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ISOTYPE WARFIELD SUBGROUPS OF GLOBAL WARFIELD GROUPS

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ABSTRACT. Using a new characterization of global Warfield groups, necessary and sufficient conditions are given for an isotype subgroup of a global Warfield group to be itself, Warfield. Our result generalizes similar theorems in the simpler contexts of totally projective p-groups and p-local Warfield groups.

1. Introduction. We shall be dealing exclusively with additively written abelian groups, hereafter referred to simply as "groups," and G will always denote such a group. We emphasize from the outset that G is allowed to be mixed.

Recall that a group is *simply presented* if it can be presented by generators and relations where each relation involves at most two generators. In the torsion and torsion-free settings, a summand of a simply presented group is again simply presented. However, for mixed groups G this is not generally the case. By definition, a *global Warfield group* is a direct summand of a simply presented group. Most of the early theory of global Warfield groups was developed by Hunter, Richman and Walker [**8, 9, 10**]. But it was not until the introduction of knice subgroups [**3**] and the attainment of an Axiom 3 characterization [**4**] that fundamental problems regarding isotype subgroups became accessible (for prime examples, see [**4**] and [**11**]).

In this paper we again demonstrate the power of the theory of knice subgroups and Axiom 3 by finding necessary and sufficient conditions for an isotype subgroup of a global Warfield group to be itself Warfield. Our result generalizes the earlier treatments of isotype subgroups of totally projective p-groups in [**2**], and of p-local Warfield groups in [**7**]. To a certain extent, our theorem and proof are modeled after the special case in [**7**]; however, the generalization from the local to

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the global case is not routine. Indeed, in addition to making further modifications to the concepts of separability and compatibility, we require the new characterization of global Warfield groups established by Theorem 2.5 below. Moreover, even when specialized to the local case, our main theorem (Theorem 5.2) is more extensive than the corresponding theorem in [**7**]. Since a torsion-free group is simply presented if and only if it is completely decomposable, Theorem 5.2 also specializes to yield a new characterization of when a pure subgroup of a completely decomposable torsion-free group is itself completely decomposable.

In the sequel, unexplained notation and terminology will be as in [**3, 4**] and [**11**]. In particular, we direct the reader to those papers for the definitions and basic properties of knice subgroups and the associated notions of primitive element and $*$-valuated coproduct. As is now customary, we use the notation $\|x\|$ for the height matrix of $x \in G$. If p is a prime, $|x|_p$ denotes the height of x at the prime p and $\|x\|_p = \{|p^i x|_p\}_{i<\omega}$ is the height sequence determined by the p-row of $\|x\|$. When necessary to avoid confusion, we affix superscripts to this notation to emphasize the group in which heights are computed. Also, the ordering of the class of ordinals, with the symbol ∞ adjoined as a maximal element, induces in a pointwise manner the lattice relation $\leq$ on height matrices and sequences. (In order to deal with groups that are not necessarily reduced, we adopt the convention $\infty < \infty$.)

2. A new characterization of global Warfield groups.

Our new characterization of global Warfield groups is a consequence of the following general result regarding the structure of knice subgroups of global k-groups. Recall that G is a (*global*) *k-group* if the trivial subgroup 0 is a knice subgroup.

Theorem 2.1. *If M is a knice subgroup of a global k-group G and if S is a finite subset of M, then there exists a $*$-valuated coproduct*

$$N = \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_s \rangle \subseteq M$$

where each y_i is a primitive element of G and $\langle S, N \rangle / N$ is finite.

Proof. Since G is a k-group, there is a $*$-valuated coproduct

$$F_0 = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_k \rangle$$

where each x_i is a primitive element of G and $\langle S, F_0\rangle/F_0$ is finite. Replacing each x_i by a nonzero multiple, if necessary, we may assume that F_0 is contained in a $*$-valuated coproduct $M \oplus B_0$ where $M \oplus B_0$ is a knice subgroup of G and B_0 is a finitely generated knice subgroup of G. With this beginning, repeated applications of [**4**, Proposition 1.6] allow us to define inductively two ascending sequences $\{T_n\}_{n<\omega}$ and $\{S_n\}_{n<\omega}$ of finite subsets of G, together with a family $\{B_n\}_{n<\omega}$ of finitely generated knice subgroups of G such that all the following conditions are satisfied for all $n < \omega$.

(1) $T_0 = \{x_1, x_2, \dots, x_k\}$, each T_n consists of primitive elements of G and $F_n = \langle T_n\rangle = \bigoplus_{x\in T_n}\langle x\rangle$ is a $*$-valuated coproduct.

(2) For each $n < \omega$, $M \oplus B_0 \oplus B_1 \oplus \cdots \oplus B_n$ is a $*$-valuated coproduct and is a knice subgroup of G that contains F_n.

(3) $F_n \subseteq \langle S_n\rangle$ and $S_n = (S_n \cap M) \cup (S_n \cap (B_0 \oplus B_1 \oplus \cdots \oplus B_n))$.

(4) $\langle S_n, F_{n+1}\rangle/F_{n+1}$ is finite.

Now let $F = \bigcup_{n<\omega} F_n$, $T = \bigcup_{n<\omega} T_n$ and $B = \bigoplus_{n<\omega} B_n$. Then clearly $F = \bigoplus_{x\in T}\langle x\rangle$ and $M \oplus B$ are $*$-valuated coproducts with F "quasi-splitting" along M and B in the sense that F is a torsion modulo $(F \cap M) \oplus (F \cap B)$.

