U$ where $e^2 = e$ and $U \subseteq S_r$. Note that the uniform dimension $\dim(U)$ of U is finite. If $\dim(U) = 0$, then $U = (0) \subseteq J(R) \cap S_r$, and we are done. Assume that, whenever $aR = eR \oplus U$ with $\dim(U) = k(\geq 0)$ where $e^2 = e$ and $U \subseteq S_r$, there exist $f^2 = f \in R$ and $V \subseteq J(R) \cap S_r$ such that $aR = fR \oplus V$. Suppose that $aR = eR \oplus U$ where $e^2 = e$, $U \subseteq S_r$ and $\dim(U) = k+1$. Since $aR = eR \oplus [aR \cap (1-e)R]$, $aR \cap (1-e)R \cong U$ and so $aR \cap (1-e)R \subseteq S_r$. We can assume that $aR \cap (1-e)R$ is not contained in $J(R)$. Thus, there exists an idempotent minimal right ideal, say I, in $aR \cap (1-e)R$. Obviously, I is a direct summand of R_R and hence of $(1-e)R$. So $eR \oplus I$ is a summand of R. Write $eR \oplus I = fR$ where $f^2 = f \in aR$. Then $aR = fR \oplus V$ where $V \subseteq S_r$ and $\dim(V) = k$. By induction hypothesis, there exist $g^2 = g \in R$ and $W \subseteq J(R) \cap S_r$ such that $aR = gR \oplus W$.

(3) $\Rightarrow$ (4). Since R/S_r is regular, $\delta_r/S_r = J(R/S_r) = \bar{0}$. So $\delta_r = S_r$. Hence, by Lemma 1.2, idempotents of R/δ_r lift to idempotents of R. Thus, by [**23**, Theorem 3.5], R is right δ_r-semiregular and $\delta_r = S_r$. To prove (4), let X be a finitely generated submodule of a projective module P. Since every projective module is a direct summand of a free module, we may assume that P is a free module, and further we can assume that P is a finitely generated free module. Then P/X is a finitely presented module. By [**23**, Theorem 3.5(2), Lemma 2.4 and Lemma 1.9], P has a decomposition $P = P_1 \oplus P_2$ such that $P_1 \subseteq X$

and $X \cap P_2 \subseteq P \cdot \delta_r = P \cdot S_r \subseteq \mathrm{Soc}\,(P)$. Thus, $X = A \oplus B$ where $A = P_1$ and $B = X \cap P_2$. $\square$

Corollary 1.7. *The following statements hold*:

(1) *Being right S_r-semiregular is a Morita invariant property of rings.*

(2) *Every right S_r-semiregular ring is right $J(R) \cap S_r$-semiregular and hence right semiregular. In this case $Z_l \subseteq S_r$, $Z_r \subseteq J(R) \subseteq S_r$ and $J(R)^2 = 0$.*

(3) *The ring R is regular if and only if every minimal right ideal is idempotent and R/S_r is regular.*

Proof. (1) This follows from Theorem 1.6(4).

(2) Theorem 1.6(2) shows that R is right $J(R) \cap S_r$-semiregular and hence right semiregular. Then it follows from [**21**, Theorem 1.2] that $Z_l \subseteq S_r$, $Z_r \subseteq S_r$ and $J(R) \subseteq S_r$. Hence, $Z_r \subseteq J(R)$ and $J(R)^2 = 0$.

(3) One direction is clear. Suppose that every minimal right ideal is idempotent and R/S_r is regular. Then $J(R) \cap S_r = 0$. By Theorem 1.6, R is right 0-semiregular, i.e., regular. $\square$

Examples 1.8. (1) A right Z_r-semiregular ring may not be right S_r-semiregular: Let $R = \{\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) : x \in \mathbf{Z}_4, y \in \mathbf{Z}_4 \oplus \mathbf{Z}_4\}$ where $\mathbf{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. Then $J(R) = Z_r = \{\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) : x \in 2\mathbf{Z}_4, y \in \mathbf{Z}_4 \oplus \mathbf{Z}_4\}$ with $J(R)^3 = 0$ and $R/J(R) \cong \mathbf{Z}_2$. So R is a Z_r-semiregular ring. But $\mathrm{Soc}\,(R) = \{\begin{pmatrix} 0 & y \\ 0 & x \end{pmatrix} : y \in 2\mathbf{Z}_4 \oplus 2\mathbf{Z}_4\}$. So $J(R)$ is not contained in $\mathrm{Soc}\,(R)$ and hence R is not right S_r-semiregular.

(2) A right S_r-semiregular ring may not be right Z_r-semiregular: Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field. Then $S_r = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $R/S_r \cong F$. So R is right S_r-semiregular. But $Z_r = 0$ with $J(R) \neq 0$. So R is not right Z_r-semiregular.

By Corollary 1.7(2), R is right S_r-semiregular if and only if R is right $J(R) \cap S_r$-semiregular and, by [**21**, Theorem 2.4], R is right Z_r-semiregular if and only if R is right $J(R) \cap Z_r$-semiregular. Next we characterize right $S_r \cap Z_r$-semiregular rings.

Corollary 1.9. *The following are equivalent for a ring R*:

(1) *R is right $S_r \cap Z_r$-semiregular.*

(2) *R is right S_r-semiregular and right Z_r-semiregular.*

(3) *For any $a \in R$, $aR = P \oplus U$ where P is projective and $U \subseteq Z_r \cap S_r$ and every principal projective right ideal is a direct summand.*

(4) *For any $a \in R$, $aR = P \oplus U$ where P is projective and $U \subseteq Z_r \cap S_r$ and R is right $C2$.*

(5) *R/S_r is a regular ring and $J(R) = Z_r$.*

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) $\Rightarrow$ (1) are obvious.

