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SHARP TYPES REVISITED

A. MADER, L.G. NONGXA AND C. VINSONHALER

Dedicated to Jim Reid on the occasion of his retirement

ABSTRACT. A generalized and canonical definition of "sharp type" is given and a decomposition theorem is proved for arbitrary almost completely decomposable groups. As an application we show that an almost completely decomposable group whose critical typeset is a garland is a direct sum of rank-one and rank-two groups.

1. Introduction. An *almost completely decomposable group* X is a finite (torsion-free abelian) extension of a completely decomposable group A of finite rank. In 1974, Lady [**4**] initiated a systematic theory of such groups based on the fundamental concept of *regulating subgroup.* The regulating subgroups can be defined as the completely decomposable subgroups of least index in an almost completely decomposable group X. This least index is the *regulating index* $\operatorname{rgi} X$. An almost completely decomposable group is *local* if its regulating index is a prime power; otherwise it is a *global* group. An accessible class of almost completely decomposable groups with arbitrary critical typeset is the class of crq-groups, namely those groups X containing a completely decomposable subgroup A such that X/A is a finite cyclic group. Campagna [**2**] showed that this is equivalent to the existence of a regulating subgroup A such that X/A is cyclic. The local and global crq-groups have been studied in [**7**] and [**3**]. For local crq-groups the direct decompositions with indecomposable summands were completely determined in [**7**]. The concept of "sharp type" was essentially involved in this determination and in [**3**] "sharp type" was defined for global crq-groups and a decomposition theorem was proved. Both definitions of sharp type were based on special representations of crq-groups. In [**6**] it was demonstrated that crq-groups are largely determined by the invariants $\operatorname{rgi} X[\tau]$ (see Theorem 3.1). This leads to a generalized concept

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of "sharp type" that is independent of any particular representation and surprisingly implies in all generality that $X^\sharp(\sigma)$ is a direct summand of X whenever σ is a sharp type of X (see Definition 3.2 and Theorem 3.5). As an application we show that an almost completely decomposable group whose typeset is a garland is a direct sum of rank-one and rank-two groups (Theorem 3.14).

Details on the theory of almost completely decomposable groups can be found in the monograph [**5**]. Rather than citing the original sources we will quote [**5**], which contains an extensive bibliography.

2. Background. The purification of a subgroup H in a torsion-free group G is denoted by H_*^G. We take it for granted that the reader is familiar with the usual type subgroups, namely the *socles* $G(\tau), G^*(\tau), G^\sharp(\tau) = G^*(\tau)_*^G$ and the *radicals* $G[\tau]$, $G^\sharp[\tau] = \cap_{\rho<\tau} G[\rho]$. A type τ is *critical* for G if $G(\tau)/G^\sharp(\tau) \neq 0$. The *critical typeset* $T_{\mathrm{cr}}(G)$ is the set of all critical types of G. If A is a completely decomposable group, then $A = \oplus_{\rho \in T_{\mathrm{cr}(A)}} A_\rho$ is always assumed to be a decomposition of A into (nonzero) ρ-homogeneous components. The *typeset* of a group G is denoted by $\mathrm{Tst}\,(G)$. If G is an almost completely decomposable group, then its typeset is the meet closure of its critical typeset. An almost completely decomposable group is *clipped* if it has no completely decomposable direct summands.

The purification lemma [**5**, Lemma 11.4.1] and [**5**, Corollary 11.2.5] is an important tool in calculating the invariants of purifications of direct summands of regulating subgroups of almost completely decomposable groups. The purification lemma is based on the so-called standard description [**5**, Chapter 11], of an almost completely decomposable group X. We can write $X = A + \vec{\mathbf{Z}} N^{-1} a^\downharpoonleft$ where N is an integral $k \times k$ matrix with $\det N \neq 0$, $a^\downharpoonleft = [a_1, \dots, a_k]^{\mathrm{tr}}$, $a_i \in A$, and juxtaposition is matrix multiplication as usual. The "base group" A need not be a regulating subgroup but can be any subgroup that has finite index in X. It may be assumed that N is in Smith normal form, $N = \mathrm{diag}\,(d_1, \dots, d_k)$ where d_i divides d_{i+1} for $i = 1, \dots, k-1$. If so, the standard description is just shorthand for $X = A + \mathbf{Z}(1/d_1)a_1 + \cdots + \mathbf{Z}(1/d_k)a_k$. Later it will be necessary to simplify the system of generators and we state here certain replacements of generators that do not change the group or N. Typically, A will be decomposed as a direct sum $A = A_1 \oplus \cdots \oplus A_r$, and correspondingly we have decompositions

$$a_i = a_{i1} + \cdots + a_{ir}.$$

So typically we have "generators" $g_i = (1/d_i)(a_{i1} + \cdots + a_{ir})$ which, together with A, generate the group X. Without changing the group, the generators can be replaced by other generators as follows.

Lemma 2.1. *Let $X = A + \mathbf{Z}(1/d_1)a_1 + \cdots + \mathbf{Z}(1/d_k)a_k$, d_i divides d_{i+1}, $A = A_1 \oplus \cdots \oplus A_r$, $a_i = a_{i1} + \cdots + a_{ir}$ and $g_i = (1/d_i)a_i$.*

1. *For $j > i$, and any integer α, g_i may be replaced by $(1/d_i)(g_i + \alpha g_j) = (1/d_i)g_i + \alpha(d_j/d_i)(1/d_j)g_j$.*

2. *If α is an integer with $\gcd(d_i, \alpha) = 1$ and $1 \le j \le m$, then $(1/d_i)(a_{i1} + \cdots + \alpha a_{ij} + \cdots + a_{ir})$ may be replaced by $(1/d_i)(\alpha a_{i1} + \cdots + a_{ij} + \cdots + v a_{ir})$, where $1 = ud_i + v\alpha$.*

We only need the following part of the purification lemma. It involves a generalized greatest common divisor $\gcd^A(N, a)$ of a nonsingular integral $k \times k$ matrix N and a column vector a of k elements of A [**5**, Chapter 11].

