ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 4, Winter 2002

# COTORSION THEORIES COGENERATED BY $\aleph_1$-FREE ABELIAN GROUPS

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ABSTRACT. Given an $\aleph_1$-free abelian group $G$ we characterize the class $\mathfrak{C}_G$ of all torsion abelian groups $T$ satisfying $\operatorname{Ext}(G,T) = 0$ assuming the special continuum hypothesis $CH$. Moreover, in Gödel's constructible universe we prove that this characterizes $\mathfrak{C}_G$ for arbitrary torsion-free abelian $G$. It follows that there exists some ugly $\aleph_1$-free abelian groups.

**1. Introduction.** In 1969 Griffith [**8**] solved Baer's splitting problem on mixed abelian groups when he proved that an abelian group $G$ is free if and only if $\operatorname{Ext}(G,T) = 0$ for all torsion abelian groups $T$. It is easy to see that an abelian group $G$ which satisfies $\operatorname{Ext}(G,T) = 0$ for all torsion abelian groups $T$ must be torsion-free and homogeneous of type $\mathbf{Z}$. Thus it was natural to ask whether or not one could extend Griffith's result to homogeneous torsion-free groups which are not necessarily of idempotent type, i.e., to ask whether a torsion-free abelian group $G$ which is homogeneous of type $R \subseteq \mathbf{Q}$ has to be completely decomposable if and only if $\operatorname{Ext}(G,T) = 0$ whenever $\operatorname{Ext}(R,T) = 0$ for all torsion groups $T$ (clearly $\operatorname{Ext}(G,T) = 0$ implies $\operatorname{Ext}(R,T) = 0$). That this is not the case was shown in [**10**] by the second author. This was a consequence of techniques and results obtained in [**12**]. Inspired by Baer's question [**1**] to characterize all pairs of torsion-free abelian $G$ and torsion abelian $T$ such that $\operatorname{Ext}(G,T) = 0$, Wallutis and the second author considered in [**12**] the torsion groups of the cotorsion class singly cogenerated by a torsion-free group $G$. Cotorsion theories were introduced by Salce in [**9**] but it was the first time in [**12**] that only the torsion groups of the cotorsion theory were considered. Recall that, for a torsion-free abelian group $G$, the class of all torsion abelian groups $T$ satisfying $\operatorname{Ext}(G,T) = 0$

2000 AMS *Mathematics Subject Classification.* 20K15, 20K20, 20K35, 20K40.
Publication 773 in the first author's list of publications. The first author was supported by Project No. G-0545-173,06/97 of the *German-Israeli Foundation for Scientific Research & Development.*
The second author was supported by a MINERVA fellowship.
Received by the editors on July 13, 2001, and in revised form on August 28, 2001.

is denoted by $\mathcal{TC}(G)$ (see [**12**]). This class is obviously closed under taking epimorphic images and contains all torsion cotorsion groups, i.e., all bounded groups. In [**12**] satisfactory characterizations of $\mathcal{TC}(G)$ were obtained for countable torsion-free abelian groups and for completely decomposable groups. In fact, it was proved in [**12**] that, for every countable torsion-free abelian group $G$, there exists a completely decomposable group $G$ such that $\mathcal{TC}(G) = \mathcal{TC}(C)$. It was later shown in [**10**] by the second author that, for every finite rank torsion-free abelian group $G$, there even exists a rational group $R \subseteq \mathbf{Q}$ such that $\mathcal{TC}(G) = \mathcal{TC}(R)$. Thus, knowing the class $\mathcal{TC}(C)$ for completely decomposable groups $C$, it was reasonable to search for groups $G$ of uncountable cardinality such that $\mathcal{TC}(G)$ equals $\mathcal{TC}(C)$ for some completely decomposable group $C$. Although a criterion was found in [**12**, Theorem 3.6] for characterizing those classes of torsion abelian groups which may appear as $\mathcal{TC}(C)$ for completely decomposable group $C$, it remained open if for instance in Gödel's universe every torsion-free abelian group is of this kind. It shall be shown in this paper that this is not the case, but it holds if we replace completely decomposable by $\aleph_1$-free of cardinality $\aleph_1$.