(2) $\Rightarrow$ (1). Let $a \in R$. Since R is right Z_r-semiregular, $aR = eR \oplus U$ with $e^2 = e$ and $U \subseteq Z_r$. Since R is right S_r-semiregular, $U = fR \oplus V$ with $f^2 = f$ and $V \subseteq S_r$. Since U is singular, $f = 0$ and so $U = V \subseteq S_r \cap Z_r$.

(1) $\Rightarrow$ (3) follows from (1) and [**21**, Lemma 2.1].

(2) $\Leftrightarrow$ (5) follows from Theorem 1.6 and [**21**, Theorem 2.4]. $\square$

The next proposition can be proved using the arguments as in the proof of [**21**, Proposition 2.2].

Proposition 1.10. *The following are equivalent for $a \in R$*:

(1) *$aR = P \oplus U$ where P is projective and $U \subseteq Z_r \cap S_r$.*

(2) *$\mathbf{r}(a)$ is the intersection of finitely many essential maximal submodules of some summand of R_R.*

Remark 1.11. For an ideal I of R, by [**21**, Theorem 1.2], the condition that (a) R is a right I-semiregular ring always implies that (b) R/I is regular and idempotents lift modulo I. (a) and (b) are equivalent when $I = J(R)$, $I = S_r$, (by Lemma 1.2 and Theorem 1.6(3)), or $I = \delta_r$ (see [**23**, Theorem 3.5]), but not equivalent in general by [**21**, Example 1.3]. From Example 2.8, we have that (b) does not imply (a) when $I = Z_r$.

As a comparison to Theorem 1.6(4), a homological characterization of right Z_r-semiregular rings is given as follows.

Proposition 1.12. *The ring R is right Z_r-semiregular if and only if, for any finitely generated submodule X of a* (*finitely generated*) *projective module P, $X = A \oplus B$ where A is a summand of P and $B \subseteq Z(P)$.*

Proof. One direction is clear. Suppose that R is right Z_r-semiregular. Let X be a finitely generated submodule of a projective module P. Since every projective module is a direct summand of a free module, we may assume that P is a free module and further we can assume that P is a finitely generated free module. Then P/X is a finitely presented module. By [**7**, Lemma 2.3], P has a decomposition $P = P_1 \oplus P_2$ such that $P_1 \subseteq X$ and $X \cap P_2 \subseteq J(P) = P \cdot J(R) = P \cdot Z_r \subseteq Z(P)$. Thus, $X = A \oplus B$ where $A = P_1$ and $B = X \cap P_2$. $\square$

2. I-Semiperfect rings. The ring R is called a *right I-semiperfect ring* if every right ideal K of R satisfies the equivalent conditions in Lemma 1.1. Clearly R is a semisimple artinian ring if and only if R is right (respectively left) (0)-semiperfect and R is semiperfect if and only if R is right (respectively left) $J(R)$-semiperfect. The right δ_r-semiperfect rings are discussed in [**23**]. The following result is well known and easy to prove.

Lemma 2.1. *The following are equivalent for a ring R*:

(1) *R is a semisimple artinian ring.*

(2) *Every simple R-module is projective.*

(3) *Every maximal right ideal of R is a direct summand of R_R.*

(4) *Every singular simple R-module is projective.*

Theorem 2.2. *A ring R is right δ_r-semiperfect if and only if R/S_r is semiperfect.*

Proof. By [**23**, Theorem 3.6], R is right δ_r-semiperfect if and only if R/δ_r is semisimple artinian and idempotents lift modulo δ_r. And the latter, by the same arguments as in the proof of Theorem 1.4, is equivalent to the condition that R/S_r is semiperfect. □

Theorem 2.3. *The following are equivalent for a ring* R:

(1) *R is right S_r-semiperfect.*

(2) *For every countably generated right ideal $K \subseteq R$, $K = eR \oplus U$ where $e^2 = e$ and $U \subseteq S_r$.*

(3) *R/S_r is semisimple artinian.*

(4) *If X is a submodule of a finitely generated projective module P, then $X = A \oplus B$ where A is a summand of P and $B \subseteq \mathrm{Soc}\,(P)$.*

(5) *There exists a complete orthogonal set of idempotents $e_1, e_2, \dots, e_n$, such that for each i, either $(e_iR)_R$ is simple or $\mathrm{Soc}\,(e_iR)$ is a maximal submodule of $(e_iR)_R$.*

(6) *For every maximal right ideal $K \subseteq R$, $K = eR \oplus U$ where $e^2 = e$ and $U \subseteq S_r$.*

Proof. (1) $\Rightarrow$ (3), (5) $\Rightarrow$ (3), (4) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1) $\Rightarrow$ (6) are obvious.

(3) $\Rightarrow$ (4). Since R/S_r is semisimple artinian, $\delta_r/S_r = J(R/S_r) = \bar{0}$. So $\delta_r = S_r$ and then idempotents of R/δ_r lift to idempotents of R by Lemma 1.2. Thus by [**23**, Theorem 3.6], R is right δ_r-semiperfect and $\delta_r = S_r$.

Let X be a submodule of a finitely generated projective module P. Then P/X is a finitely generated module. By [**23**, Theorem 3.6(2), Lemma 2.4 and Lemma 1.9], P has a decomposition $P = P_1 \oplus P_2$ such that $P_1 \subseteq X$ and $X \cap P_2 \subseteq P \cdot \delta_r = P \cdot S_r \subseteq \mathrm{Soc}\,(P)$. Thus, $X = A \oplus B$ where $A = P_1$ and $B = X \cap P_2$.

(6) $\Rightarrow$ (3). Condition (6) implies that every maximal right ideal of R/S_r is a direct summand. Thus, by Lemma 2.1, R/S_r is semisimple artinian.