Lemma 2.2 (Purification lemma). *Let X be an almost completely decomposable group and A a subgroup of finite index in X. Suppose that $A = B \oplus C$ and $X = A + \vec{\mathbf{Z}}N^{-1}a^{\downharpoonleft}$ with $a^{\downharpoonleft} = b^{\downharpoonleft} + c^{\downharpoonleft}$ where $a^{\downharpoonleft} \in A^{\downharpoonleft}$, $b^{\downharpoonleft} \in B^{\downharpoonleft}$ and $c^{\downharpoonleft} \in C^{\downharpoonleft}$. Then $B_*^X = B + \vec{\mathbf{Z}}N_B^{-1}b^{\downharpoonleft}$ where $N_B = \gcd^A(N, c^{\downharpoonleft})$. If A is a regulating subgroup of X, then B is regulating in B_*^X.*

Let A be a regulating subgroup of the almost completely decomposable group X. Lemma 2.2 implies the well-known facts that $A(\tau)$ is regulating in $X(\tau)$, $A^\sharp(\tau)$ is regulating in $X^\sharp(\tau)$, $A[\tau]$ is regulating in $X[\tau]$, $A^\sharp[\tau]$ is regulating in $X^\sharp[\tau]$ but also less evidently that $A[\tau](\sigma)$ is regulating in $X[\tau](\sigma)$, $A^\sharp(\tau)[\tau]$ is regulating in $X^\sharp(\tau)[\tau]$ and more.

Recall that a completely decomposable subgroup $A = \oplus_{\rho \in T_{\mathrm{cr}}(A)} A_\rho$ of an almost completely decomposable group X is regulating in X if and only if $X(\tau) = A_\tau \oplus X^\sharp(\tau)$ for every $\tau \in T_{\mathrm{cr}}(X)$. An almost completely decomposable group X decomposes if and only if there is a regulating subgroup A and a decomposition $A = B \oplus C$ such that $X = B_*^X \oplus C_*^X$.

To employ this idea we need to get a grip on the totality of regulating subgroups of an almost completely decomposable group.

Proposition 2.3 [**5**, Proposition 4.1.11]. *Let X be an almost completely decomposable group and $A = \oplus_{\rho\in T_{\mathrm{cr}}(X)} A_\rho$ a regulating subgroup of X. Then the regulating subgroups of X are exactly the groups*

$$\bigoplus_{\rho\in T_{\mathrm{cr}}(X)} A_\rho(1+\phi_\rho), \quad \phi_\tau \in \mathrm{Hom}\,(A_\tau, X^\sharp(\tau)).$$

This means that we need to deal with $\mathrm{Hom}\,(A_\tau, X^\sharp(\tau))$ which, since A_τ is τ-homogeneous completely decomposable, reduces to the study of $\mathrm{Hom}\,(\langle v\rangle_*^X, X^\sharp(\tau))$ where $v \in A_\tau$. The existence of maps is a matter of characteristics. In place of the characteristic or height sequence of an element x in a torsion-free group G we will use the *coefficient group* $\mathbf{Q}_x^G = \{r \in \mathbf{Q} : rx \in G\}$. Some of the important properties of coefficient groups are listed below (see [**5**, Lemma 2.1.5]).

Lemma 2.4. *Let G be a torsion-free group and $0 \neq x \in G$. Then the following hold.*

1. $\mathbf{Q}_x^G \cong \mathbf{Q}_x^G x$ *and* $\mathbf{Q}_x^G x = \langle x\rangle_*^G$.
2. *If H is a pure subgroup of G and $x \in H$, then $\mathbf{Q}_x^H = \mathbf{Q}_x^G$.*
3. $\mathbf{Q}_{rx}^G = r^{-1}\mathbf{Q}_x^G$ *for* $0 \neq r \in \mathbf{Q}$.
4. *If $x, y \in \mathbf{Q}G$, then $\mathbf{Q}_{x+y}^G \supseteq \mathbf{Q}_x^G \cap \mathbf{Q}_y^G$.*
5. *Suppose that $G = \oplus_i G_i$ and $x = \sum_i x_i \in G$ where $x_i \in G_i$. Then $\mathbf{Q}_x^G = \cap_i \mathbf{Q}_{x_i}^{G_i}$.*
6. *If $\phi \in \mathrm{Hom}\,(G, H)$, then $\mathbf{Q}_x^G \subseteq \mathbf{Q}_{x\phi}^H$.*
7. *Let $y \in H$. There is a well-defined homomorphism $\phi : \mathbf{Q}_x^G x \to H : x\phi = y$ if and only if $\mathbf{Q}_x^G \subseteq \mathbf{Q}_y^H$.*

Remark 2.5. Consider $\mathrm{Hom}\,(\langle v\rangle_*^X, X^\sharp(\sigma))$ where X is an almost completely decomposable group with regulating index n and the type of $\langle v\rangle_*^X$ is σ. We can write $\langle v\rangle_*^X = \sigma v$ where $\sigma = \mathbf{Q}_v^X$. So σ is now a subgroup of the additive group of rationals containing $\mathbf{Z}$, a *rational* *group* for short, while σ at the same time is used to designate its type (isomorphism class). We may assume that v is n-adjusted; i.e., for a prime divisor p of n, either $p\sigma = \sigma$ or $p^{-1} \notin \sigma$. This can be achieved by replacing v by v' where $v = p^e v'$ and $\operatorname{hgt}_p(v') = 0$. Let $x \in X^\sharp(\sigma)$. We wish to map v to x. While the type of x is $\geq \sigma$ it need not be true that $\sigma \subseteq \mathbf{Q}_x^X$. However, since v is n-adjusted, there exists an integer m relatively prime to n such that $\sigma \subseteq m^{-1}\mathbf{Q}_x^X = \mathbf{Q}_{mx}^X$. By Lemma 2.4.7 there is a well-defined map $\sigma v \to X^\sharp(\sigma) : v \mapsto mx$.

3. Sharp types. In the setting of clipped p-local crq-groups the indecomposable decompositions were described by means of special critical types called "sharp types" [**5**, Section 6.5]. The definition was based on a special representation of the group in which certain p-powers appeared that could be shown with some effort to be invariants of the group. The following theorem from [**6**] shows that the obvious invariants $\operatorname{rgi} X[\tau]$ largely determine the crq-group X.

Theorem 3.1. *Two* crq-*groups* X *and* Y *are nearly isomorphic if and only if*

1. $R(X) \cong R(Y)$
2. $\operatorname{rgi}(X) = \operatorname{rgi}(Y)$
3. $\operatorname{rgi} X[\tau] = \operatorname{rgi} Y[\tau]$ *for every* $\tau \in T_{\mathrm{cr}}(X) = T_{\mathrm{cr}}(Y)$.

This result caused us to search for a canonical definition of "sharp type" which is given below. It generalizes the concept in two ways: the following definition makes sense for any almost completely decomposable group and any type.

Definition 3.2. Let X be any almost completely decomposable group and σ any type. Then σ is called *sharp* if $\operatorname{rgi} X^\sharp(\sigma) = \operatorname{rgi} X[\sigma]$.