Assuming $CH$ we give in Section 2 a construction of $\aleph_1$-free abelian groups $G$ of size $\aleph_1$ having a strange class $\mathcal{TC}(G)$. It shall be proved that, for every ideal $I$ in the set of primes (more general in the set of all powers of primes) containing all finite subsets of the set of primes, there exists an $\aleph_1$-free abelian group $G$ of size $\aleph_1$ such that $\bigoplus_{p \in P} \mathbf{Z}(p) \in \mathcal{TC}(G)$ if and only if $P \in I$. It follows in Section 3 that in Gödel's constructible universe $(V = L)$ every torsion-free abelian group $G$ satisfies $\mathcal{TC}(G) = \mathcal{TC}(H)$ for some $\aleph_1$-free group $H$ of size $\aleph_1$. Thus, we obtain a characterization of the class $\mathcal{TC}(G)$ for all torsion-free abelian groups $G$ in Gödel's universe and prove that the structure of the group $G$ is not very much affected by the class $\mathcal{TC}(G)$. This solves Baer's problem in $V = L$ and contrasts a result from [**7**] in which it was shown that the cotorsion theory singly cogenerated by $G$ determines the group $G$ up to isomorphism in many cases.

All groups under consideration are abelian. The notations are standard and, for unexplained notions in abelian group theory and set theory, we refer to [**6**] and [**5**].

**2. The construction.** In this section we construct some $\aleph_1$-free abelian groups having special properties. Let us first recall a definition from [**12**]. For a torsion-free group $G$ we denote by $\mathcal{TC}(G)$ the class of all torsion groups $T$ satisfying $\operatorname{Ext}(G,T) = 0$. Obviously, the class $\mathcal{TC}(G)$ is closed under taking epimorphic images and contains all torsion cotorsion groups, i.e., all bounded groups. Recall that a torsion-free group $G$ is called $\aleph_1$-*free* if all its countable subgroups are free. Let $\Pi$ be the set of natural primes. By $\overline{\Pi}$ we denote the set of all powers of natural primes, i.e., $\overline{\Pi} = \{p^n : p \in \Pi, n < \omega\}$. Moreover, for an infinite subset $P \subseteq \overline{\Pi}$ we define $T_P = \bigoplus_{p \in P} \mathbf{Z}(p)$, where $\mathbf{Z}(p)$ denotes the cyclic group of order $p$. The reader should keep in mind that here $p$ is not necessarily a prime but could be a prime power. We begin with a compactness result for countable torsion-free groups (see [**10**, Lemma 3.1]).

**Lemma 2.1** ([**10**]). *Let $G$ be a countable torsion-free group and $T$ a torsion group. Then $T \in \mathcal{TC}(G)$ if and only if $T \in \mathcal{TC}(H)$ for all finite rank pure subgroups $H$ of $G$.*

*Proof*. The proof can be found in [**10**, Lemma 3.1] and is based on the fact that, for countable $G$ and any pure subgroup $H \subseteq G$ of finite rank, $T \in \mathcal{TC}(G)$ implies $T \in \mathcal{TC}(G/H)$ ($T$ a torsion group). $\square$

Recall that, for a torsion-free group $G$ of size $\kappa$, a $\kappa$-*filtration* of $G$ is a continuous ascending chain of pure subgroups of cardinality less than $\kappa$ such that its union equals $G$.

**Proposition 2.2** ($CH$). *Let $G$ be a torsion-free group of cardinality $\aleph_1$ and $P$ an infinite subset of $\overline{\Pi}$. If $\langle G_\alpha : \alpha < \omega_1 \rangle$ is an $\omega_1$-filtration of $G$, then $T_P \notin \mathcal{TC}(G)$ if and only if one of the following conditions holds*:

(i) *$T_P \notin \mathcal{TC}(H)$ for some finite rank pure subgroup $H$ of $G$ or*;

(ii) *$\{\delta < \omega_1 : G/G_\delta$ contains a finite rank pure subgroup $L_\delta \subseteq G/G_\delta$ such that $T_P \notin \mathcal{TC}(L_\delta)\}$ is stationary in $\omega_1$.*

*Proof.* Let

$$(2.1) \quad S = \big\{\delta < \omega_1 : G/G_\delta \text{ contains a finite rank pure subgroup } L_\delta \subseteq G/G_\delta \text{ such that } T_P \notin \mathcal{TC}(L_\delta)\big\}.$$