(1) $\Rightarrow$ (5). For any module M, let $\delta(M) = \cap\{N \subseteq M : M/N$ is a singular simple module$\}$. By [**23**, Lemma 1.9], for any projective module P, $\delta(P)$ is the intersection of all essential maximal submodules of P. Suppose that (1) holds. Then R is right δ_r-semiperfect and

$\delta_r = S_r$. By [**23**, Theorem 3.6], there exists a complete orthogonal set of idempotents $e_1, e_2, \ldots, e_n$ such that, for each i, either $(e_iR)_R$ is simple or $(e_iR)_R$ has a unique essential maximal submodule. The latter means that $\delta(e_iR)$ is an essential maximal submodule of e_iR. But, by [**23**, Corollary 1.7], $\delta_r = \delta(R_R)$. So $S_r = \delta(R_R)$. It follows from [**23**, Lemma 1.5] that $\mathrm{Soc}\,(e_iR) = \delta(e_iR)$ for all i. Thus (5) follows.

(2) $\Rightarrow$ (1). Suppose (2) holds. Then R is right S_r-semiregular and hence R/S_r is regular by Theorem 1.6. So, $\delta_r/S_r = J(R/S_r) = \bar{0}$. Thus $\delta_r = S_r$. Moreover, by [**23**, Theorem 3.6], (2) implies that R is right δ_r-semiperfect. □

Remark 2.4. Clearly, if R/S_r is semisimple artinian, then S_r is essential in R_R.

Theorem 2.5. *The following are equivalent for a ring R:*

(1) *R is right Z_r-semiperfect.*

(2) *R is semiperfect and $J(R) = Z_r$.*

(3) *If X is a submodule of a finitely generated projective module P, then $X = A \oplus B$ where A is a summand of P and $B \subseteq Z(P)$.*

(4) *For every maximal right ideal $K \subseteq R$, $K = eR \oplus U$ where $e^2 = e$ and $U \subseteq Z_r$.*

Proof. (1) $\Rightarrow$ (2). Because of (1), every right ideal of R/Z_r is a direct summand and so R/Z_r is semisimple artinian. Moreover, by [**21**, Theorem 2.4], $Z_r = J(R)$ and idempotents of R/Z_r lift to idempotents of R.

(2) $\Rightarrow$ (3). Let X be a submodule of a finitely generated projective module P. Then P/X is finitely generated and hence has a projective cover. By [**7**, Lemma 2.3], P has a decomposition $P = P_1 \oplus P_2$ such that $P_1 \subseteq X$ and $X \cap P_2 \subseteq J(P)$. But $J(P) = P \cdot J(R) = P \cdot Z_r \subseteq Z(P)$. Thus, $X = A \oplus B$ where $A = P_1$ and $B = X \cap P_2$.

(3) $\Rightarrow$ (1) $\Rightarrow$ (4). These are obvious.

(4) $\Rightarrow$ (2). By (4), every maximal right ideal of R/Z_r is a direct summand. Then by Lemma 2.1, R/Z_r is semisimple artinian and hence $J(R) \subseteq Z_r$. Suppose $Z_r \neq J(R)$. There exists $x \in Z_r$ and a

maximal right ideal K of R such that $x \notin K$. Then $R = K + xR$. By (4), $K = eR + U$ where $e^2 = e \in R$ and $U \subseteq Z_r$. Clearly $e \neq 1$. It follows that $R = eR + Z_r + xR = eR + Z_r$. This shows that $(1-e)R \cong R/eR \cong Z_r/(Z_r \cap eR)$ is singular and projective. By [**21**, Lemma 2.1], $1 - e = 0$. This is a contradiction. So $Z_r = J(R)$. Thus Condition (4) implies that every simple R-module has a projective cover and hence R is semiperfect. $\square$

In view of Theorem 2.3 and Theorem 2.5, the next corollary is immediate.

Corollary 2.6. *Being a right S_r-semiperfect (respectively right Z_r-semiperfect) ring is a Morita invariant.*

Examples 2.7. (1) A right S_r-semiperfect ring may not be semiperfect: Let $Q = \Pi_{i=1}^{\infty} F_i$ where $F_i = \mathbf{Z}_2$ and T the subring of Q generated by $\oplus_{i=1}^{\infty} F_i$ and 1_Q. Then T is right S_r-semiperfect, but is not semiperfect and hence not right Z_r-semiperfect.

(2) A right Z_r-semiperfect ring may not be right S_r-semiperfect: Let $R = \{\left(\begin{smallmatrix} a & x \\ 0 & a \end{smallmatrix}\right)\} : a, x \in \mathbf{Z}_4\}$. Then $S_r = \{\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)\} : x \in 2\mathbf{Z}_4\}$ and $Z_r = J(R) = \{\left(\begin{smallmatrix} a & x \\ 0 & a \end{smallmatrix}\right)\} : a \in 2\mathbf{Z}_4, x \in \mathbf{Z}_4\}$. R is clearly (right) Z_r-semiperfect but is not right S_r-semiperfect.

(3) Every right Z_r-semiperfect ring is semiperfect. The ring R in Example 1.8(2) is semiperfect but is not right Z_r-semiperfect.

(4) Every right S_r-semiperfect ring is right S_r-semiregular. The ring R in Examples 1.8(1) is right S_r-semiregular but not right S_r-semiperfect.

(5) Every right Z_r-semiperfect ring is right Z_r-semiregular. The ring T in (1) is right Z_r-semiregular, but not right Z_r-semiperfect.

For an ideal I, the condition (a) "R is right I-semiperfect" is equivalent to the condition (b) "R/I is semisimple artinian and idempotents of R/I lift to idempotents of R" when $I = S_r$ (see Theorem 2.3(3) and Lemma 1.2). But the next example shows that (a) is not equivalent to (b) if $I = Z_r$.