By [**4**, Corollary 2.5], there is a $*$-valuated coproduct

$$F' = \langle z_1\rangle \oplus \langle z_2\rangle \oplus \cdots \oplus \langle z_k\rangle \oplus M' \oplus B'$$

where F/F' is torsion, $M' \subseteq M$, $\operatorname{rank}((F \cap M)/M') \le k$ and the $z_1, z_2, \dots, z_k$ are nonzero multiples of $x_1, x_2, \dots, x_k$, respectively. Let $F_0' = \langle z_1\rangle \oplus \langle z_2\rangle \oplus \cdots \oplus \langle z_k\rangle$ and observe that, since both F_0/F_0' and $\langle S, F_0\rangle/F_0$ are finite, it follows that $\langle S, F_0'\rangle/F_0'$ is finite. Note also that $F' \cap M = M' \oplus A$ where $A = M \cap (F_0' \oplus B')$. Since $S \subseteq M$, $\langle S\rangle \cap A = \langle S\rangle \cap (F_0' \oplus B')$ and the finiteness of

$$\langle S, A\rangle/A \cong \langle S, F_0' \oplus B'\rangle/(F_0' \oplus B')$$

is a consequence of $\langle S, F_0'\rangle/F_0'$ being finite. Moreover, $s = \operatorname{rank}(A) \le K$ because $A \cong (F' \cap M)/M'$. Using the fact that F/F' is torsion, it is easily verified that F' is also "quasi-splitting" along M and B. Hence, F' is torsion modulo

$$(F' \cap M) \oplus (F' \cap B) = M' \oplus A \oplus (F' \cap B)$$

and therefore, by [**4**, Proposition 2.2, Corollary 2.4] applies to yield a $*$-valuated coproduct $N = \langle y_1\rangle \oplus \langle y_2\rangle \oplus \cdots \oplus \langle y_s\rangle$ where each y_i is a primitive element of G contained in $A \subseteq M$ and A/N is torsion. Finally, since $\langle S, A\rangle/A$ is finite and A/N is torsion, it follows that $\langle S\rangle/(\langle S\rangle \cap N)$ is both torsion and finitely generated; that is, $\langle S, N\rangle/N$ is finite as desired. □

Recall that a subgroup N of G is a *nice* subgroup if, for all primes p and ordinals σ, the cokernel of the canonical map

$$(p^\sigma G + N)/N \longmapsto p^\sigma(G/N)$$

contains no element of order p. Both here and in Section 4, it will be convenient to have the following characterization of knice subgroups.

Lemma 2.2 [**4**, Proposition 1.7]. *A subgroup N of a group G is a knice subgroup if and only if the following conditions are satisfied.*

(a) *N is a nice subgroup of G.*

(b) *G/N is a global k-group.*

(c) *To each $g \in G$ there corresponds positive integer m such that the coset $mg + N$ contains an element x with $\|x\|^G = \|mg + N\|^{G/N}$.*

Corollary 2.3. *Suppose that N is a pure knice subgroup of G. If A is any subgroup of N, then N/A is a pure knice subgroup of G/A.*

Proof. Since N is both knice and pure, [**4**, Corollary 1.10] says that $p^\sigma(G/N) = (p^\sigma G + N)/N$ for all primes p and ordinals σ. It is now routine to verify that N/A is nice in G/A. Indeed, even more is true:

$$p^\sigma((G/A)/(N/A)) = \frac{p^\sigma(G/A) + (N/A)}{N/A}$$

for all primes p and ordinals σ. Moreover, $(G/A)/(N/A) \cong G/N$ is a k-group by Lemma 2.2. Also, N/A inherits from N the properties required by condition (c) of Lemma 2.2 since $\|mg + A + (N/A)\|^{(G/A)/(N/A)} = \|mg + N\|^{G/N}$ for all $g \in G$ and positive integers m. Because it is clear that N/A is pure in G/A, the proof is complete. □

As a crucial step in the proof of Theorem 2.5, we show that, for *pure* knice subgroups M, Theorem 2.1 generalizes to include *countable* sets S.

Proposition 2.4. *If M is a pure knice subgroup of a global k-group G, and if S is a countable subset of M, then there exists a $*$-valuated coproduct*

$$N = \bigoplus_{y \in Y} \langle y \rangle \subseteq M$$

where Y is a countable set of primitive elements of G and $\langle S/N \rangle / N$ is torsion.

Proof. Select an ascending family

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq \cdots \quad (n < \omega)$$

of finite subsets of S with $S = \bigcup_{n<\omega} S_n$. It suffices to show that there exists an ascending sequence

$$Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_n \subseteq \cdots \quad (n < \omega)$$

where, for each $n < \omega$, Y_n is a finite set of primitive elements of G,

$$N_n = \bigoplus_{y \in Y_n} \langle y \rangle \subseteq M$$

is a $*$-valuated coproduct and $\langle S_n, N_n \rangle / N_n$ is finite. Proceeding by induction, we assume that Y_n has been constructed with the desired properties and demonstrate how to construct a suitable Y_{n+1}. Let $T = S_{n+1} \setminus S_n$ and $\overline{T} = \{t + N_n : t \in T\}$. Note that M/N_n is a knice subgroup of G/N_n by Corollary 2.3 and, since N_n is knice in G, G/N_n is a k-group by Lemma 2.2. Therefore, Theorem 2.1 implies that there is a finite subset $\{y_1', y_2', \dots, y_s'\}$ of M such that $\overline{N} = \bigoplus_{i=1}^{s} \langle y_i' + N_n \rangle$ is a $*$-valuated coproduct in G/N_n with each $y_i' + N_n$ primitive in G/N_n and $\langle \overline{T}, \overline{N} \rangle / \overline{N}$ finite. As noted in the proof of Theorem 3.7 in [**3**], N_n being knice in G allows us to replace each $y_i' + N_n$ by a positive multiple and then select an appropriate y_i in each coset so that each y_i

is a primitive element of G with $N_{n+1} = N_n \oplus \langle y_1\rangle \oplus \langle y_2\rangle \oplus \cdots \oplus \langle y_s\rangle$ a $*$-valuated coproduct. Set $Y_{n+1} = Y_n \cup \{y_1, y_2, \dots, y_s\}$. Clearly then

$$N_{n+1} = \bigoplus_{y\in Y_{n+1}} \langle y\rangle \subseteq M$$

is a $*$-valuated coproduct and $\langle S_{n+1}, N_{n+1}\rangle/N_{n+1}$ is finite. □

We are now in a position to establish the main result of this section.