If X is a crq-group, then $\operatorname{rgi} X^\sharp(\sigma) = \exp(X^\sharp(\sigma)/R(X^\sharp(\sigma))) = \beta_\sigma^X$, the Burkhardt invariant, and in the setting of [**7**], $\operatorname{rgi} X[\sigma] = s_\sigma$ provided that σ is critical. Hence, for clipped local crq-groups, the critical type σ is sharp if and only if $\beta_\sigma^X = s_\sigma$ which is the definition in [**7**] except that σ was required to be nonmaximal in $T_{\mathrm{cr}}(X)$ in order to avoid that $X^\sharp(\sigma) = 0$.

There are some immediate consequences if a type is sharp. By $\sigma \perp \tau$ we mean that σ and τ are incomparable.

Proposition 3.3. *Let $A = \oplus_{\rho \in T_{\rm cr}(X)} A_\rho$ be a regulating subgroup of the almost completely decomposable group X. Then the following hold.*

1. *σ is a sharp type of X if and only if*

$$X[\sigma] = \oplus\{A_\rho : \rho \perp \sigma\} \oplus X^\sharp(\sigma).$$

2. *If σ_1 and σ_2 are incomparable sharp types, then $\operatorname{rgi} X^\sharp(\sigma_1) = \operatorname{rgi} X^\sharp(\sigma_2)$ and, moreover,*

$$X[\sigma_1] = \Big(\bigoplus\{A_\rho : \rho \not\leq \sigma_1, \rho \not> \sigma_2\}\Big) \oplus X^\sharp(\sigma_2),$$

and

$$X[\sigma_2] = \Big(\bigoplus\{A_\rho : \rho \not\leq \sigma_2, \rho \not> \sigma_1\}\Big) \oplus X^\sharp(\sigma_1).$$

3. *If σ_1 and σ_2 are incomparable sharp types, then*

$$X^\sharp(\sigma_1) = \Big(\bigoplus\{A_\rho : \rho \perp \sigma_2, \rho > \sigma_1\}\Big) \oplus (X^\sharp(\sigma_1) \cap X^\sharp(\sigma_2)),$$

and

$$X^\sharp(\sigma_2) = \Big(\bigoplus\{A_\rho : \rho \perp \sigma_1, \rho > \sigma_2\}\Big) \oplus (X^\sharp(\sigma_1) \cap X^\sharp(\sigma_2)).$$

Proof. 1. Suppose that σ is sharp. We note that $A[\sigma] = B \oplus A^\sharp(\sigma)$ where $B = \oplus\{A_\rho : \rho \perp \sigma\}$, $A[\sigma]$ is regulating in $X[\sigma]$, and also $B \oplus A^\sharp(\sigma)$ is regulating in $B \oplus X^\sharp(\sigma)$. By the product formula [**5**, Proposition 4.2.14], and the assumption that σ is a sharp type we have

$$\begin{aligned}\operatorname{rgi} X[\sigma] = [X[\sigma] : A[\sigma]] &= [X[\sigma] : B \oplus X^\sharp(\sigma)][B \oplus X^\sharp(\sigma) : B \oplus A^\sharp(\sigma)] \\ &= [X[\sigma] : B \oplus X^\sharp(\sigma)]\operatorname{rgi}(B \oplus X^\sharp(\sigma)) \\ &= [X[\sigma] : B \oplus X^\sharp(\sigma)]\operatorname{rgi}(X^\sharp(\sigma)) \\ &= [X[\sigma] : B \oplus X^\sharp(\sigma)]\operatorname{rgi}(X[\sigma]).\end{aligned}$$

Consequently $[X[\sigma] : B \oplus X^\sharp(\sigma)] = 1$ which establishes the claim.

The converse is an immediate consequence of the product formula for regulating indices [**5**, Proposition 4.2.14]:

$$\operatorname{rgi} X[\sigma] = \operatorname{rgi}(\oplus\{A_\rho : \rho \perp \sigma\})\operatorname{rgi} X^\sharp(\sigma) = \operatorname{rgi} X^\sharp(\sigma).$$

2. Suppose that σ_1 and σ_2 are incomparable types. Then $X^\sharp(\sigma_1) \subseteq X[\sigma_2]$. If $A_\sigma \subseteq X^\sharp(\sigma_1)$, then $\sigma \not\leq \sigma_2$ and this implies that $A_\sigma \subseteq \oplus_{\rho\not\leq\sigma_2} A_\rho = A[\sigma_2]$. Thus $A[\sigma_2] = A^\sharp(\sigma_1) \oplus C$ where $C = \oplus\{A_\rho : \rho \not\leq \sigma_2,\, \rho \not> \sigma_1\}$. Note that $C \oplus A^\sharp(\sigma_1)$ is regulating in $C \oplus X^\sharp(\sigma_1)$ and

$$\begin{aligned}[X[\sigma_2] : A[\sigma_2]] &= [X[\sigma_2] : C \oplus X^\sharp(\sigma_1)][C \oplus X^\sharp(\sigma_1) : C \oplus A^\sharp(\sigma_1)]\\ &= [X[\sigma_2] : C \oplus X^\sharp(\sigma_1)][X^\sharp(\sigma_1) : A^\sharp(\sigma_1)].\end{aligned}$$

Hence, $\operatorname{rgi}(X^\sharp(\sigma_1))$ divides $\operatorname{rgi}(X[\sigma_2])$ and by symmetry $\operatorname{rgi}(X^\sharp(\sigma_2))$ divides $\operatorname{rgi}(X[\sigma_1])$.

Now if σ_1 and σ_2 are sharp types, then by the previous paragraph, $\operatorname{rgi}(X^\sharp(\sigma_i))$ divides $\operatorname{rgi}(X[\sigma_j]) = \operatorname{rgi}(X^\sharp(\sigma_j))$ where $i, j \in \{1, 2\}$ and $i \neq j$. This means that $\operatorname{rgi} X^\sharp(\sigma_1)) = \operatorname{rgi}(X^\sharp(\sigma_2))$. The claimed decompositions are established as was the decomposition in part 1, mutatis mutandis.