Example 2.8 [Bergman's example]. The ring R in this example is given in detail in [**12**, Example 1.36]. Let W be the set of all surjective real-valued analytic functions f of a real variable such that f has positive derivative and $f(x+1) = f(x) + 1$ for all x. Then W is a group with respect to the compositions of functions. As shown in [**12**, p. 28], there exists a real number p such that, for $f, g \in W$, $f(p) = g(p) \Leftrightarrow f = g$. Let G be the subgroup of W generated by all elements f of W which are given by $f(x) = x + g(x)$ for all x with g a truncated Fourier series of period 1 with rational coefficients, i.e., $g = \sum_{k=0}^{n}[a_k \cos(2\pi kx) + b_k \sin(2\pi kx)]$ for some $n \geq 0$ where the a_k and b_k are rationals. Let $c \in W$ be given by $c(x+1) = x+1$ for all x and $S = \{g \in G : g(p) \geq p\}$. Then S is a sub-semigroup of G and c is a central element of S. Let K be a field and then c will be a central element of the semigroup algebra KS. Now set $R = KS/cKS$. As shown in [**12**, pp. 28–30], R is right primitive (and so $J(R) = 0$) and $Z_r \neq 0$. So R is not right Z_r-semiregular (and hence not right Z_r-semiperfect). But it can be proved from the construction of R given in [**12**, p. 29] that Z_r is a maximal right ideal of R. Thus R/Z_r is a division ring and hence idempotents of R lift modulo Z_r.

By Theorem 2.5, R is right Z_r-semiperfect if and only if R is right $J(R) \cap Z_r$-semiperfect. But in contrast to Corollary 1.7(2), a right S_r-semiperfect ring may not be right $J(R) \cap S_r$-semiperfect: The ring T in Example 2.7(1) provides such an example. Next we consider right $J(R) \cap S_r$-semiperfect and right $S_r \cap Z_r$-semiperfect rings.

Corollary 2.9. *The following are equivalent for a ring R:*

(1) *R is right $J(R) \cap S_r$-semiperfect.*

(2) *R is semiperfect and right S_r-semiperfect.*

(3) *R is semiprimary with $J(R) \subseteq S_r$.*

Proof. (1) $\Rightarrow$ (3). Clearly, (1) implies that R is right S_r-semiperfect. So R is right S_r-semiregular. Thus, $J(R) \subseteq S_r$ by Corollary 1.7(2) and so $J(R)^2 = 0$. (1) also implies that R is semiperfect, so R is semiprimary.

(3) $\Rightarrow$ (2). R is clearly semiperfect, i.e., right $J(R)$-semiperfect. Since $J(R) \subseteq S_r$, it follows that R is right S_r-semiperfect.

(2) $\Rightarrow$ (1). Let K be a right ideal of R. Since R is semiperfect, $K = eR \oplus U$ with $e^2 = e$ and $U \subseteq J(R)$. Since R is right S_r-semiperfect, $U = fR \oplus V$ with $f^2 = f$ and $V \subseteq S_r$. Since $U \subseteq J(R)$, $f = 0$ and so $U = V \subseteq J(R) \cap S_r$. □

Lemma 2.10. *Let $e^2 = e \in R$ such that $\mathrm{Soc}\,(eR)$ is a maximal submodule of $(eR)_R$. If $K \subseteq eR$ is an idempotent right ideal, then $eR = K \oplus fR$ where $f^2 = f$ and $\mathrm{Soc}\,(fR)$ is a maximal submodule of $(fR)_R$.*

Proof. We can write $K = gR$ where $g^2 = g$. Then $eR = gR \oplus [(1-g)R \cap eR]$. Write $(1-g)R \cap eR = fR$ where $f^2 = f$. Then $eR = K \oplus fR$ and $\mathrm{Soc}\,(eR) = K \oplus \mathrm{Soc}\,(fR)$ is maximal in $K \oplus fR$. It follows that $\mathrm{Soc}\,(fR)$ is maximal in $(fR)_R$. □

A ring R is right Kasch if every simple right R-module embeds in R_R or, equivalently $\mathbf{1}(K) \neq 0$ for every maximal right ideal K. Analogously, one defines left Kasch rings.

Theorem 2.11. *The following are equivalent for a ring R:*

(1) *R is right $S_r \cap Z_r$-semiperfect.*

(2) *R is both right S_r-semiperfect and right Z_r-semiperfect.*

(3) *R is semiprimary and $J(R) = Z_r \subseteq S_r$.*

(4) *$R = S \oplus T$ where S is a semisimple artinian ring and T is a semiprimary ring with $J(T) = Z(T_T) = \mathrm{Soc}\,(T_T)$.*

In this case, $Z_l \subseteq Z_r = J(R) \subseteq S_r \subseteq S_l$, R is left Kasch, $J(R)^2 = 0$ and R satisfies ACC on left annihilators and ACC on right annihilators.

Proof. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (1). Let K be a right ideal of R. Since R is right Z_r-semiperfect, $K = eR \oplus U$ with $e^2 = e$ and $U \subseteq Z_r$. Since R is right S_r-semiperfect, $U = fR \oplus V$ with $f^2 = f$ and $V \subseteq S_r$. Since U is singular, $f = 0$ and so $U = V \subseteq S_r \cap Z_r$.

(2) $\Leftrightarrow$ (3). It follows from Corollary 2.9 and Theorem 2.5.