Theorem 2.5. *If*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots \quad (\alpha < \mu)$$

is a smooth chain of pure knice subgroups of the group G such that $|M_{\alpha+1}/M_\alpha| < \aleph_0$ for all α and $\bigcup_{\alpha<\mu} M_\alpha = G$, then G is a global Warfield group.

Proof. We may assume that μ is the first ordinal of cardinality $|G|$ and, to avoid trivialities, that each quotient $M_{\alpha+1}/M_\alpha$ has infinite torsion subgroup. Note from the hypotheses that G is at least a k-group since we are given that $M_0 = 0$ is a knice subgroup.

We shall show that the chain of M_α's that can be refined to a smooth chain of knice subgroups

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\beta \subseteq \cdots \qquad (\beta < \mu)$$

with the additional property that, for each β, either $N_{\beta+1}/N_\beta$ is cyclic of prime order, or $N_{\beta+1} = N_\beta \oplus \langle x_\beta\rangle$ is a $*$-valuated coproduct with x_β a primitive element of G. Once this is achieved, we may apply Theorem 3.2 in [**4**] to conclude that G is a global Warfield group. The refinement will be accomplished with each $M_\alpha = N_\beta$ for some limit ordinal β and the proof requires an explanation of how, for each α, one interpolates between M_α and $M_{\alpha+1}$ an ascending sequence of knice subgroups

$$M_\alpha = N_\beta \subseteq N_{\beta+1} \subseteq \cdots \subseteq N_{\beta+n} \subseteq \cdots \quad (n < \omega)$$

satisfying the requisite properties and with $M_{\alpha+1} = \bigcup_{n<\omega} N_{\beta+n}$.

We consider first the interpolation between M_0 and M_1. By Proposition 2.4, there is a countable set Y of primitive elements such that $N = \bigoplus_{y\in Y}\langle y\rangle$ is a $*$-valuated coproduct and M_1/N is a countable torsion group. If Y is finite, say $Y = \{x_1, x_2, \dots , x_s\}$, then the subgroups $N_0 = 0$, $N_1 = \langle x_1\rangle$, $N_2 = N_1 \oplus \langle x_2\rangle, \dots,$ $N_s = N_{s-1} \oplus \langle x_s\rangle$ are knice since G is a k-group, and we select an ascending sequence

$$N_{s+1} \subseteq N_{s+2} \subseteq \cdots \subseteq N_{s+n} \subseteq \cdots$$

of subgroups with $M_1 = \bigcup_{n<\omega} N_n$ and N_{i+1}/N_i cyclic of prime order for $i \geq s$. Since finite extensions of knice subgroups are knice, this establishes the desired interpolation in the special case where Y is finite. Suppose, however, that $Y = \{y_1, y_2, \dots , y_n, \dots\}$ is infinite and choose an ascending sequence of subgroups

$$0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n \subseteq \cdots \quad (n < \omega)$$

having M_1 as its union and such that, for each $n \geq 1$, B_n contains the finitely generated knice subgroup $A_n = \langle y_1, y_2, \dots , y_n\rangle$ with B_n/A_n finite. Thus, for each n, B_n is a knice subgroup of G and, by Proposition 2.7 in [**3**], there is a positive multiple x_n of y_n such that $B_{n-1}\oplus\langle x_n\rangle$ is a $*$-valuated coproduct. We then choose an ascending chain of subgroups $N_0 = 0$, $N_1 = \langle x_1\rangle$, $N_2, \dots , N_{k_1} = B_1$ where the successive quotients N_{i+1}/N_i are cyclic of prime order for $i \geq 1$. Then $N_{k_1+1} = B_1 \oplus \langle x_2\rangle$ is $*$-valuated with x_2 primitive. Continuing in this manner, we generate an ascending sequence of knice subgroups

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$$

having M_1 as its union and an associated increasing sequence of positive integers

$$1 < k_1 < k_2 < \cdots < k_i < \cdots$$

such that, for each i, $N_{k_i} = B_i$, $N_{k_i+1} = B_i \oplus \langle x_{i+1}\rangle$ is a $*$-valuated coproduct with x_{i+1} primitive and N_{n+1}/N_n is cyclic of prime order whenever n is distinct from each k_i. Finally, by remaining each x_{i+1} as x_{k_i}, we have the desired interpolation between M_0 and M_1 in the case where Y is infinite.