3. We have by 2 that $X^\sharp(\sigma_1) \subset X[\sigma_2]$ and hence by 1 with $\sigma = \sigma_2$, that $X^\sharp(\sigma_1) = (\bigoplus\{A_\rho : \rho \perp \sigma_2, \rho > \sigma_1\} \oplus \bigoplus\{A_\rho : \rho > \sigma_2, \rho > \sigma_1\})_*$ where the purification is the same whether taken in X or $X[\sigma_2]$. Now $\oplus\{A_\rho : \rho \perp \sigma_2, \rho > \sigma_1\}$ is a summand of $X[\sigma_2]$ and hence pure in $X[\sigma_2]$. It follows that $X^\sharp(\sigma_1) = \oplus\{A_\rho : \rho \perp \sigma_2, \rho > \sigma_1\} \oplus (\oplus\{A_\rho : \rho > \sigma_2, \rho > \sigma_1\})_* = \oplus\{A_\rho : \rho \perp \sigma_2, \rho > \sigma_1\} \oplus X^\sharp(\sigma_1) \cap X^\sharp(\sigma_2)$. The second claim follows with roles of σ_1 and σ_2 exchanged. $\square$

The equality $\operatorname{rgi} X^\sharp(\sigma_1) = \operatorname{rgi} X^\sharp(\sigma_2)$ generalizes the first part of [**5**, Lemma 6.5.3.1].

The new definition of sharp type allows for a comparison of sharp types of summands and sums.

Proposition 3.4. *Let X be an almost completely decomposable group and suppose that $X = Y \oplus Z$. Then σ is a sharp type of X if and only if σ is a sharp type for both Y and Z.*

Proof. Observe that $\operatorname{rgi} X^\sharp(\sigma)$ divides $\operatorname{rgi} X[\sigma]$. Furthermore, $X^\sharp(\sigma) = Y^\sharp(\sigma) \oplus Z^\sharp(\sigma)$ and $X[\sigma] = Y[\sigma] \oplus Z[\sigma]$; hence, by the product rule for regulating indices [**5**, Proposition 4.2.14], $\operatorname{rgi} X^\sharp(\sigma) = (\operatorname{rgi} Y^\sharp(\sigma)) \cdot (\operatorname{rgi} Z^\sharp(\sigma))$ and $\operatorname{rgi} X[\sigma] = (\operatorname{rgi} Y[\sigma])(\operatorname{rgi} Z[\sigma])$. Thus $\operatorname{rgi} X^\sharp(\sigma) = \operatorname{rgi} X[\sigma]$ if and only if $\operatorname{rgi} Y^\sharp(\sigma) = \operatorname{rgi} Y[\sigma]$ and $\operatorname{rgi} Z^\sharp(\sigma) = \operatorname{rgi} Z[\sigma]$. □

We will now prove a general decomposition theorem. Note that the decomposition may be trivial either because $X^\sharp(\sigma) = 0$ or because $X^\sharp(\sigma) = X$.

Theorem 3.5. *Let X be an almost completely decomposable group and suppose that σ is a sharp type of X. Then $X = Y \oplus X^\sharp(\sigma)$ for some* (*almost completely decomposable*) *subgroup Y.*

The proof requires a number of steps. In particular, we first settle the local case ($\operatorname{rgi} X$ is a prime power), which is then used to prove the general case. An almost completely decomposable group is a finite extension of a regulating subgroup. We will first replace the regulating subgroup by a larger "base group" and inspect the consequences.

Changing the base group. Let σ be a type. For the time being σ need not be sharp. Let $A = \oplus_{\rho \in T_{\mathrm{cr}}(A)} A_\rho$ be a regulating subgroup of X and set $A_{\leq\sigma} = \oplus_{\rho\leq\sigma} A_\rho$. Then X is a finite essential extension of the base group $W = A_{\leq\sigma} \oplus X[\sigma] \geq A$. There is a standard description

$$X = W(A) + \vec{\mathbf{Z}} N^{-1}(a^{\downarrow}_{\leq\sigma} + x^{\downarrow}_{[\sigma]}), \quad W(A) = A_{\leq\sigma} \oplus X[\sigma], \tag{3.6}$$

and without loss of generality we may assume that $\gcd^W(N, a^{\downarrow}_{\leq\sigma} + x^{\downarrow}_{[\sigma]}) = I$ and that $N = \operatorname{diag}(d_1, \dots, d_k)$ is in Smith normal form with $d_1 > 1$. The purification lemma and the fact that $X[\sigma]$ is pure in X imply that $\gcd^W(N, a^{\downarrow}_{\leq\sigma}) = I$. This property has important implications that we will exploit in more detail. For example, $a^{\downarrow}_{\leq\sigma}$ cannot be a column of zeros since $N \neq I$ by assumption.

Set $n = \det N$. Write [**5**, Lemma 2.4.8] $A_{\leq\sigma} = \sigma_1 v_1 \oplus \cdots \oplus \sigma_s v_s$ as a direct sum of rank-one groups; i.e., the $v_i \in A_{\leq\sigma}$ form a set of linearly independent elements and $\sigma_i = \mathbf{Q}^A_{v_i}$. We assume without loss of generality ([**5**, Section 11.3]) that

1. $\{v_1, \dots, v_s\}$ is an n-basis; i.e., for every prime p dividing n, the p-height of v_i is 0 or ∞.
2. $a^{\downharpoonleft}_{\leq\sigma} = M_{\leq\sigma}[v_1, \dots, v_s]^{\mathrm{tr}}$ for some integral $k \times s$-matrix $M_{\leq\sigma}$,
3. $\gcd^W(N, a^{\downharpoonleft}_{\leq\sigma}) = \gcd(N, M_{\leq\sigma})$.

The nontrivial item 3 uses [**5**, Theorem 11.3.3]. We have that $\gcd(N, M_{\leq\sigma}) = \gcd^W(N, a^{\downharpoonleft}_{\leq\sigma}) = I$ and, by [**5**, Theorem 11.2.6], there exist integral matrices U_1, U_2 such that $I = NU_1 + M_{\leq\sigma}U_2$. Recall that N is a diagonal matrix with d_1 dividing all other diagonal entries. It follows that $M_{\leq\sigma}U_2 \equiv I \bmod d_1$ or, equivalently, that $M_{\leq\sigma}U_2 = I$ considered as matrices over the ring $\mathbf{Z}/d_1\mathbf{Z}$. By [**8**, Main Theorem], there exist $k \times k$ submatrices of $M_{\leq\sigma}$ whose determinants generate $\mathbf{Z}/d_1\mathbf{Z}$ or, in other words, for every prime divisor p of d_1 there is a $k \times k$ submatrix of $M_{\leq\sigma}$ whose determinant is relatively prime to p.

The local case. The type σ is still an arbitrary type but we assume that $n = \det N$ is a power of a prime p. Then $M_{\leq\sigma}$ contains a $k \times k$ submatrix whose determinant is relatively prime to p. Certainly, by relabeling if necessary, we may assume that the submatrix M_0 formed by the first k columns of $M_{\leq\sigma}$ has determinant relatively prime to p. Write $x^{\downharpoonleft}_{[\sigma]} = [x_1, \dots, x_k]^{\mathrm{tr}}$ and $M_{\leq\sigma} = [m_{ij}]$. Then the elements $g_i = (1/d_i)(x_i + \sum_{j=1}^s m_{ij}v_j)$ generate X together with the base group W.