(2) and (3) $\Rightarrow$ (4). Since R is right S_r-semiperfect, by Theorem 2.3, there exists a decomposition $R = e_1R \oplus \cdots \oplus e_sR \oplus e_{s+1}R \oplus \cdots \oplus e_nR$ where $e_i^2 = e_i$ for all i, $(e_iR)_R$ is simple for $i = 1, \dots, s$, and $\mathrm{Soc}\,(e_iR)$ is maximal in $(e_iR)_R$ for $i = s+1, \dots, n$. Clearly (3) implies that R is semiprimary with $J(R)^2 = 0$. So, by [**19**, Lemma 4.10], R_R has ACC on direct summands. Therefore, because of Lemma 2.10, we can assume that, for each $s+1 \leq i \leq n$, $\mathrm{Soc}\,(e_iR)$ is nilpotent. So, $\mathrm{Soc}\,(e_iR) \subseteq J(R) \cap e_iR = J(e_iR) = e_iJ(R) = e_iZ_r = Z(e_iR)$. Since $\mathrm{Soc}\,(e_iR)$ is maximal in $(e_iR)_R$, $\mathrm{Soc}\,(e_iR) \supseteq J(e_iR)$. So, $\mathrm{Soc}\,(e_iR) = J(e_iR) = Z(e_iR)$ for $i = s+1, \dots, n$. Write $R = S \oplus T$ where $S = e_1R \oplus \cdots \oplus e_sR$ and $T = e_{s+1}R \oplus \cdots \oplus e_nR$. Then $Z_r = Z(e_{s+1}R) \oplus \cdots \oplus Z(e_nR)$ and $T = Z_2(R_R)$ is the second right singular ideal of R. Clearly $S \cdot Z_2(R_R) = Z_2(R_R) \cdot S = 0$. So, $R = S \oplus T$ is a ring direct sum and S is a semisimple artinian ring. Clearly $J(T_R) = J(T_T)$ and $\mathrm{Soc}\,(T_R) = \mathrm{Soc}\,(T_T)$ and it can be easily checked that $Z(T_R) = Z(T_T)$. Since $J(T_R) = Z(T_R) = \mathrm{Soc}\,(T_R)$, we have $J(T_T) = Z(T_T) = \mathrm{Soc}\,(T_T)$. So, $J(T_T)^2 = 0$. As seen above, $T/J(T) = T/\mathrm{Soc}\,(T_T)$ is semisimple artinian. Thus T is semiprimary.

To see the last statement, we have $Z_l \subseteq Z_r$ by [**21**, Theorem 1.2] since R is right Z_r-semiperfect. By (4), $Z_r = J(R) \subseteq S_r$ and R is semiprimary. Hence $J(R)^2 = 0$ and $S_l = \mathbf{r}(J(R)) = \mathbf{r}(Z_r) \supseteq S_r$. Thus, S_l is essential in R_R. By [**20**, Lemma 3.11], R is left Kasch. And it follows from [**19**, Lemma 4.10] that R has ACC on left annihilators and ACC on right annihilators. $\square$

Examples 2.12. (1) For any semisimple artinian ring S, $R = \{\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) : x, y \in S\}$ is an artinian ring with $J(R) = Z_r = S_r$, but R is not semisimple artinian.

(2) Let $Q = \Pi_{i=1}^{\infty} F_i$ where $F_i = \mathbf{Z}_4$ and R be the subring of Q generated by $\oplus_{i=1}^{\infty} 2F_i$ and 1_Q. R is semiprimary but not right artinian, and $\mathrm{Soc}\,(R) = J(R) = Z_r = (\oplus_{i=1}^{\infty} 2F_i) + 2\mathbf{Z} \cdot \mathbf{1}_Q$.

(3) Every right $S_r \cap Z_r$-semiperfect ring is right $J(R) \cap S_r$-semiperfect. The ring R in Example 1.8(2) is right $J(R) \cap S_r$-semiperfect, but is not right $S_r \cap Z_r$-semiperfect.

A ring R is a QF-*ring* if and only if R is left (or right) self-injective and left (or right) artinian. A ring R is called a *right* CS-*ring* if every right ideal is essential in a direct summand of R_R and a right CS-ring R is called *right continuous* if R is right C2, i.e., any right ideal isomorphic to a direct summand of R_R is itself a direct summand of R_R (see [**17**]). A right self-injective (respectively a left and right continuous) ring R such that R/S_r is right artinian or right noetherian is QF (see [**3**], [**4**] and [**14**]). Also right CS-rings R such that R/S_r is right artinian or right noetherian have been studied in [**10**]. Motivated by these results, we characterize below the right CS, right S_r-semiperfect rings. Following [**15**], a ring R is called a *right* CEP-*ring* if every cyclic right R-module can be essentially embedded in a projective module.

Theorem 2.13. *The following are equivalent for a ring R:*

(1) *R is right CS and R/S_r is semisimple artinian.*

(2) *R is right continuous, right artinian with $J(R)^2 = 0$.*

(3) *R is a right CEP-ring with $J(R)^2 = 0$.*

(4) *There exists a complete orthogonal set of idempotents $e_1, e_2, \dots, e_n$ such that all e_iR are indecomposable modules of composition length at most 2 and, for $i \neq j$, every isomorphism $\mathrm{Soc}\,(e_iR) \to \mathrm{Soc}\,(e_jR)$ extends to an isomorphism $e_iR \to e_jR$.*

(5) *$R = S \oplus T$ where S is a semisimple artinian ring and there exists a complete orthogonal set of idempotents $t_1, t_2, \dots, t_k$ in T such that all $(t_iT)_T$ are indecomposable modules of composition length 2 and, for $i \neq j$, every isomorphism $\mathrm{Soc}\,(t_iT)_T \to \mathrm{Soc}\,(t_jT)_T$ extends to an isomorphism $(t_iT)_T \to (t_jT)_T$.*

Proof. (1) $\Rightarrow$ (2). By [**10**, Lemma 4 and Corollary 6], R is right artinian. Then by Theorem 2.11, R is left Kasch and $J(R)^2 = 0$ and so R is a right C2-ring (see [**21**, Examples (7)]).