We assume now that, for some limit ordinal $\lambda < \mu$, we have obtained knice subgroups $\{N_\beta\}_{\beta<\lambda}$ with the requisite properties and such that

$\bigcup_{\beta<\lambda} N_\beta = M_\alpha$ for some $\alpha < \mu$. It remains to explain how we continue the process by interpolating between M_α and $M_{\alpha+1}$. Since $M_{\alpha+1}/M_\alpha$ is a pure knice subgroup of G/M_α by Corollary 2.3, taking $N_\lambda = M_\alpha$ and applying the construction of the preceding paragraph in the k-group G/N_λ, we obtain an ascending sequence

$$N_\lambda \subseteq N_{\lambda+1} \subseteq N_{\lambda+2} \subseteq \cdots \subseteq N_{\lambda+n} \subseteq \cdots \quad (n < \omega)$$

of subgroups of G such that $\bigcup_{n<\omega} N_{\lambda+n} = M_{\alpha+1}$, each $N_{\lambda+n}/N_\lambda$ is a knice subgroup of G/N_λ and, for each n, either

(1) $(N_{\lambda+n+1}/N_\lambda)/(N_{\lambda+n}/N_\lambda) \cong N_{\lambda+n+1}/N_{\lambda+n}$ is cyclic of prime order, or else

(2) $N_{\lambda+n+1}/N_\lambda = (N_{\lambda+n}/N_\lambda) \oplus \langle x_{\lambda+n} + N_\lambda\rangle$ is a $*$-valued coproduct with $x_{\lambda+n} + N_\lambda$ a primitive element of G/N_λ.

It follows from [**3**, Theorem 3.7] that each $N_{\lambda+n}$ is a knice subgroup of G. Therefore, as in the proof of Proposition 2.4, in case (2) we have a $*$-valuated coproduct $N' = N_{\lambda+n} \oplus \langle x'_{\lambda+n}\rangle$, where $x'_{\lambda+n}$ is a primitive element of G with $x'_{\lambda+n} + N_\lambda$ a positive multiple of $x_{\lambda+n} + N_\lambda$. Thus, by interpolating when necessary, a finite number of subgroups with successive quotients of prime order between each N' and the corresponding $N_{\lambda+n+1}$, we obtain (after reindexing as needed) the desired knice subgroups $\{N_\beta\}_{\lambda\leq\beta<\lambda+\omega}$ between M_α and $M_{\alpha+1}$. Finally the requirement that the N_β form a smooth chain completes the construction. □

Remarks. Certainly the converse of Theorem 2.5 is true. That this is so is an immediate consequence of [**4**, Theorem 3.2] and the fact that each countable subgroup of G is contained in a countable pure subgroup. Hence, it will remain a mystery as to why the hypotheses of Theorem 2.5 do not appear among the six conditions of [**4**, Theorem 3.2], all of which are equivalent to G being a global Warfield group. We should also mention that the simpler p-local versions of Theorems 2.1 and 2.5 appear as [**6**, Theorem 3.1] and [**7**, Proposition 9], respectively.

3. Separability.

Definition 3.1. Call a subgroup H of G strongly separable in G if, for each $g \in G$, there is a corresponding countable subset $\{h_n\}_{n<\omega} \subseteq H$

with the following property: if $h \in H$, there exists an $n < \omega$ such that $\|g+h\| \leq \|g+h_n\|$.

Lemma 3.2. *Suppose that H is an isotype subgroup of G and that H contains a valuated coproduct $H' = \bigoplus_{i \in I} H_i$ with $|H_i| \leq \aleph_0$ for all $i \in I$. Then H' is a strongly separable subgroup of G.*

Proof. Suppose to the contrary that H' is not strongly separable in G. Therefore, we can select and fix a $g \in G$ with the property that, for each countable subset C of H', there exists $h_C \in H'$ such that the inequality $\|g+h_C\| \leq \|g+x\|$ fails for all $x \in C$.

We claim that there exist subsets $J(n)$ of I that satisfy the following two conditions.

(1) $J(0) = \varnothing$, $J(n)$ is a countable subset of I for each $n < \omega$, and

$$J(0) \subseteq J(1) \subseteq J(2) \subseteq \cdots \subseteq J(n) \subseteq \cdots \quad (n < \omega)$$

(2) If $h' \in \bigoplus_{i \in J(n)} H_i$ for some $n < \omega$ and if $|p^k(g+h)|_p \gneqq |p^k(g+h')|_p$ for some $h \in H'$, prime p and nonnegative integer k, there exists $h'' \in \bigoplus_{i \in J(n+1)} H_i$ such that $|p^k(g+h'')|_p \gneqq |p^k(g+h')|_p$.

To verify the claim, suppose we have constructed a finite sequence

$$J(0) \subseteq J(1) \subseteq \cdots \subseteq J(m)$$

of countable subsets of I so that condition (2) holds for all $n < m$. Now to extend the sequence to $J(m+1)$, let $\mathcal{S}$ be the set of all triples $s = (h', p, k)$ such that $h' \in \bigoplus_{i \in J(m)} H_i$, p is a prime, k a nonnegative integer, and the inequality $|p^k(g+h)|_p \gneqq |p^k(g+h')|_p$ holds for some $h \in H'$. For each such s, select and fix a single $h_s \in H'$ such that $|p^k(g+h_s)|_p \gneqq |p^k(g+h')|_p$. Since g is fixed and $\mathcal{S}$ is countable, there is a countable subset J' of I such that $h_s \in \bigoplus_{i \in J'} H_i$ for all $s \in \mathcal{S}$. Observe now that $J(m+1) = J' \cup J(m)$ has the desired properties to establish the claim.