Now suppose that σ is sharp, i.e., $X[\sigma] = A_{\perp\sigma} \oplus X^\sharp(\sigma)$ (Proposition 3.3). Writing $y_i = a_i + x_i$ where $a_i \in A_{\perp\sigma}$ and $x_i \in X^\sharp(\sigma)$, (3.6) becomes

$$X = W + \vec{\mathbf{Z}}N^{-1}\left(M_{\leq\sigma}\begin{bmatrix} v_1 \\ \vdots \\ v_s \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} + \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}\right).$$

Using permitted row transformations (Lemma 2.1) and permutations of columns, we will achieve the form

$$M_{\le\sigma} = \begin{bmatrix} 1 & 0 & \cdots & 0 & m_{1,k+1} & \cdots & m_{1s} \\ m_{21} & 1 & \cdots & 0 & m_{2,k+1} & \cdots & m_{2s} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ m_{k1} & m_{k2} & \cdots & 1 & m_{k,k+1} & \cdots & m_{ks} \end{bmatrix},$$

as follows. Since $\det M_0$ is relatively prime to p, the last row of M_0 contains an entry relatively prime to p and by a column exchange we can move it to the kth column. The entry can now be turned into a 1 and, by row transformations, can be used to produce 0 entries above the 1. The process can be repeated from the bottom to the top to achieve the desired form.

Combining Lemma 2.1 and Remark 2.5 we may assume that, for all i and j, we have $\mathbf{Q}^X_{v_i} \subset \mathbf{Q}^X_{x_j}$. Hence there is a well-defined basis change $w_1 = v_1 + x_1$ that produces a new regulating subgroup $A_1 = \sigma_1 w_1 \oplus \cdots$ and, with respect to $W(A_1) = \sigma_1 w_1 \oplus \sigma_2 v_2 \oplus \cdots \oplus \sigma_s v_s \oplus A_{\perp\sigma} \oplus X^\sharp(\sigma)$, we have the description

$$X = W_1 + \vec{\mathbf{Z}} N^{-1} \left(M_{\le\sigma} \begin{bmatrix} w_1 \\ v_2 \\ \vdots \\ v_s \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 - m_{21}x_1 \\ \vdots \\ x_k - m_{k1}x_1 \end{bmatrix} \right).$$

The next basis change (permissible since $\mathbf{Q}^X_{v_i} \subset \mathbf{Q}^X_{x_j}$) is $w_2 = v_2 + x_2 - m_{21}x_1$ and, with respect to $A_2 = \sigma_1 w_1 \oplus \sigma_2 w_2 \oplus \cdots$ and $W(A_2) = \sigma_1 w_1 \oplus \sigma_2 w_2 \oplus \sigma_3 v_3 \oplus \cdots \oplus \sigma_s v_s \oplus A_{\perp\sigma} \oplus X^\sharp(\sigma)$, we have the description

$$X = W_2 + \vec{\mathbf{Z}} N^{-1} \cdot \left(M_{\le\sigma} \begin{bmatrix} w_1 \\ w_2 \\ v_3 \\ \vdots \\ v_s \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 - (m_{31} - m_{32}m_{21})x_1 - m_{32}x_2 \\ \vdots \\ x_k - (m_{k1} - m_{k2}m_{21})x_1 - m_{k2}x_2 \end{bmatrix} \right).$$

Continuing in this fashion we finally arrive at $A_k = \sigma_1 w_1 \oplus \cdots \oplus \sigma_k w_k \oplus \cdots$, $W(A_k) = \sigma_1 w_1 \oplus \cdots \oplus \sigma_k w_k \oplus \sigma_{k+1} v_{k+1} \oplus \cdots \oplus \sigma_s v_s \oplus X[\sigma]$, and the description

$$X = W(A_k) + \vec{\mathbf{Z}}N^{-1}\left(M_{\le\sigma}\begin{bmatrix} w_1 \\ \vdots \\ w_k \\ v_{k+1} \\ \vdots \\ v_s \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right).$$

Setting $V_k = [w_1, \dots, w_k, v_{k+1}, \dots, v_s]^{\mathrm{tr}}$, it is now clear that

$$\begin{aligned} X &= W(A_k) + \vec{\mathbf{Z}}N^{-1}(M_{\le\sigma}V_k + a^{\downharpoonleft}_{\perp\sigma}) \\ &= [(M_{\le\sigma}V_k \oplus A_{\perp\sigma}) + \vec{\mathbf{Z}}N^{-1}(M_{\le\sigma}V_k + a^{\downharpoonleft}_{\perp\sigma})] \oplus X^{\sharp}(\sigma) \\ &= \langle (A_k)_{\le\sigma} \oplus A_{\perp\sigma}\rangle^X_* \oplus X^{\sharp}(\sigma). \qquad \square \end{aligned}$$

We will derive the global theorem from the local result. To do so we need two preparatory lemmas. We denote by $\mathbf{Z}_q$ the localization of the ring of integers at the prime q and for any abelian group G we have its localization $G_q = \mathbf{Z}_q \otimes G$. Note that for a torsion-group T the localization T_q is isomorphic to the q-primary component of T.

Lemma 3.7. *Let G be a torsion-free group and H a subgroup of G such that G/H is a torsion group. Then $(G/H)_q \cong G_q/H_q$. Furthermore, if G is torsion-free and G/H is bounded by q^d, then $G_q/H_q = (G_q \cap q^{-d}H_q)/H_q$.*

Proof. Since $\mathbf{Z}_q$ is torsion-free, hence flat, the short exact sequence $H \mapsto G \twoheadrightarrow G/H$ implies the short exact sequence $H_q \mapsto G_q \twoheadrightarrow (G/H)_q$ and the rest follows easily. $\square$

Lemma 3.8. *Let A be any torsion-free group, $\mathbf{Q}A$ some divisible hull of A, and X a group with $A \le X \le \mathbf{Q}A$ and X/A finite. Let $|X/A| = mn$ be a factorization of $|X/A|$ into relatively prime factors. Suppose further that $A = B \oplus Y$ and $\phi \in \mathrm{Hom}\,(B, Y)$. Then $A = B \oplus Y = B(1 + m\phi) \oplus Y$ and for a prime factor q of m,*

$$\left| \left(\frac{X}{B(1+m\phi)^X_* \oplus Y} \right)_q \right| \quad \textit{divides} \quad \left| \left(\frac{X}{B^X_* \oplus Y} \right)_q \right|.$$