(2) $\Rightarrow$ (4). Suppose that (2) holds. Then R is semiperfect and so $R = e_1R \oplus \cdots \oplus e_nR$ where each $(e_iR)_R$ is indecomposable and $J(e_iR)$ is maximal in $(e_iR)_R$. It follows that $(e_iR)_R$ is uniform since R is right continuous. Thus, each $\mathrm{Soc}\,(e_iR)$ is simple since R is right artinian. Note that, since R is right artinian with $J(R)^2 = 0$, $J(R) \subseteq \mathbf{1}(J(R)) = S_r$. So $J(e_iR) \subseteq \mathrm{Soc}\,(e_iR) \subseteq e_iR$. If $\mathrm{Soc}\,(e_iR) = e_iR$, then e_iR has composition length 1. If $\mathrm{Soc}\,(e_iR) \neq e_iR$, then

$\operatorname{Soc}(e_iR) = J(e_iR)$ is maximal in e_iR. So e_iR has composition length 2. Let $f : \operatorname{Soc}(e_iR) \to \operatorname{Soc}(e_jR)$ be an R-isomorphism where $i \neq j$. Since R is right continuous, f extends to an R-homomorphism $g : e_iR \to e_jR$ and f^{-1} extends to an R-homomorphism $h : e_jR \to e_iR$ by [**17**, Proposition 2.10]. Both maps g and h must be one-to-one since f is an isomorphism. Since e_iR has composition length at most 2, g is an isomorphism.

(4) $\Rightarrow$ (5). Let e_i, $i = 1, \dots, n$, be as in (4). Set $S = \oplus\{e_iR : e_iR$ is simple$\}$ and $T = \oplus\{e_jR : e_jR$ has composition length 2$\}$. It can easily be proved that, if e_iR is simple (i.e., of composition length 1) and e_jR is of composition length 2, then $e_iR \cdot e_jR = 0 = e_jR \cdot e_iR$ and hence $R = S \oplus T$ is a direct sum of rings. The rest of (5) is clear.

(5) $\Rightarrow$ (4) is clear and (3) $\Rightarrow$ (2) is by [**20**, Theorem 5.8].

(4) $\Rightarrow$ (1). Suppose (4) holds. Then R is right S_r-semiperfect by Theorem 2.3. (4) also implies that, for $i \neq j$, e_iR is e_jR-injective. It follows from [**17**, Corollary 2.14] that R is right CS.

(2) and (4) $\Rightarrow$ (3). By [**20**, Theorem 5.8], it suffices to show that every right ideal of R is an annihilator. First we show that R is right Kasch. Let $\{e_1, \dots, e_n\}$ be given as in (4). Then, since R is semiperfect, it contains a basic set of idempotents, say $\{e_1, \dots, e_m\}$ where $m \leq n$. Thus, $e_iR \ncong e_jR$ if $i \neq j$ and $1 \leq i, j \leq m$. By (4), $\operatorname{Soc}(e_iR) \ncong \operatorname{Soc}(e_jR)$ if $i \neq j$ and $1 \leq i, j \leq m$. Hence, $\{\operatorname{Soc}(e_1R), \dots, \operatorname{Soc}(e_mR)\}$ is an irredundant set of representatives of the simple right R-modules. This shows that R is right Kasch. Let L be a maximal right ideal. Then R/L is isomorphic to a minimal right ideal of R. Thus, $(R/L) \cdot \mathbf{r}(S_r) = \bar{0}$, i.e., $\mathbf{r}(S_r) \subseteq L$ for any maximal right ideal L. Thus, $\mathbf{r}(S_r) \subseteq J(R)$. The other inclusion is clear. Therefore, $J(R) = \mathbf{r}(S_r)$. Next we show that every right ideal contained in $J(R)$ is an annihilator. Let K be such a right ideal. Since R is right CS, K is essential in eR where $e^2 = e \in R$. Then $\mathbf{rl}(K) \subseteq \mathbf{rl}(eR) = eR$. From $K \subseteq J(R)$, we see that $\mathbf{rl}(K) \subseteq \mathbf{rl}(J(R)) = \mathbf{r}(S_r) = J(R)$. But $J(R) \subseteq S_r$ by (2). It follows that $K \leq_e \mathbf{rl}(K) \subseteq S_r$. It must be that $K = \mathbf{rl}(K)$. Now we let I be a right ideal of R. Since R is semiperfect, $I = eR \oplus U$ where $e^2 = e \in R$ and $U \subseteq J(R)$. Then $\mathbf{rl}(I) = \mathbf{r}(R(1-e) \cap l(U)) \supseteq I$. If $x \in \mathbf{r}(R(1-e) \cap l(U))$, then $\mathbf{l}((1-e)U) \subseteq \mathbf{l}(1-e)x)$ and so $(1-e)U = \mathbf{rl}((1-e)U) \supseteq (1-e)xR$ (note $(1-e)U \subseteq J(R)$). Write $(1-e)x = (1-e)u$ where $u \in U$. Then

$x = e(x-u) + u \in I$. Therefore, $I = \mathbf{rl}\,(I)$. $\square$

We call a module M *socle-injective* if any homomorphism $f : S_r \to M$ extends to R or equivalently for any semisimple right ideal K of R, any homomorphism $f : K \to M$ extends to R.

Lemma 2.14. *Let R/S_r be semisimple artinian. Then a module M is socle-injective if and only if M is injective.*

Proof. Let M be socle-injective, and let $f : K \to M$ be an R-homomorphism where K is a right ideal of R. By Theorem 2.3, R is right S_r-semiperfect, and so $K = eR \oplus U$ where $e^2 = e$ and $U \subseteq S_r$. Write $K = eR \oplus V$ where $V = (1-e)R \cap K \cong U$ is semisimple. By the socle-injectivity, there exists $g : R_R \to R_R$ such that $g(x) = f(x)$ for all $x \in V$. Let $h : R_R \to R_R$ be defined by $h(er + (1-e)t) = f(er) + g((1-e)t)$. Then h extends f and thus M is injective. $\square$

Corollary 2.15. *The following are equivalent for a ring R:*

(1) *R is a QF-ring with $J(R)^2 = 0$.*

(2) *$(R \oplus R)_R$ is CS and R/S_r is semisimple artinian.*

(3) *R_R is socle-injective and R/S_r is semisimple artinian.*

(4) *R is right self-injective and R is a direct sum of indecomposable right ideals of composition length at most* 2.