To complete the proof, set $J(\omega) = \bigcup_{n<\omega} J(n)$ and $H_\omega = \bigoplus_{i \in J(\omega)} H_i$. Since H_ω is a countable subgroup of H', there is an $h \in H'$ such that the inequality $\|g+h\| \leq \|g+x\|$ fails for all $x \in H_\omega$. In particular

$\|g+h\| \leq \|g+h'\|$ fails where h' is the projection of h onto H_ω. Therefore,

$$|p^k(g+h)|_p \gneqq |p^k(g+h')|_p$$

for some prime p and nonnegative integer k. Since $h' \in \bigoplus_{i \in J(n)} H_i$ for some n, condition (2) above implies that there exists $h'' \in H_\omega$ such that

$$|p^k(g+h'')|_p \gneqq |p^k(g+h')|_p.$$

We now have

$$\begin{aligned} |p^k(h'-h'')|_p &= |p^k(g+h') - p^k(g+h'')|_p \\ &= |p^k(g+h')_p \lneqq |p^k(g+h) - p^k(g+h'')|_p \\ &= |p^k(h-h'')|_p. \end{aligned}$$

Thus, since H is isotype in G,

$$|p^k(h'-h'')|_p^H \lneqq |p^k(h-h'')|_p^H.$$

However this last inequality is absurd. Indeed, as a consequence of the facts that $p^k(h'-h'')$ is the projection of $p^k(h-h'') \in H'$ onto H_ω and that

$$H' = H_\omega \oplus \Big(\bigoplus_{i \in I \setminus J(\omega)} H_i \Big)$$

is a valuated coproduct in H, we obtain $|p^k(h'-h'')|_p^H \geq |p^k(h-h'')|_p^H$. $\square$

Recall that a subset X of a group H is a *decomposition basis* for H if each $x \in X$ has infinite order and $\langle X \rangle = \bigoplus_{x \in X} \langle x \rangle$ is a valuated coproduct for which $H/\langle X \rangle$ is torsion. Since each $\langle x \rangle$ is countable, the next result follows immediately from Lemma 3.2.

Corollary 3.3. *Suppose H is an isotype subgroup of G. If H has a decomposition basis X, then $\langle X \rangle$ is a strongly separable subgroup of G.*

Definition 3.4. Let p be a prime. Call a subgroup H of G p-separable if, for each $g \in G$, there is a corresponding countable subset $\{h_n\}_{n<\omega}$ of H with the following property: if $h \in H$, there is an $n < \omega$ such that $|g+h|_p \leq |g+h_n|_p$.

If a subgroup H of G is p-separable in G for all primes p, we say that H is *locally separable* in G.

Definition 3.5. Suppose H is a subgroup of G. Call H *almost strongly separable* in G if H is locally separable in G and, for each $g \in G$, there is a corresponding countable subset $\{h_n\}_{n<\omega}$ of H with the following property: if $h \in H$ there is an $n < \omega$ and a positive integer m such that $\|m(g+h)\| \leq \|m(g+h_n)\|$.

In contrast to the torsion-free and p-local settings, it is shown in [**5**] that a global Warfield group is not necessarily a strongly separable subgroup of every group in which it appears as an isotype subgroup. However, we do have the following positive result.

Proposition 3.6. *Suppose H is an isotype subgroup of G. If H is a global Warfield group, then H is almost strongly separable in G.*

Proof. By a result of [**12**], H is at least locally separable in G. To complete the proof, recall that any global Warfield group has a decomposition basis. In particular, we can select a decomposition basis X for H. If $g \in G$, Corollary 3.3 implies that there is a countable subset D of $\langle X \rangle$ such that, for any positive integer k and $y \in \langle X \rangle$, there exists $d \in D$ with $\|kg + y\| \leq \|kg + d\|$.

For each positive integer k, set $D_k = \{h' \in H : kh' \in D\}$. Define an equivalence relation on D_k by decreeing that h'_1 and h'_2 in D_k are equivalent if and only if $kh'_1 = kh'_2$. Let C_k be a set of representatives for the distinct equivalence classes in D_k. Since D is countable, so are C_k and $C = \bigcup_{1\leq k<\omega} C_k$. Now, for a given $h \in H$, there is a positive integer m with $mh \in \langle X \rangle$ and so there is a $d \in d$ such that $\|m(g+h)\| \leq \|mg + d\|$. Clearly $d \in mG \cap H = mH$. Thus, there exists $d' \in H$ with $md' = d$; hence, $d' \in D_m$ so that $md' = mx$ for some $x \in C_m \subseteq C$. Therefore, $\|m(g+h)\| \leq \|m(g+x)\|$ with $x \in C$. We conclude that H is almost strongly separable in G. $\square$

Remark. It is also true that an isotype global Warfield subgroup H of G is strongly locally separable. That is, for each $g \in G$ and prime p, there is a countable subset $\{h_n\}_{n<\omega} \subseteq H$ such that, for every $h \in H$, there is an $n < \omega$ with $\|g+h\|_p \leq \|g+h_n\|_p$. However, this fact will not be needed.

4. Compatibility.

Definition 4.1. Let H and N be subgroups of C, and suppose that p is a prime. If, for each pair $(h,x) \in H \times N$ there is a corresponding $x' \in H \cap N$ with $|h+x|_p \leq |h+x'|_p$, then H and N are called *p-compatible*.

If the subgroups H and N of G are p-compatible for all primes p, we say that H and N are *locally compatible*.

Lemma 4.2. *Suppose that K is a subgroup of G and that N is a pure nice subgroup of G. If K and N are locally compatible, then the following conditions are satisfied.*

(i) *If M is a subgroup of G that contains N, then M/N is locally compatible with $(K+N)/N$ if and only if M is locally compatible with K.*

(ii) *If K is isotype in G, then $(K+N)/N$ is isotype in G/N and $K \cap N$ is knice in K.*

Proof. Since N is both knice and pure, [**4**, Corollary 1.10] says that $p^\sigma(G/N) = (p^\sigma G + N)/N$ for all primes p and ordinals σ. It then follows that the lemma essentially reduces to the local case where only one prime is relevant; and, in the local case, (i) and (ii) are well known. □

Definition 4.3. Let H and N be subgroups of G. If H and N are locally compatible and, if for each pair $(h,x) \in H \times N$ there is a corresponding $x' \in H \cap N$ and a positive integer m with $\|m(h+x)\| \leq \|mh+x'\|$, then we say that H and N are *almost strongly compatible*.