By Lemma 2.1, $S = \{\delta < \omega_1 : T_P \notin \mathcal{TC}(G_\beta/G_\delta)$ for some $\delta < \beta < \omega_1\}$. Now it is easy to see that $S$ stationary implies that the relative $\Gamma$-invariant $\Gamma_{T_P}(G) \neq 0$. Since we are assuming $CH$ the weak diamond $\Phi_{\aleph_1}$ holds (see [**3**]). Thus (i) or (ii) imply $T_P \notin \mathcal{TC}(G)$ by [**5**, Proposition XII.1.15]. Conversely, assume that $T_P \notin \mathcal{TC}(G)$ but (i) and (ii) do not hold. Then the relative $\Gamma$-invariant $\Gamma_{T_P}(G) = 0$ and hence [**5**, Theorem XII.1.14] shows that $T_P \in \mathcal{TC}(G)$ – a contradiction. □

*Remark* 2.3. If we assume $V = L$, then we could extend Proposition 2.2 to larger cardinalities using techniques as, for instance, developed in [**2**, Theorem 3.1] and using an appropriate filtration, but it is not needed here.

Let $S$ be a stationary subset of $\omega_1$ consisting of limit ordinals, i.e., for all $\alpha \in S$, cf $(a) = \omega$. Recall the following definition.

**Definition 2.4.** A ladder system $\bar{\eta}$ on $S$ is a family of functions $\bar{\eta} = \langle \eta_\delta : \delta \in S\rangle$ such that $\eta_\delta : \omega \to \delta$ is strictly increasing with $\sup(\mathrm{rg}\,(n_\delta)) = \delta$ where $\mathrm{rg}\,(\eta_\delta)$ denotes the range of $\eta_\delta$. We call the ladder system *tree-like* if, for all $\delta, \nu \in \bar{\eta}$ and every $\alpha, \beta \in \omega$, $\eta_\delta(\alpha) = \eta_\nu(\beta)$ implies $\alpha = \beta$ and $\eta_\delta(\rho) = \eta_\nu(\rho)$ for all $\rho \leq \alpha$.

**Proposition 2.5** ($CH$). *Let $\langle P_\alpha : \alpha < \omega_1\rangle$ be a sequence of infinite subsets of $\overline{\Pi}$. Then there exists an $\aleph_1$-free torsion-free group $G$ of cardinality $\aleph_1$ such that, for any infinite subset $P$ of $\overline{\Pi}$, $T_P \notin \mathcal{TC}(G)$ if and only if $\{\delta < \omega_1 : |P \cap P_\delta| = \aleph_0\}$ is stationary.*

*Proof.* Since we are assuming $CH$, the weak diamond $\Phi_{\aleph_1}$ holds. Let $S$ be a stationary subset of $\omega_1$ such that $\Phi_{\aleph_1}(S)$ holds. Since $\lim(\omega_1)$ is a cub in $\omega_1$, we may assume without loss of generality that $S = \lim(\omega_1)$, i.e., $S$ consists of all limit ordinals of $\omega_1$. Choose a tree-like ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ such that $\eta_\delta(\alpha)$ is a successor ordinal for all $\alpha < \omega$ and $\delta \in S$ (see [**5**, p. 386, Exercise 17]). We enumerate the sets $P_\alpha$ by $\omega$ without repetitions, e.g., $P_\alpha = \{p_{\alpha,n} : n < \omega\}$. Now let $F$ be the free group generated by the elements $\{x_\nu : \nu < \omega_1\} \cup \{y_{\delta,n} : \delta \in S, n < \omega\}$. Let $z_{\delta,-1} = y_{\delta,0}/p_{\delta,0}$ and, for $n \geq 0$,

$$z_{\delta,n} = (y_{\delta,0} - w_{\delta,n}) \Big/ \Big( \prod_{i=0}^{n+1} p_{\delta,i} \Big),$$

where $\omega_{\delta,n} = \sum_{i=0}^{n} (\prod_{j=0}^{i} p_{\delta,j}) x_{\eta_\delta(i)}$. Let $G$ be the subgroup of $F$ generated by the elements $\{x_\nu : \nu < \omega_1\} \cup \{z_{\delta,n} : \delta \in S, n < \omega\}$. Then the only relations between the generators of $G$ are