Proof. By Lemma 3.7 we have that

$$\left(\frac{X}{B_*^X \oplus Y}\right)_q \cong \frac{X_q}{(B_*^X \oplus Y)_q}$$

and

$$\left(\frac{X}{B(1+m\phi)_*^X \oplus Y}\right) \cong \frac{X_q}{(B(1+m\phi)_*^X \oplus Y)_q}.$$

Hence it suffices to show that $(B_*^X \oplus Y)_q \subset (B(1+m\phi)_*^X \oplus Y)_q$. Let $x \in (B_*^X \oplus Y)_q$. Then $x = q^{-d}\alpha b + \beta y$ where $\alpha, \beta \in \mathbf{Z}_q$, $b \in B$, $y \in Y$ and d is a nonnegative integer. Moreover, since $[X/(B_*^X \oplus Y)]_q \cong [X_q/(B_*^X \oplus Y)_q]$ is bounded by m, we may assume without loss of generality that q^d divides m. This implies that $\beta y - q^{-d}\alpha b m\phi \in Y_q$ and we conclude that $x = q^{-d}\alpha b(1 + m\phi) + (\beta y - q^{-d} m\alpha b\phi) \in (B(1+m\phi)_*^X \oplus Y)_q$. $\square$

The global case. We assume that σ is a sharp type of X so that there is a decomposition $X[\sigma] = A_{\perp\sigma} \oplus X^\sharp(\sigma)$ (Proposition 3.3). Recall that A is a regulating subgroup of X and put $W(A) = A_{\leq\sigma} \oplus A_{\perp\sigma} \oplus X^\sharp(\sigma) \geq A$ and $W_*(A) = (A_{\leq\sigma} \oplus A_{\perp\sigma})_*^X \oplus X^\sharp(\sigma)$. We claim that there is another regulating subgroup B such that $W_*(B) = X$ as desired to complete the proof.

Suppose that $(X/W_*(A))_p \neq 0$. Write $|X/W(A)| = p^e m$ such that $e \geq 1$ and p and m are relatively prime. We will replace the regulating subgroup A by a regulating subgroup A' in such a way that $(X/W_*(A'))_p = 0$, and $|X/W_*(A')|$ divides $|X/W_*(A)|$ so that $|X/W_*(A')|$ contains fewer prime factors (at least p is eliminated) than $|X/W_*(A)|$. Then an induction on the number of prime factors in $|X/W_*(A)|$ establishes the theorem. Let $X(p)$ be the subgroup of X determined by the identity $X(p)/W(A) = (X/W(A))_p$. Using that σ is sharp, the description (3.6) can be refined to read

$$X = W(A) + \vec{\mathbf{Z}} N^{-1}(a_{\leq\sigma}^{\downharpoonleft} + a_{\perp\sigma}^{\downharpoonleft} + x^{\downharpoonleft}),$$

where the entries of $a_{\perp\sigma}^{\downharpoonleft}$ are from $A_{\perp\sigma}$ and $x^{\downharpoonleft} = [x_1, \dots, x_k]^{\mathrm{tr}}$ with $x_i \in X^\sharp(\sigma)$. It is easy to see that ($d_i(p)$ denotes the highest p-power factor of d_i)

$$X(p) = W(A) + \vec{\mathbf{Z}} N_p^{-1}(a_{\leq\sigma}^{\downharpoonleft} + a_{\perp\sigma}^{\downharpoonleft} + x^{\downharpoonleft}) \tag{3.9}$$

where

$$N_p = \operatorname{diag}(d_1(p), \dots, d_k(p)).$$

Then A is regulating in $X(p)$, $X(p) \supset X[\sigma] \supset X^\sharp(\sigma)$, $X(p)[\sigma] = X[\sigma]$, $X(p)^\sharp(\sigma) = X^\sharp(\sigma)$ and $X(p)[\sigma] = A_{\perp\sigma} \oplus X(p)^\sharp(\sigma)$, so σ is a sharp type of $X(p)$. Write $A_{\leq\sigma} = \sigma_1 v_1 \oplus \cdots \oplus \sigma_s v_s$. Using Lemma 2.1 we can replace the generator $x^\downharpoonleft$ in (3.9) by $cx^\downharpoonleft$ such that m divides c and $\sigma_j \subseteq \mathbf{Q}^X_{cx_i}$ for all i, j and obtain that

$$X(p) = W(A) + \vec{\mathbf{Z}} N_p^{-1}(a^\downharpoonleft_{\leq\sigma} + a^\downharpoonleft_{\perp\sigma} + cx^\downharpoonleft) \tag{3.10}$$

By the local case there is a basis change of the form

$$w_i = v_i + cy_i, \quad y_i \in X^\sharp(\sigma), \quad i = 1, \dots, k,$$

such that $X(p) = \langle A'_{\leq\sigma} \oplus A_{\perp\sigma}\rangle_*^{X(p)} \oplus X^\sharp(\sigma)$ where $A' = \sigma_1 w_1 \oplus \cdots \oplus \sigma_k w_k \oplus \sigma_{k+1} v_{k+1} \oplus \cdots \oplus \sigma_s v_s \oplus A_{\perp\sigma} \oplus A^\sharp(\sigma)$ is regulating in $X, W(A') = A'_{\leq\sigma} \oplus A_{\perp\sigma} \oplus X^\sharp(\sigma) = W(A)$ and $W_*(A') = (A'_{\leq\sigma} \oplus A_{\perp\sigma})^X_*$. Since $X(p) \subseteq W_*(A')$, $(X/W_*(A'))_p = 0$ and, by Lemma 3.8 (with $B = A_{\leq\sigma} \oplus A_{\perp\sigma}$, $Y = X^\sharp(\sigma)$, $\phi = 0$ on $A_{\perp\sigma}$, $v_i\phi = w_i = v_i + cy_i$ for $i = 1, \dots, k$, and $v_i\phi = 0$ otherwise), $|(X/W_*(A'))_q|$ divides $|(X/W_*(A))_q|$ for the prime factors q of m. We have achieved our goal of decreasing the number of different prime factors of $|X/W_*(A)|$ and, by induction, the proof is complete. □

Corollary 3.11. *Let X be an almost completely decomposable group and σ a type that is comparable with every type in* $\operatorname{Tst}(X)$. *Then $X = Y \oplus X^\sharp(\sigma)$ for some subgroup Y.*

