ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 4, Winter 2002

ON TORSION AND MIXED MINIMAL ABELIAN GROUPS

B. GOLDSMITH AND S. Ó hÓGÁIN

ABSTRACT. An abelian group is said to be minimal if it is isomorphic to all its subgroups of finite index. We obtain a complete characterization of such groups in the torsion case; in the case of mixed groups of rank 1 we obtain a characterization for some large classes of such groups.

1. Introduction. There are many possible ways of considering the concept of minimality within a general categorical setting. Some approaches centered on categories of topological spaces have recently yielded interesting results (see, e.g., [**5**] and [**6**]). An intrinsic part of the difficulty in deciding on what approach to take appears to arise from the fact that the obvious orderings available are only reflexive and transitive and are not anti-symmetric. In recent work (see [**4**]) the authors and Wallutis have investigated categories of abelian groups from a similar standpoint to that applied by McMaster et al. in [**6**] and [**7**] for topological spaces. The present work presents a related but quite different approach. An abelian group G is said to be *minimal* if G is isomorphic to every subgroup H of finite index in G. Interestingly, an identical concept has been the subject of recent investigation in non-abelian group theory under the title "hc-groups"; the work of Robinson and Timm (see [**11**]) shows the connection between such groups in the finitely generated case and certain aspects of connectedness for manifolds.

The classification of all minimal abelian groups seems to be a difficult problem. However, by restricting our considerations to torsion groups and certain classes of mixed groups, it is possible to obtain fairly complete classifications in terms of Ulm invariants.

Our notation is standard and largely in accord with Fuchs [**2**] and [**3**], which contains all undefined terms used herein; an exception is that we

---

1991 AMS *Mathematics Subject Classification.* Primary 20K21, 20K10.
Received by the editors on July 28, 2001, and in revised form on October 15, 2001.

write mappings on the right and use the notation $A \sqsubset B$ to denote that A is a direct summand of B. In the sequel we shall use the term "group" to denote an abelian group.

2. General minimal groups. As mentioned above, a group G is said to be minimal if G is isomorphic to all its subgroups of finite index. Clearly a minimal group is finite if and only if it is the trivial group and so it makes sense to consider infinite groups only. It is not difficult to see that free groups are minimal as are elementary groups of infinite order. We begin by deriving some simple properties of minimal groups. Since a group cannot be simultaneously divisible and finite unless it is the trivial group we have immediately:

Lemma 2.1. *Every divisible group is minimal.*

Not surprisingly, we get a reduction to reduced groups:

Proposition 2.2. *Let $G = D \oplus M$ where D is the maximal divisible subgroup of G and M is reduced. Then G is minimal if and only if M is minimal.*

Proof. Straightforward.

If p is any prime we can define a local version of minimality at p as follows.

Definition 2.3. We say a group G is p-minimal if G is isomorphic to all its subgroups of index p.

Our next result shows that, in investigating the minimality of a group, it suffices to consider the local case.

Theorem 2.4. (i) *A group G is minimal if and only if G is p-minimal for all primes p;*

(ii) *A group G is p-minimal if and only if G is isomorphic to all its* *subgroups of index a power of* p;

(iii) *A* p*-group is minimal if and only if it is* p*-minimal.*

Proof. (i) If G is minimal, then obviously G is p-minimal for all p. Conversely, suppose that G is p-minimal for all p and that H is any finite index subgroup of G. We show that G is isomorphic to H by induction on the order of G/H. If $|G/H| = 1$, then $G = H$. Now assume that G is isomorphic to all its subgroups of index less than some integer n and let $|G/H| = n$. If n is prime, then $G \cong H$ by assumption. If n is not prime, then $n = r \cdot s$ for some $r, s < n$. Now G has a subgroup G_1 containing H such that $|G_1/H| = s$. Then $|G/G_1| = r$ and so the induction hypothesis gives us $G \cong G_1$ and hence G_1 is isomorphic to all its subgroups of index less than n since this property is an isomorphic invariant. Therefore $G_1 \cong H$ and so $G \cong H$, i.e., G is minimal.

(ii) The same argument as in (i), where the induction now is on the power of p.

(iii) This follows from (ii) and the fact that a p-group has no subgroups of index q for any prime $q \neq p$ since it is q-divisible.

Finally we show that, in the case of torsion minimal groups, we may restrict ourselves to the study of p-groups.

Theorem 2.5. *A torsion group is minimal if and only if all of its primary components are minimal.*

Proof. Let $G = \oplus_{p\in\Pi} G_p$. If G is minimal and H_p is of finite index in G_p, then $\oplus_{q\neq p} G_q \oplus H_p$ is of finite index in G and so $\oplus_{q\neq p} G_q \oplus H_p \overset{\phi}{\cong} G$. Therefore $\phi \restriction H_p : H_p \to G_p$ is an isomorphism.