(5) *$R = S \oplus T$ where S is a semisimple ring, T is right self-injective and is a direct sum of indecomposable right ideals of composition length* 2.

Since (1) *is left-right symmetric, these are also equivalent to the left versions of conditions* (2), (3), (4) *and* (5).

Proof. (1) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (3). By Theorem 2.13.

(3) $\Rightarrow$ (2). By Lemma 2.14, R_R is an injective, and so $(R \oplus R)_R$ is CS.

(2) $\Rightarrow$ (1). By Theorem 2.13, R is right artinian, right continuous with $J(R)^2 = 0$. Then by [**21**, Corollary 2.7], R is right self-injective.

Thus R is QF. $\square$

Next, we give another characterization of QF-rings R with $J(R)^2 = 0$. A ring R is said to satisfy (P1) if R_R is indecomposable of composition length 2 such that $(R/S_r)_R \cong (S_r)_R$. Clearly, such a ring is right self-injective if and only if every isomorphism $(S_r)_R \to (S_r)_R$ extends to an isomorphism $R_R \to R_R$. The ring $\mathbf{Z}_4$ satisfies (P1). A ring R is said to satisfy (P2) if $R = e_1R \oplus \cdots \oplus e_nR$ where $n > 1$ such that $e_iR \cong e_jR$ only if $i = j$ and, for each $1 \leq i \leq n$, $(e_iR)_R$ is an indecomposable module of composition length 2, and $e_iR/\mathrm{Soc}\,(e_iR) \cong \mathrm{Soc}\,(e_{\sigma(i)}R)$ where σ is an n-cycle. Clearly again, such a ring is right self-injective if and only if, for each i, every isomorphism $\mathrm{Soc}\,(e_iR) \to \mathrm{Soc}\,(e_iR)$ extends to an isomorphism $e_iR \to e_iR$. Note that there exist QF-rings R satisfying (P2) such that $J(R)^2 = 0$ (see [**16**, Examples (16.19), (5) and (6)]).

Corollary 2.16. *The following are equivalent for a ring R:*

(1) *R is a QF-ring with $J(R)^2 = 0$.*

(2) *R is Morita equivalent to a ring direct product $R_0 \oplus R_1 \oplus R_2$ where each R_i is right self-injective, R_0 is a direct sum of division rings, R_1 is a direct sum of rings satisfying* (P1) *and R_2 is a direct sum of rings satisfying* (P2).

Proof. Only need to show that (1) implies (2). Suppose that (1) holds. Since being a QF-ring with $J(R)^2 = 0$ is a Morita invariant and every semiperfect ring is Morita equivalent to its basic ring, it suffices to show that a basic ring S of a QF-ring R with $J(R)^2 = 0$ has the ring decomposition described as in (2). Since the ring S is basic, i.e., the identity is the sum of a basic set of primitive idempotents and is QF with $J(S)^2 = 0$, without loss of generality we can assume that R is itself a basic ring. So by (4) of Corollary 2.15, $R = e_1R \oplus \cdots \oplus e_mR$ where each e_iR is an indecomposable module of composition length at most 2 and $e_iR \cong e_jR$ only if $i = j$. By the injectivity and projectivity of these e_iR, we have

(a) $\mathrm{Soc}\,(e_iR) \cong \mathrm{Soc}\,(e_jR)$ if and only if $e_iR \cong e_jR$ if and only if $e_iR/\mathrm{Soc}\,(e_iR) \cong e_jR/\mathrm{Soc}\,(e_jR)$ and

(b) for $i \neq j$, $e_iR \cdot e_jR \neq 0$ implies $e_jR/\mathrm{Soc}\,(e_jR) \cong \mathrm{Soc}\,(e_iR)$.

Let $R_1 = \oplus\{e_iR : e_iR \text{ is simple}\}$, $R_2 = \oplus\{e_iR : e_iR/\mathrm{Soc}\,(e_iR) \cong \mathrm{Soc}\,(e_iR)\}$ and $R_3 = \oplus\{e_iR : e_iR \text{ is not simple and } e_iR/\mathrm{Soc}\,(e_iR) \ncong \mathrm{Soc}\,(e_iR)\}$. By (a) and (b), R_1, R_2 and R_3 all are ideals of R and so $R = R_1 \oplus R_2 \oplus R_3$ is a ring direct product. By (a), every e_iR in R_1 is an ideal of R_1 and so $R_1 = \oplus\{e_iR : e_iR \text{ is simple}\}$ is a ring direct sum with each e_iR a division ring.

By (a) and (b), every e_iR in R_2 is an ideal of R_2 and so $R_2 = \oplus\{e_iR : e_iR/\mathrm{Soc}\,(e_iR) \cong \mathrm{Soc}\,(e_iR)\}$ is a ring direct sum with each e_iR a ring satisfying (P1).