Note that almost strong compatibility is a symmetric relation. Moreover, it is inductive in the sense that if $N = \bigcup_{\alpha<\mu} N_\alpha$ is an ascending union of subgroups of G such that each N_α is almost strongly compatible with a fixed subgroup H, then N is almost strongly compatible with H. Our next three propositions are the generalized global versions of local results established in [**7**].

Proposition 4.4. *Suppose that H is an isotype subgroup of G and that N is a pure knice subgroup of G. If $H/(H \cap N)$ is a global k-group and if H and N are almost strongly compatible, then $H \cap N$ is a knice subgroup of H.*

Proof. By Lemma 4.2, $H \cap N$ is a nice subgroup of H. Since $H/(H \cap N)$ is a k-group, by hypothesis it remains to show condition (c) of Lemma 2.2; that is, if $h \in H$, then there is a positive integer m such that the coset $mh + (H \cap N)$ contains an element x with $\|x\|^H = \|mh + (H \cap N)\|^{H/(H\cap N)}$.

So now suppose that $h \in H$. Since N is knice in G, by Lemma 2.2 there exists a positive integer n and an element $y \in nh + N$ such that $\|y\|^G = \|nh + N\|^{G/N}$. Write $y = nh + z'$ for some $z' \in N$. Clearly $z' \in nG$. Thus, because N is pure in G, $z' = nz$ for some $z \in N$. We now apply the hypothesis that H and N are almost strongly compatible to obtain $x' \in H \cap N$ and a positive integer k such that $\|kn(h + z)\|^G \le \|knh + x'\|^G = \|knh + x'\|^H$. We now set $x = knh + x' \in H$ and $m = kn$. Then certainly $x \in mh + (H \cap N)$. Moreover, $\|mh + (H \cap N)\|^{H/(H\cap N)} \le \|mh + N\|^{G/N} = \|ky\|^G = \|mh + kz'\|^G = \|kn(h + z)\|^G \le \|knh + x'\|^H = \|x\|^H$. Therefore, $\|x\|^H = \|mh + (H \cap N)\|^{H/(H\cap N)}$, as desired. □

Proposition 4.5. *Suppose that H is a subgroup of G and that N is a pure knice subgroup of G that is almost strongly compatible with H. If M is a subgroup of G that contains N and if M/N is almost strongly compatible with $(H+N)/N$ in G/N, then M and H are almost strongly compatible in G.*

Proof. By Lemma 4.2(i), at least H and M are locally compatible. Now suppose that $h \in H$ and $x \in M$. To complete the proof we need to show that there exist a positive integer m and an $x' \in H \cap M$ such that $\|m(h + x)\| \le \|mh + x'\|$.

Because $(H+N)/N$ is almost strongly compatible with M/N, $\|k(h+x) + N\| \le \|(kh + c) + N\|$ for some positive integer k and $c \in$

$(H+N)\cap M = (H\cap M)+N$. Thus, without loss, we may assume that $c \in H\cap M$. Since N is knice in G, Lemma 2.2 applies to produce a $y \in N$ such that $\|n(kh+c)+y\| = \|n(kh+c)+N\|$ for some positive integer n. Note that $y \in nG$. Therefore, since N is pure in G, $y = ny'$ for some $y' \in N$. We now have

$$\begin{aligned}\|nk(h+x)\| \le \|nk(h+x)+N\| &\le \|n(kh+c)+N\| \\ &= \|n(kh+c)+y\| = \|n((kh+c)+y')\|.\end{aligned}$$

Since $kh+c \in H$ and $y' \in N$, the fact that H and N are almost strongly compatible implies that $\|l((kh+c)+y')\| \le \|l(kh+c)+z\|$ for some $z \in H\cap N$ and positive integer l. Hence, $\|nl((kh+c)+y')\| \le \|n(l(kh+c)+z\|$. If we now take $m = nlk$ and $x' = nlc + nz$, then $x' \in H\cap M$ and

$$\begin{aligned}\|m(h+x)\| = \|l(nk(h+x))\| &\le \|l(n((kh+c)+y'))\| \\ &\le \|n(l(kh+c)+z)\| = \|mh+x'\|\end{aligned}$$

as desired. □

As the final result of this section, we establish a link between the notions of almost strong compatibility and almost strong separability.

Proposition 4.6. *Suppose H is almost strongly separable in G. If A is a countable subgroup of G, then there is a countable subgroup B of G that contains A and is almost strongly compatible with H.*

Proof. Since H is almost strongly separable in G, for each $a \in A$ there is a countable subgroup $C_a \subseteq H$ such that the following conditions are satisfied.

(1) For each prime p and $h \in H$, there exists $c_p \in C_a$ such that $|a+h|_p \le |a+c_p|_p$.

(2) For each $h \in H$, there exists $c' \in C_a$ such that $\|m(a+h)\| \le \|m(a+c')\|$ for some positive integer m.

Set $C = (\bigcup_{a\in A} C_a)$ and $A_1 = \langle A, C\rangle$. Notice that A_1 is countable and, for each $a \in A$, the following conditions are satisfied.

(1′) For each prime p and $h \in H$, there exists $c \in H\cap A_1$ such that $|a+h|_p \le |a+c|_p$.