Proof. In this case

$$\begin{aligned} X[\sigma] &= \Big(\bigoplus\{A_\rho : \rho \in T_{\mathrm{cr}}(X), \rho \not\leq \sigma\}\Big)^X_* \\ &= \Big(\bigoplus\{A_\rho : \rho \in T_{\mathrm{cr}}(X), \rho > \sigma\}\Big)^X_* = X^\sharp(\sigma). \end{aligned}$$

In particular, σ is a sharp type and Theorem 3.5 applies. □

Corollary 3.12. *Suppose that σ_1 and σ_2 are incomparable sharp types of the clipped group X. Then $X^\sharp(\sigma_1) = X^\sharp(\sigma_2)$.*

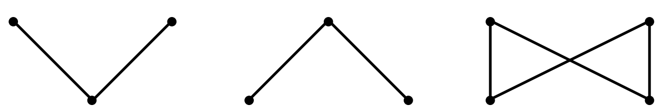
Proof. By Theorem 3.5 and Proposition 3.3 there is a group Y and a completely decomposable group Z such that $X = Y \oplus X^\sharp(\sigma_1) = Y \oplus Z \oplus X^\sharp(\sigma_1) \cap X^\sharp(\sigma_2)$. Since X is clipped, it follows that $Z = 0$ and so $X^\sharp(\sigma_1) = X^\sharp(\sigma_1) \cap X^\sharp(\sigma_2) = X^\sharp(\sigma_2)$. $\square$

We proceed to an application.

Definition 3.13. A poset T is called a *garland* if it is not a chain and every subset of T with exactly three elements either has a largest element or a smallest element.

Theorem 3.14. *If X is an almost completely decomposable group whose critical typeset $T_{\mathrm{cr}}(X)$ is a garland, then X decomposes into a direct sum of rank-one and rank-two groups.*

Proof. The proof is by induction on the depth, $\operatorname{depth}(T_{\mathrm{cr}}(X))$, of the critical typeset of X. If $\operatorname{depth}(T_{\mathrm{cr}}(X)) = 0$, then $T_{\mathrm{cr}}(X)$ consists of two incomparable types and the group is a direct sum of rank one and rank two groups by [**5**, Theorem 12.3.4]. The claim is true also if $\operatorname{depth}(T_{\mathrm{cr}}(X)) = 1$ as will be seen next. In this case $T_{\mathrm{cr}}(X)$ is one of the following



In the first case we have the Butler decomposition $X = X(\mu) = A_\mu \oplus X^\sharp(\mu)$ where μ is the least critical type. Here $X^\sharp(\mu)$ is an almost completely decomposable group with two critical types and therefore a direct sum of rank-one and rank-two groups by [**5**, Theorem 12.3.4]. In the second case, by [**5**, Proposition 7.2.9], X decomposes as $X = Y \oplus X(\nu)$ where ν is the largest critical type. In the third case let σ_1 and σ_2 denote the two minimal types. Observe that $X^\sharp(\sigma_1) = X^\sharp(\sigma_2)$ and that $X[\sigma_1] = X(\sigma_2)$. It follows that $\operatorname{rgi} X[\sigma_1] = \operatorname{rgi} X(\sigma_2) = \operatorname{rgi} X^\sharp(\sigma_2) = \operatorname{rgi} X^\sharp(\sigma_1)$. This says that σ_1 is a sharp type of X and by Theorem 3.5 we have a decomposition $X = Y \oplus X^\sharp(\sigma_1)$. Both summands have two critical types and decompose into a direct sum of rank-one and rank-two groups.

Assume now that the result holds for all Y for which $T_{\mathrm{cr}}(Y)$ is a garland and $\operatorname{depth}(T_{\mathrm{cr}}(Y)) < \operatorname{depth}(T_{\mathrm{cr}}(X))$. If X has a smallest critical type μ, then $X = X(\mu) = A_\mu \oplus X^\sharp(\mu)$ where A_μ is completely decomposable and $X^\sharp(\mu)$ is a direct sum of rank-one and rank-two by induction hypothesis. Otherwise, since $T_{\mathrm{cr}}(X)$ is a garland, it contains two minimal types. Let σ_1 and σ_2 denote the two minimal types. Note that, as above, $X^\sharp(\sigma_1) = X^\sharp(\sigma_2)$ and that $X[\sigma_1] = X(\sigma_2)$. Again it follows that $\operatorname{rgi} X[\sigma_1] = \operatorname{rgi} X(\sigma_2) = \operatorname{rgi} X^\sharp(\sigma_2) = \operatorname{rgi} X^\sharp(\sigma_1)$ which says that σ_1 is a sharp type of X. By Theorem 3.5 $X = Y \oplus X^\sharp(\sigma_1)$ where Y has two critical types, namely σ_1, σ_2, and $X^\sharp(\sigma_1)$ is a direct sum of rank-one and rank-two groups by induction. $\square$

We mention that the local version of Theorem 3.14 is a corollary of [**1**, Proposition 4.1.6]. In fact, for an almost completely decomposable group X whose critical typeset is a garland, the poset $S = T_{\mathrm{cr}}(X)^{\mathrm{opp}}$ is also a garland and, in the local case, Richman's category equivalence [**1**, Corollary 4.3.2] applies.

The converse of Theorem 3.5 does not hold as the following example illustrates.

Example 3.15. There exist crq-group X and a type $\tau \in \operatorname{Tst}(X)$ such that $X = Y \oplus X^\sharp(\tau)$ nontrivially but τ is not a sharp type.

Proof. Let T_1 and T_2 be finite sets of types each with at least two minimal elements such that for any τ in the meet closure of T_1 and any σ in the meet closure of T_2 we have that τ and σ are incomparable. Let τ_i be the infimum of the elements of T_i, $i = 1, 2$. For $i = 1, 2$, choose any clipped crq-groups X_i with $T_{\mathrm{cr}}(X_i) = T_i$ and $\operatorname{rgi}(X_i) = n_i$, where $n_1, n_2 > 1$ and relatively prime. Then $X = X_1 \oplus X_2$ is a crq-group with $\operatorname{rgi}(X) = n_1 n_2$. Note that $X^\sharp(\tau_1) = X_1$ is a summand of X but $\operatorname{rgi}(X^\sharp(\tau_1)) = n_1 \neq n_1 n_2 = \operatorname{rgi}(X) = \operatorname{rgi}(X[\tau_1])$. $\square$

If X is an almost completely decomposable group and λ is a type that is less than every critical type of X, then λ is trivially sharp. The corresponding decomposition is trivial since $X^\sharp(\lambda) = X$. The following proposition shows that, for local clipped crq-groups, only critical sharp types matter, in contrast to global crq-groups, Example 3.17.