$$p_{\delta,n+1} z_{\delta,n} = z_{\delta,n-1} - x_{\eta_\delta(n)} \tag{2.2}$$

for $\delta \in S$ and $n \geq 0$. Now for $\nu < \omega_1$, let $G_\nu$ be the pure closure in $G$ of $G \cap (\{x_\mu : \mu < \nu\} \cup \{z_{\delta,n} : \delta \in S \cap \nu, n < \omega\})$. Then the sequence $\langle G_\nu : \nu < \omega_1 \rangle$ forms an $\omega_1$-filtration of $G$. Moreover, for $\nu \in S$ we have

$$G_{\nu+1}/G_\nu \cong F_\nu \oplus H_\nu, \tag{2.3}$$

where $F_\nu$ is the free group on the generator $x_\nu + G_\nu$ and $H_\nu \cong \langle 1/p_{\nu,n} : n < \omega \rangle =: R_{P_\nu} \subseteq \mathbf{Q}$. Finally $G$ is $\aleph_1$-free by Pontryagin's criterion. Indeed, if $J_0$ is a finite subset of $G$, then the pure closure of $J_0$ is contained in the pure closure of a finite subset $J_1$ of $\{x_\nu : \nu < \omega_1\} \cup \{y_{\delta,n} : \delta \in S, n < \omega\}$. By enlarging $J_1$ we may assume that there exists $m$ such that, for all $y_{\delta,0} \in J_1$, $x_{\eta_\delta(n)} \in J_1$ if and only if $n \leq m$. Then the equations (2.2) show that the pure closure of $J_1$ is free (compare [**5**, Example VIII 1.1]).

Finally, let $P$ be an infinite subset of $\overline{\Pi}$. Since $G$ is $\aleph_1$-free there exists no finite rank pure subgroup $H$ of $G$ such that $T_P \notin \mathcal{TC}(H)$, hence Proposition 2.2 shows that $T_P \notin \mathcal{TC}(G)$ if and only if the set

$$N = \big\{ \delta < \omega_1 : G/G_\delta \textit{ contains a finite rank pure subgroup } L_\delta \subseteq G/G_\delta \textit{ such that } T_P \notin \mathcal{TC}(L_\delta) \big\} \tag{2.4}$$

is stationary in $\omega_1$. Since for $\delta \in S$ we have $G_{\delta+1}/G_\delta \cong R_{P_\delta}$ it is now easy to see that $N$ is stationary if and only if $U = \{\delta < \omega_1 : |P \cap P_\delta| = \aleph_0\}$ is stationary. Note that $S$ is a cub in $\omega_1$. $\square$

If $G$ is a torsion-free group, then it is not hard to see that the set

$$\{P \subseteq \overline{\Pi} : T_P \in \mathcal{TC}(G)\} \tag{2.5}$$

forms an ideal of $\mathcal{P}(\overline{\Pi})$ containing all finite subsets of $\overline{\Pi}$. In fact, the next theorem shows that every such ideal may appear. To avoid additional notation let us allow an ideal in $\mathcal{P}(\overline{\Pi})$ to contain $\overline{\Pi}$ itself.

**Theorem 2.6** $(CH)$. *Let $I \subseteq \mathcal{P}(\overline{\Pi})$ be an ideal containing all finite subsets of $\overline{\Pi}$. Then there exists an $\aleph_1$-free group $G$ of cardinality $\aleph_1$ such that for every $P \subseteq \overline{\Pi}$, $T_P \in \mathcal{TC}(G)$ if and only if $P \in I$.*

*Proof.* Let $I$ be given. If $\overline{\Pi} \in I$, then we choose $G$ to be free of cardinality $\aleph_1$, and we are done. Therefore, assume that $\overline{\Pi} \notin I$. Choose a continuous increasing sequence of boolean subalgebras $\langle B_\alpha \subseteq \mathcal{P}(\overline{\Pi}) : \alpha < \omega_1\rangle$ such that each $B_\alpha$ is countable and contains all finite subsets of $\overline{\Pi}$. Note that this is possible since we are assuming $CH$. Let $\alpha < \omega_1$ and put

$$I \cap B_\alpha = \{I^-_{\alpha,i} : i < \omega\} \tag{2.6}$$

and

$$B_\alpha \setminus I = \{I^+_{\alpha,i} : i < \omega\}, \tag{2.7}$$

where we assume that each $I^+_{\alpha,i}$ is repeated infinitely many times. Choose for $\alpha < \omega_1$ and $i < \omega$

$$p_{\alpha,i} \in I^+_{\alpha,i} \setminus \Big(\bigcup\{I^-_{\alpha,j} : j < i\} \cup \{i_{\alpha,j} : j < i\}\Big). \tag{2.8}$$