(2′) For each $h \in H$, there exists $c' \in H \cap A_1$ such that $\|m(a+h)\| \leq \|ma + c'\|$ for some positive integer m.

By repeated applications of the above construction, we obtain an ascending sequence

$$A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots \quad (n < \omega)$$

of countable subsets of G with the property that, for any $a \in A_n$, $h \in H$ and prime p, there exist elements c and c' in $H \cap A_{n+1}$ such that $|a+h|_p \leq |a+c|_p$ and $\|m(a+h)\| \leq \|ma+c'\|$ for some positive integer m. Then $B = \bigcup_{n<\omega} A_n$ is countable and is almost strongly compatible with H. □

Remark 4.7. For later use, we note that Proposition 4.6 is also true if "almost strongly separable" and "almost strongly compatible" are replaced by "locally separable" and "locally compatible," respectively.

5. The main theorem. Recall that a collection $\mathcal{C}$ of subgroups of G is an Axiom 3 system for G if the following three conditions are satisfied.

($\mathcal{G}0$) $\mathcal{C}$ contains the trivial subgroup 0.

($\mathcal{G}1$) $\mathcal{C}$ is closed under unions of ascending chains.

($\mathcal{G}2$) For each $N \in \mathcal{C}$ and countable subgroups A of G, there exists $M \in \mathcal{C}$ such that $N + A \subseteq M$ and M/N is countable.

As is shown in [**4**, Theorem 3.2], a group G is a global Warfield group if and only if G has an Axiom 3 system $\mathcal{C}$ consisting of knice subgroups.

Definition 5.1. Suppose that H is a subgroup of a global Warfield group G.

(I) If there exists an Axiom 3 system $\mathcal{C}$ of knice subgroups of G such that, for each $N \in \mathcal{C}$, $(H + N)/N$ is a global k-group that is almost strongly separable in G/N, then we say that G *satisfies Axiom* 3 *over* H *with respect to almost strongly separable* k*-groups.*

(II) If there exists an Axiom 3 system $\mathcal{C}$ of knice subgroups of G such that, for each $N \in \mathcal{C}$, $(H + N)/N$ is locally separable in G/N, then we say that G satisfies Axiom 3 over H with respect to locally separable groups.

Theorem 5.2. *For an isotype subgroup H of a global Warfield group G, the following statements are equivalent.*

(1) *H is a global Warfield group.*

(2) *G satisfies Axiom* 3 *over H with respect to almost strongly separable k-groups.*

(3) *H has a decomposition basis and G satisfies Axiom 3 over H with respect to locally separable groups.*

Proof. We begin by showing that (1) implies (2). Since, assuming (1), both G and H are Warfield groups, there exist an Axiom 3 system $\mathcal{C}_G$ of knice subgroups for G and an Axiom 3 system $\mathcal{C}_H$ of knice subgroups for H. Now introduce $\mathcal{C}$ as that family containing all $N \in \mathcal{C}_G$ that satisfy the following three conditions:

(a) N is a pure subgroup of G.

(b) $H \cap N \in \mathcal{C}_H$.

(c) N is almost strongly compatible with H.

We shall show that $\mathcal{C}$ is the desired Axiom 3 system satisfying condition (I) of Definition 5.1. Indeed, since H/M is a global Warfield group for all $M \in \mathcal{C}_H$ and $(H+N)/N \cong H/(H \cap N)$, it follows that $(H+N)/N$ a Warfield group for each $N \in \mathcal{C}$. In particular, $(H+N)/N$ is a k-group. Moreover, $(H+N)/N$ is isotype in G/N by Lemma 4.2(ii), and hence $(H+N)/N$ is almost strongly separable in G/N by Proposition 3.6. It remains to verify that $\mathcal{C}$ is an Axiom 3 system for G. But the collection of all Axiom 3 systems for G is closed under finite intersections [**1**, Lemma 1.2] and so we may deal with the three conditions (a), (b) and (c) separately. As the set of all pure subgroups of G is easily seen to be an Axiom 3 system, there is no difficulty with condition (a). Likewise, the proof of Lemma 1.5 in [**1**] shows that the collection of all $N \in \mathcal{C}_G$ with $H \cap N \in \mathcal{C}_H$ is an Axiom 3 system for G. Finally, since almost strong compatibility is an inductive relation, it suffices to show that if $N \in \mathcal{C}$ and A is a countable subgroup of G, there is an M in $\mathcal{C}_G$ such that M contains $A+N$, M/N is countable and M is almost strongly compatible with H. By a simple interlacing argument using property ($\mathcal{G}$2) and Proposition 4.6, we obtain an $M \in \mathcal{C}_G$ satisfying the first two of these conditions and with M/N almost strongly compatible with $(H+N)/N$. Then, since N is almost strongly compatible with H, and we may now assume condition (a), M is almost strongly compatible with H by Proposition 4.5.

We next turn to the proof that (2) implies (1). So suppose that $\mathcal{C}$ is an Axiom 3 system of knice subgroups for G with the property that, for each $N \in \mathcal{C}$, $(H+N)/N$ is a k-group that is almost strongly separable in G/N. We restrict attention to the subfamily $\mathcal{C}_G$ consisting of all $N \in \mathcal{C}$ such that

(a) N is a pure subgroup of G.

(b) $H \cap N$ is pure in H.

(c) N is almost strongly compatible with H.