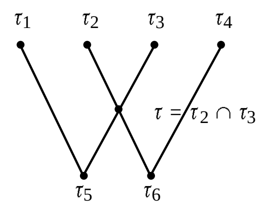
Proposition 3.16. *Let X be a local clipped* crq-*group. If σ is a sharp type of X with $X^\sharp(\sigma) \neq X$, then $X^\sharp(\sigma) = X^\sharp(\tau)$ for some sharp critical type τ of X.*

Proof. Let $X = A + \mathbf{Z}p^{-n}\sum_{\rho \in T_{\mathrm{cr}}(X)} s_\rho v_\rho$ be a cyclic representation of the clipped p-local crq-group X (see [**5**, Section 6.4]). The properties of cyclic representations that we need here are that $s_\tau = \gcd^A(p^n, s_\tau v_\tau)$ and that $s_\tau < s_\sigma$ for $\tau > \sigma$. By assumption $X^\sharp(\sigma) \neq X$, and this means that $\{\rho \in T_{\mathrm{cr}}(X) : \rho \not> \sigma\} \neq \varnothing$. Hence $\operatorname{rgi} X^\sharp(\sigma) = \beta_\sigma^X = \min\{s_\rho : \rho \not> \sigma\} < p^n$. As σ is sharp, $\operatorname{rgi} X[\sigma] = \beta_\sigma^X < p^n$, so that $X[\sigma] \neq X$ and $\{\rho \in T_{\mathrm{cr}}(X) : \rho \leq \sigma\} \neq \varnothing$. Let $\tau \in T_{\mathrm{cr}}(X)$ be such that $\tau \leq \sigma$ and $s_\tau = \min\{s_\rho : \rho \leq \sigma\}$. Then $X^\sharp(\sigma) \subseteq X^\sharp(\tau)$. Also, σ being sharp, $s_\tau = \operatorname{rgi} X^\sharp(\sigma) = \min\{s_\rho : \rho \not> \sigma\}$. Suppose that $\lambda \in T_{\mathrm{cr}}(X)$ and $\lambda > \tau$. If $\lambda \not> \sigma$, then $s_\lambda < s_\tau = \min\{s_\rho : \rho \not> \sigma\} \leq s_\lambda$ which is a contradiction. Hence $\lambda > \sigma$. We conclude that $X^\sharp(\tau) \subseteq X^\sharp(\sigma)$ which establishes equality. Finally $\operatorname{rgi} X[\tau] = \min\{s_\rho : \rho \leq \tau\} = s_\tau = \operatorname{rgi} X[\sigma] = \operatorname{rgi} X^\sharp(\sigma) = \operatorname{rgi} X^\sharp(\tau)$ which shows that τ is sharp. $\square$

The following example shows that, in contrast to Proposition 3.16, in global crq-groups, noncritical sharp types need to be taken into account.

Example 3.17. There exists a clipped global crq-group X that has a nonmaximal noncritical sharp type τ; hence $X^\sharp(\tau)$ is a nontrivial summand of X, but X has no nonmaximal critical sharp types, i.e., $X^\sharp(\sigma) = 0$ for every critical sharp type of X.

Proof. Let $\{\tau_i : 1 \leq i \leq 6\}$ be a set of rational groups containing 1, with the following Hasse diagram as types. Suppose that there exist distinct primes p, q such that $p^{-1}, q^{-1} \notin \tau_i$, $1 \leq i \leq 6$. Let $A = \oplus_{i=1}^6 \tau_i a_i$, $X = A + \mathbf{Z}(pq)^{-3}a$, where $a = pqa_1 + a_2 + a_3 + pqa_4 + p^2qa_5 + pq^2a_6$.



The following table contains the essential information about X.

σ	$\operatorname{rgi} X[\sigma]$	$\operatorname{rgi} X^\sharp[\sigma]$	$\operatorname{rgi} X(\sigma)$	$\operatorname{rgi} X^\sharp(\sigma)$
τ_1	pq	p^2q	1	1
τ_2	1	pq	1	1
τ_3	1	pq	1	1
τ_4	pq	pq^2	1	1
τ_5	p^2q	p^3q^3	pq	pq
τ_6	pq^2	p^3q^3	pq	pq

It follows that A is regulating in X and that only the maximal critical types τ_2 and τ_3 are sharp. Since $\operatorname{rgi}(X[\sigma]) \neq \operatorname{rgi}(X^\sharp[\sigma])$ for all $\sigma \in T_{\rm cr}(X)$, X is clipped by [**6**, Proposition 4.4.2].

Further $\operatorname{rgi}(X[\tau]) = pq = \operatorname{rgi}(X^\sharp(\tau))$ which implies that τ is a sharp type. $\square$

REFERENCES

1. D.M. Arnold, *Abelian groups and representations of finite partially ordered sets*, vol. 2, CMS Books in Math., Springer, New York, 2000.

2. M.J. Campagna, *Single-relation almost completely decomposable groups*, Commun. Algebra **28** (2000), 83–92.

3. U. Dittmann, A. Mader and O. Mutzbauer, *Almost completely decomposable groups with a cyclic regulating quotient*, Commun. Algebra **25** (1997), 769–784.

4. E.L. Lady, *Almost completely decomposable torsion-free abelian groups*, Proc. Amer. Math. Soc. **45** (1974), 41–47.

5. A. Mader, *Almost completely decomposable groups*, vol. 13, Algebra Logic Appl., Gordon and Breach Sci. Publ., New York, 2000.

6. A. Mader, L.G. Nongxa and M. Ould-Beddi, *Invariants of global crq-groups*, presented at Abelian Groups and Modules, Internat. Conf. (Perth, July 2000), Contemp. Math., vol. 273, 2001, pp. 209–222.

7. A. Mader and C. Vinsonhaler, *Almost completely decomposable groups with a cyclic regulating quotient*, J. Algebra **177** (1995), 463–492.

8. R. Bruce Richter and William P. Wardlaw, *Good matrices*: *Matrices that preserve ideals*, Amer. Math. Monthly **104** (1997), 932–938.

Department of Mathematics, University of Hawaii, 2565 The Mall, Honolulu, HI 96822, USA
E-mail address: adolf@math.hawaii.edu

Office of Deputy Vice-Chancellor (Research), University of the Witwatersrand, Private Bag 3, WITS, 2050, South Africa
E-mail address: 160nong@atlas.wits.ac.za

Department of Mathematics, U-9, University of Connecticut, Storrs, CT 06269, USA
E-mail address: vinson@math.uconn.edu