Choose $e_{i_1}R \subseteq R_3$. Again because of (a) and (b), there exists $e_{i_j}R \subseteq R_3$, $j = 1, \dots, t$, such that $e_{i_j}R/\mathrm{Soc}\,(e_{i_j}R) \cong \mathrm{Soc}\,(e_{i_{j+1}}R)$ for $j = 1, \dots, t-1$ and $e_{i_t}R/\mathrm{Soc}\,(e_{i_t}R) \cong \mathrm{Soc}\,(e_{i_1}R)$. If $A = \oplus\{e_{i_j}R : j = 1, \dots, t\}$ and $B = \oplus\{e_iR : e_iR \subseteq R_3 \text{ but } i \neq i_j \text{ for } j = 1, \dots, t\}$. From (a) and (b), $R_3 = A \oplus B$ is a ring direct product and A satisfies (P2). If $B \neq 0$, then a ring satisfying (P2) splits from B using the same process. And this process will ensure that R_3 is a direct sum of rings satisfying (P2). $\square$

Example 2.17 [**8**, p. 70]. Given a field F and an isomorphism $a \mapsto \bar{a}$ from $F \to \overline{F} \subseteq F$, let R be the right F-space on basis $\{1, t\}$ with multiplication given by $t^2 = 0$ and $at = t\bar{a}$ for all $a \in F$. Then R is a local ring, and the only right ideals are $0, J(R)$ and R. Hence R is a local, right artinian, right continuous, right dual ring (i.e., every right ideal is a right annihilator). It follows that $J(R) = Z_r = Z_l = S_r = S_l$ and that R/S_r is semisimple artinian. Moreover, R is right CEP by Theorem 2.13. But R is not left continuous if $\dim_{\overline{F}}(F) \geq 2$. Indeed, if R were left continuous, then, being local, it would be left uniform. But if X and Y are nonzero $\overline{F}$-subspaces of F with $X \cap Y = 0$, then $P = tX$ and $Q = tY$ are nonzero left ideals with $P \cap Q = 0$. R is left artinian when $\dim_{\overline{F}}(F) < \infty$ but is not left finitely dimensional when $\dim_{\overline{F}}(F) = \infty$.

Example 2.18 [**9**, p. 36]. Let $R = \mathbf{Z}_2[x_1, x_2, \dots]$ where $x_i^3 = 0$ for all i, $x_ix_j = 0$ for all $i \neq j$ and $x_i^2 = x_j^2 = m \neq 0$ for all i and j. Then R is a commutative local ring with $J(R) = \mathrm{span}\,\{m, x_1, x_2, \dots\}$, and R has a simple essential socle $J(R)^2 = \mathbf{Z}_2m$. In particular, R is uniform and so is CS; C2 also holds because $r(a) = 0$, $a \in R$, implies that a is a unit. Hence R is continuous. Thus, R is a commutative, local, continuous, semiprimary ring with $J(R)^3 = 0$, but R is not finite dimensional. Note that $\mathrm{Soc}\,(R) \subseteq J(R) = Z(R)$.

3. I-Perfect rings. Let I be an ideal of a ring R. Then R is called a *right I-perfect ring* if, for any submodule X of a projective module P, X has a decomposition $X = A \oplus B$ where A is a summand of P and $B \subseteq P \cdot I$. Note that R is right perfect if and only if R is right $J(R)$-perfect and R is semisimple artinian if and only if R is right (0)-perfect. The right δ_r-perfect rings are discussed in [**23**].

The next theorem is an improvement of [**23**, Theorem 3.8].

Theorem 3.1. *A ring R is right δ_r-perfect if and only if R/S_r is right perfect.*

Proof. By [**23**, Theorem 3.8], R is right δ_r-perfect if and only if R/S_r is right perfect and idempotents lift modulo δ_r and the latter is equivalent to the condition that R/S_r is right perfect by Lemma 1.3. $\square$

The next corollary is an interesting contrast to the fact that a semiperfect ring is not necessarily right perfect.

Corollary 3.2. *The following are equivalent for a ring R:*

(1) *R is right S_r-perfect.*

(2) *Every submodule X of a projective module P has a decomposition $X = A \oplus B$ where A is a summand of P and $B \subseteq \mathrm{Soc}\,(P)$.*

(3) *R is right S_r-semiperfect.*

Proof. (1) $\Leftrightarrow$ (2). This is because of the fact that $P \cdot S_r = \mathrm{Soc}\,(P)$ for any projective module P.

(2) $\Rightarrow$ (3). This is obvious.

(3) $\Rightarrow$ (2). Suppose that (3) holds. Then R/S_r is semisimple artinian by Theorem 2.3. Thus, R is right δ_r-perfect by Theorem 3.1 and $\delta_r = S_r$. So R is right S_r-perfect. $\square$

Proposition 3.3. *The following are equivalent for a ring R:*

(1) *R is right Z_r-perfect.*

(2) *Every submodule X of a projective module P has a decomposition $X = A \oplus B$ where A is a summand of P and $B \subseteq Z(P)$.*

(3) *R is right perfect and $J(R) = Z_r$.*

Proof. (1) $\Leftrightarrow$ (2). This is because of the fact that $P \cdot Z_r = Z(P)$ for any projective module P.

(3) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). By Theorem 2.5, $J(R) = Z_r$ and thus R is right perfect. $\square$

Examples 3.4. (1) Every right Z_r-perfect ring is right perfect. The ring R is Example 2.7(2) is right perfect but is not right Z_r-perfect.

(2) Every right Z_r-perfect ring is right Z_r-semiperfect. Let R be a dual ring which is not self-injective. Such rings exist by [**13**, Example 6.1]. By [**11**, Theorem 13], R is not right perfect. But clearly $Z_r = J(R)$ and, by [**13**, Theorem 3.9], R is semiperfect. So there exist right Z_r-semiperfect rings which are not right perfect, and hence not right Z_r-perfect.

Finally we note that R is right $J(R) \cap Z_r$-perfect if and only if R is right Z_r-perfect (by Proposition 3.3), R is right $J(R) \cap S_r$-perfect if and only if R is right $J(R) \cap S_r$-semiperfect (by Corollary 2.9), and R is right $S_r \cap Z_r$-perfect if and only if R is right $S_r \cap Z_r$-semiperfect (by Theorem 2.11).

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