Note that this is possible since $I^+_{\alpha,i}$ is infinite and $\bigcup\{I^-_{\alpha,j} : j < i\} \in I$. Let

$$P_\alpha = \{p_{\alpha,i} : i < \omega\} \tag{2.9}$$

and let $G$ be the group from Proposition 2.5 for $\langle P_\alpha : \alpha < \omega_1\rangle$. Then $G$ is $\aleph_1$-free and of cardinality $\aleph_1$ and, by Proposition 2.5, it suffices to prove that for a subset $P \subseteq \overline{\Pi}$ we have $P \in I$ if and only if there exists $\gamma < \omega_1$ such that, for all $\delta > \gamma$, $P \cap P_\alpha$ is finite. Thus, let $P \subseteq \overline{\Pi}$

and assume that $P \in I$. Then there exists $\gamma < \omega_1$ such that, for all $\delta > \gamma$, $P \in I \cap B_\delta$. Fix $\delta > \gamma$. Then $P = I^-_{\delta,i}$ for some $i < \omega$. Hence, for all $j > i$ we obtain $p_{\delta,j} \notin I^-_{\delta,i}$. Thus $P \cap P_\delta \subseteq \{p_{\delta,j} : j \leq i\}$ which is finite. Conversely, assume that $P \notin I$. Then there exists $\gamma < \omega_1$ such that, for all $\delta \in \gamma$, $P \in B_\delta \setminus I$. Fix $\delta > \gamma$, then $P = I^+_{\delta,j}$ for infinitely many $j < \omega$ by the choice of the $I^+_{\delta,i}$s. But, if $P = I^+_{\delta,j}$, then $p_{\delta,j} \in P_\delta \cap (P \setminus \{p_{\delta,i} : i < j\})$ and hence $P \cap P_\delta$ is infinite. This finishes the proof. □

We are now able to characterize the class $\mathcal{TC}(G)$ for torsion-free groups of cardinality $\aleph_1$ assuming $CH$.

**3. The characterization.** In [**12**, Theorem 3.6 (v)] a characterization of all classes $\mathfrak{C}$ of torsion groups was given which could satisfy $\mathfrak{C} = \mathcal{TC}(C)$ for some completely decomposable group $C$. We shall show next that we can drop condition [**12**, Theorem 3.6 (v)] if we assume $CH$ and replace completely decomposable by $\aleph_1$-free cardinality $\aleph_1$. Recall that condition [**12**, Theorem 3.6 (v)] says the following

(3.1) *If $P$ is an infinite set of primes such that $T_P \notin \mathfrak{C}$, then there exists an infinite subset $P'$ of $P$ such that for all infinite subsets $X$ of $P'$, $T_X \notin \mathfrak{C}$.*

**Theorem 3.1** ($CH$). *Let $\mathfrak{C}$ be a class of torsion groups. Then $\mathfrak{C} = \mathcal{TC}(G)$ for some ($\aleph_1$-free) torsion-free group $G$ of cardinality less than or equal to $\aleph_1$ if and only if the following conditions are satisfied*:

(i) $\mathfrak{C}$ *is closed under epimorphic images*;

(ii) $\mathfrak{C}$ *contains all torsion cotorsion groups*;

(iii) *If $p$ is a natural prime, then $\bigoplus_{n<\omega} \mathbf{Z}(p^n) \in \mathfrak{C}$ if and only if $\mathfrak{C}$ contains all $p$-groups*;

(iv) *If $P$ is an infinite subset of* $\Pi$, *then $\bigoplus_{p\in P} \mathbf{Z}(p) \in \mathfrak{C}$ if and only if $\bigoplus_{p\in P} H_p \in \mathfrak{C}$ for all $p$-groups $H_p \in \mathfrak{C}$, $p \in P$.*