To verify that $\mathcal{C}_G$ is an Axiom 3 system, we may once again deal with the three conditions (a), (b) and (c) separately. But (a) and (c) are handled exactly as in the preceding paragraph; while condition (b) again follows from the proof of Lemma 1.5 in [**1**] and the fact that pure subgroups of H form an Axiom 3 system for H. Now, from the Axiom 3 system $\mathcal{C}_G$, we extract a smooth chain

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots \quad (\alpha < \mu)$$

where $G = \bigcup_{\alpha<\mu} N_\alpha$ and, for all $\alpha < \mu$, $N_\alpha \in \mathcal{C}_G$ and $|N_{\alpha+1}/N_\alpha| \leq \aleph_0$. Then, for all α, $H/(H \cap N_\alpha) \cong (H+N_\alpha)/N_\alpha$ is a k-group and N_α is a pure knice subgroup of G that is almost strongly compatible with H. It then follows from Proposition 4.4 that each $H \cap N_\alpha$ is knice in H. Also, each $H \cap N_\alpha$ is pure in H by condition (b). Therefore,

$$0 = H \cap N_0 \subseteq H \cap N_1 \subseteq \cdots \subseteq H \cap N_\alpha \subseteq \cdots \quad (\alpha < \mu)$$

is a smooth chain of pure knice subgroups of H, where each link in the chain has countable index in the next and $H = \bigcup_{\alpha<\mu}(H \cap N_\alpha)$. Theorem 2.5 now applies to show that H is a global Warfield group.

At this point it should be clear that (2) implies (3). Indeed, if (2) holds, the argument above shows that H is a global Warfield group and hence has a decomposition basis. Moreover, condition (2) implies directly that G satisfies Axiom 3 over H with respect to locally separable groups.

Finally we complete the proof of Theorem 5.2 by showing that condition (3) implies (1). Assuming (3), select a decomposition basis

X for H, and let $\mathcal{C}$ be an Axiom 3 system of knice subgroups for G that witnesses to the fact that G satisfies Axiom 3 over H with respect to locally separable groups. We first introduce the family $\mathcal{C}'$ consisting of all $N \in \mathcal{C}$ that satisfy to four conditions:

(a) N is a pure subgroup of G.

(b) $H \cap N$ is pure in H.

(c) N is locally compatible with H.

(d) $(H \cap N)/\langle X_N \rangle$ is torsion, where $X_N = (H \cap N) \cap X = N \cap X$.

We claim that $\mathcal{C}'$ is an Axiom 3 system for G. Conditions (a) and (b) can be dealt with exactly as in the proof that (2) implies (1); while (c) can be handled by an interlacing argument using property $(\mathcal{G}2)$, Lemma 4.2 and Remark 4.7. As it is routine to verify that the family of all subgroups N of G that satisfy condition (d) forms an Axiom 3 system for G, we conclude that $\mathcal{C}'$ is indeed an Axiom 3 system for G.

Throughout this paragraph, N will denote a *fixed* group in C' that is locally compatible with $\langle X \rangle$. Note that Lemma 4.2 implies that $H \cap N$ is a nice subgroup of H. We maintain that $H \cap N$ is actually knice in H. To see this, it is enough to show that $(H \cap N) \oplus \langle Y_N \rangle$ is a $*$-valuated coproduct in H where $Y_N = X \setminus X_N$. But this will be the case if $(H \cap N) \oplus \langle Y_N \rangle$ is a valuated coproduct since $\langle X_N \rangle \oplus \langle Y_N \rangle$ is a $*$-valuated coproduct and $(H \cap N)/\langle X_N \rangle$ is torsion. However, that $(H \cap N) \oplus \langle Y_N \rangle$ is a valuated coproduct is an easy consequence of the fact that $H \cap N$ is locally compatible in H with $\langle X \rangle$ (where the latter follows from H being isotype in G and $\langle X \rangle \subseteq H$). Therefore, as claimed, $H \cap N$ is knice in H. Furthermore, by Remark 1.8 in [**4**], $H/(H \cap N)$ has a decomposition basis consisting of all the cosets $y + (H \cap N)$ with $y \in Y_N$. Then, by the canonical isomorphism $H/(H \cap N) \cong (H + N)/N$, the corresponding cosets $y + N$ form a decomposition basis for the group $(H + N)/N$. Since $(H + N)/N$ is an isotype subgroup of G/N by Lemma 4.2, $\langle y + N : y \in Y_N \rangle = (\langle X \rangle + N)/N$ is strongly separable in G/N by Corollary 3.3. Since $\mathcal{C}'$ is an Axiom 3 system, a simple interlacing argument exploiting Remark 4.7 yields an $M \in \mathcal{C}'$ that contains N and any given countable subgroup A of G where M/N is countable and locally compatible with $(\langle X \rangle + N)/N$. Because N is locally compatible with $\langle X \rangle$, the same applies to M by Lemma 4.2.

Now take $\mathcal{C}_G$ to be the family of all $N \in \mathcal{C}'$ that are locally compatible with $\langle X \rangle$. Since local compatibility is an inductive relation, the argument in the preceding paragraph allows us to conclude that $\mathcal{C}_G$ is an Axiom 3 system for G. To complete the proof that condition (3) implies (1), extract from the Axiom 3 system $\mathcal{C}_G$ a smooth chain

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots \quad (\alpha < \mu)$$

where $G = \bigcup_{\alpha<\mu} N_\alpha$ and, for all $\alpha < \mu$, $N_\alpha \in \mathcal{C}_G$ and $|N_{\alpha+1}/N_\alpha| \leq \aleph_0$. As shown above, $N_\alpha \in \mathcal{C}_G$ implies that $H \cap N_\alpha$ is knice in H. Moreover, by condition (b), each $H \cap N_\alpha$ is pure in H. Thus, after intersecting each link in the chain with H, the conclusion that H is a global Warfield group follows from Theorem 2.5. □

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