*Proof*. Let us first show that (i)–(iv) hold for $\mathcal{TC}(G)$ for any torsion-free group $G$ of cardinality less than or equal to $\aleph_1$. Clearly (i) and (ii) are true. Moreover, if $G$ is countable, then [**12**, Corollary 3.7] shows that (iii) and (iv) hold for $G$. Thus assume that $G$ is of cardinality $\aleph_1$, and let $\langle G_\alpha : \alpha < \omega_1 \rangle$ be an $\omega_1$-filtration of $G$. Let $p$ be a prime and assume that $\bigoplus_{n<\omega} \mathbf{Z}(p^n) \in \mathcal{TC}(G)$. Moreover, assume that $T$ is a $p$-group and $T \notin \mathcal{TC}(G)$. By Proposition 2.2 there exists either a finite rank pure subgroup $H$ of $G$ such that $T \notin \mathcal{TC}(H)$ or the set $Q = \{\delta < \omega_1 : G/G_\delta$ *contains a finite rank pure subgroup* $L_\delta \subseteq G/G_\delta$ *such that* $T \notin \mathcal{TC}(L_\delta)\}$ is stationary in $\omega_1$. If $H$ exists, then [**12**, Theorem 3.6] shows that $\bigoplus_{n<\omega} \mathbf{Z}(p^n) \notin \mathcal{TC}(H)$, contradicting the fact that $\bigoplus_{n<\omega} \mathbf{Z}(p^n) \in \mathcal{TC}(G) \subseteq \mathcal{TC}(H)$. Thus assume that $Q$ is stationary in $\omega_1$. Again by [**12**, Theorem 3.6], it follows that for $\delta \in Q$ also $\bigoplus_{n<\omega} \mathbf{Z}(p^n) \notin \mathcal{TC}(L_\delta)$ since all $G_\alpha$'s are countable. Thus,

$$Q = \Big\{\delta < \omega_1 : G/G_\delta \textit{ contains a finite rank pure subgroup } L_\delta \subseteq G/G_\delta \textit{ such that } \bigoplus_{n<\omega} \mathbf{Z}(p^n) \notin \mathcal{TC}(L_\delta)\Big\} \tag{3.2}$$

and Proposition 2.2 shows that $\bigoplus_{n<\omega} \mathbf{Z}(p^n) \notin \mathcal{TC}(G)$ – a contradiction. Thus (iii) holds since the converse implication is trivial.

It is straightforward to see that also (iv) holds for $\mathcal{TC}(G)$ using similar arguments as above.

Finally, assume that $\mathfrak{C}$ satisfies (i) to (iv). We identify $\omega$ with $\overline{\Pi}$ by a bijection $i : \omega \to \overline{\Pi}$. Let $I = \{X \subseteq \omega : \bigoplus_{p \in i(X)} \mathbf{Z}(p) \in \mathfrak{C}\}$. Then it is easy to see that $I$ is an ideal on $\omega$ containing all finite subsets of $\omega$. Thus by Theorem 2.6 there exists an $\aleph_1$-free group $G$ of cardinality $\aleph_1$ such that, for every subset $P \subseteq \overline{\Pi}$, $\bigoplus_{p\in P} \mathbf{Z}(p) \in \mathcal{TC}(G)$ if and only if $i^{-1}(P) \in I$. Since we have already shown that $\mathcal{TC}(G)$ satisfies (i) to (iv), it is now obvious that $\mathfrak{C} = \mathcal{TC}(G)$. □

Since it was shown in [**11**, Theorem 2.7] and [**12**, Corollary 3.9] that in Gödel's universe for every torsion-free group $G$, Theorem 3.1 (i)–(iv) are satisfied for $\mathfrak{C} = \mathcal{TC}(G)$, we immediately get the following result.

**Corollary 3.2** ($V = L$). *For every torsion-free group $G$, there exists an $\aleph_1$-free group $H$ of cardinality $\aleph_1$ such that $\mathcal{TC}(G) = \mathcal{TC}(H)$.*

Moreover, we obtain the existence of some ugly torsion-free groups showing that the $\mathcal{TC}$-conjecture from [**11**, $\mathcal{TC}$-Conjecture 2.12] does not hold. In [**11**] it was conjectured that in $V = L$ for every torsion-free group $G$, there exists a completely decomposable group $C$ such that $\mathcal{TC}(G) = \mathcal{TC}(C)$, hence condition (3.1) would be satisfied for all torsion-free groups $G$. This is not the case.

**Corollary 3.3** ($CH$). *For every infinite set of primes $P$, there exists an $\aleph_1$-free torsion-free group $G$ of cardinality $\aleph_1$ satisfying $T_P \notin \mathcal{TC}(G)$ such that for every infinite subset $Q \subseteq P$, there exists an infinite subset $Q_1 \subseteq Q$ such that $T_{Q_1} \in \mathcal{TC}(G)$. Thus $\mathcal{TC}(G) \neq \mathcal{TC}(C)$ for every completely decomposable group $C$.*

*Proof.* Let $P$ be the given infinite set of primes. It was shown by Eda in [**4**, Theorem 5, Proof] that there exists a strictly decreasing chain of subsets $P_\alpha \subseteq P$, $\alpha < \omega_1$, such that

(i) $P_\alpha$ is infinite;

(ii) $\alpha < \beta$ implies $P_\beta$ is almost contained in $P_\alpha$;

(iii) $\alpha < \beta$ implies $|P_\alpha \setminus P_\beta|$ is infinite;

(iv) $\bigcap_{\alpha<\omega_1} P_\alpha$ is infinite.

Let $U$ be the ultrafilter generated by $\overline{P} = \{P_\alpha : \alpha < \omega_1\}$ and let $G$ be the group from Proposition 2.5 for $\overline{P}$. If $Q$ is an infinite subset of $P$, then divide $Q$ into two disjoint infinite subsets, e.g., $Q = Q_1 \cup Q_2$. Since $U$ is an ultrafilter, it follows that without loss of generality $Q_1 \notin U$. Hence, there exists $\alpha < \omega_1$ such that $|P_\alpha \cap Q_1|$ is finite. Thus, for every $\alpha \leq \beta$, we obtain $|P_\beta \cap Q_1|$ is finite. Therefore the set $\{\delta < \omega_1 : |P \cap P_\delta| = \aleph_0\}$ is not stationary in $\omega_1$ and Proposition 2.5 implies that $T_{Q_1} \in \mathcal{TC}(G)$.

Finally $\mathcal{TC}(G) \neq \mathcal{TC}(C)$ for any completely decomposable group $C$ since $\mathcal{TC}(G)$ violates [**12**, Theorem 3.2 (v)] which is our condition (3.1). □

## REFERENCES

**1.** R. Baer, *The subgroup of the elements of finite order of an abelian group*, Ann. of Math. **37** (1936), 766–781.

**2.** T. Becker, L. Fuchs and S. Shelah, *Whitehead modules over domains*, Forum Math. **1** (1989), 53–68.

**3.** K.J. Devlin and S. Shelah, *A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$*, Israel J. Math. **29** (1978), 239–247.

**4.** K. Eda, *A characterization of $\aleph_1$-free abelian groups and its application to the Chase radical*, Math. Ann. **261** (1982), 359–385.

**5.** P.C. Eklof and A.H. Mekler, *Almost free modules—Set-theoretic methods*, North-Holland, Amsterdam, 1990.

**6.** L. Fuchs, *Infinite abelian groups*, Vol. I and II, Academic Press, New York, 1970 and 1973.

**7.** R. Göbel, S. Shelah and S.L. Wallutis, *On the lattice of cotorsion theories*, J. Algebra **238** (2001), 292–313.

**8.** P. Griffith, *A solution to the splitting mixed group problem of Baer*, Trans. Amer. Math. Soc. **139** (1969), 261–269.

**9.** L. Salce, *Cotorsion theories for abelian groups*, Sympos. Math., vol. 23, 1979, Acad. Press, London, pp. 11–32.

**10.** L. Strüngmann, *Torsion groups in cotorsion theories*, Rend. Sem. Mat. Univ. Padova **107** (2002).

**11.** ———, *On problems by Baer and Kulikov using $V = L$*, submitted.

**12.** L. Strüngmann and S.L. Wallutis, *On the torsion groups in cotorsion classes*, AGRAM 2000 Conference (Perth, West. Australia), Contemp. Math., vol. 273, Amer. Math. Soc., Providence, RI, 2001, pp. 269–283.

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