Conversely, suppose that G_p is minimal for all p and H is of index q in G for some prime q. Then $H = \oplus_{p\in\Pi} H_p$ with $H_p = H \cap G_p$ for all p and $\mathbf{Z}(q) \cong G/H \cong \oplus_{p\in\Pi}(G_p/H_p)$. Therefore $H_p = G_p$ for all $p \neq q$ and $\mathbf{Z}(q) \cong G_q/H_q$. Now since G_q is minimal, we get that $G_q \cong H_q$ and so $H \cong G$. Hence G is minimal, by Theorem 2.4.

In light of Proposition 2.2 and Theorem 2.5 we concentrate on developing a characterization of reduced minimal p-groups. Not surprisingly, this characterization is in terms of the Ulm invariants of the group. We need the following result which is well known.

Lemma 2.6. *If G is a reduced p-group and B is a basic subgroup of G, then*

(i) $f_n(G) = f_n(G/p^\omega G)$ *for all* $n < \omega$;

(ii) *if B is bounded, then* $B = G$;

(iii) $f_n(B) = f_n(G)$ *for all* $n < \omega$.

Proof. See [**12**, Proposition 7.4 and Section 23].

Now we establish some important properties of the Ulm invariants of minimal p-groups.

Proposition 2.7. *Let G be a minimal p-group such that $f_n(G) = 0$ for some $n < \omega$. Then $f_m(G) = 0$ for all $m \geq n$ and G is bounded.*

Proof. Let B be a basic subgroup of G. By Lemma 2.6, $f_m(B) = f_m(G)$ for all m. So if $f_{n+1}(G) \neq 0$, then B has a summand $\mathbf{Z}(p^{n+2})$ and, since $B \leq_* G$ so has G, $G = \mathbf{Z}(p^{n+2}) \oplus G_1$, say. Now $H = p\mathbf{Z}(p^{n+2}) \oplus G_1$ is a finite index subgroup of G and so $H \cong G$. We have $0 = f_n(G) = f_n(H) = 1 + f_n(G_1) = 1 + f_n(G) = 1$, which is a contradiction. Therefore, $f_{n+1}(G) = 0$. A simple induction argument now gives that $f_m(G) = 0$ for all $m \geq n$. Also $f_m(B) = 0$ for all $m \geq n$ so B is bounded. Therefore, $G = B$, again by Lemma 2.6.

Proposition 2.8. *Let G be a minimal p-group such that $f_n(G) \neq 0$ for some $n < \omega$. Then $f_n(G)$ is infinite.*

Proof. Let B be a basic subgroup of G. Then $f_n(B) \neq 0$. Therefore B, and hence G, has a summand $\mathbf{Z}(p^{n+1})$, i.e., $G = \mathbf{Z}(p^{n+1}) \oplus G_1$. Since G/G_1 is finite, $G_1 \cong G$ and so $f_n(G_1) = f_n(G)$; this means that $f_n(G) = 1 + f_n(G)$ and so $f_n(G)$ must be infinite.

3. The characterization of minimal p-groups. The following result is the main tool used in characterizing minimal p-groups. We could, of course, have restricted our considerations to subgroups of index p, but we shall need the more general result later when we consider mixed groups.

Proposition 3.1. *If G is a p-group and H is a finite index subgroup of G, then there exists $K \leq H$ such that $K \sqsubset G$ and G/K is finite.*

Proof. See Pierce [**9**, Lemma 16.5].

We are now ready to state and prove the characterization.

Theorem 3.2. *If G is a reduced p-group, then G is minimal if and only if there exists an ordinal $\lambda \leq \omega$ such that $f_n(G)$ is infinite for all $n < \lambda$ and, if $\lambda < \omega$, $f_n(G) = 0$ for all $n \geq \lambda$.*

Proof. If G is minimal the necessity of the condition is given by Propositions 2.7 and 2.8. For sufficiency consider H, a subgroup of index p in G. Proposition 3.1 tells us that there exists $K \leq H$ such that $G = K \oplus A$ where A is cyclic. Then $H = K \oplus (H \cap A) = K \oplus B$, say. We get $f_n(G) = f_n(K) + f_n(A)$ and $f_n(H) = f_n(K) + f_n(B)$ so $f_n(K)$ and $f_n(H)$ are both infinite for $n < \lambda$, by assumption. Also, if $\lambda < \omega$, then $0 = f_n(G) = f_n(K) + f_n(A)$ so $f_n(K) = f_n(A) = 0$ for all $n \geq \lambda$ and $f_n(H) = 0 + f_n(B) = 0$ for all $n \geq \lambda$ since $B \leq A$ and A is cyclic. If B_K is a basic subgroup of K, then $f_n(B_K) = f_n(K) = f_n(G)$ for all n, so $B_K = \bigoplus_{I_1} \mathbf{Z}(p) \oplus \bigoplus_{I_2} \mathbf{Z}(p^2) \oplus \cdots \oplus \bigoplus_{I_n} \mathbf{Z}(p^n) \oplus \cdots$ with I_j infinite for all $j < \lambda$ and I_j empty for all $j \geq \lambda$ if $\lambda < \omega$. Since A is cyclic we get that $A = \mathbf{Z}(p^n)$ for some $n < \lambda$. B_K has a bounded summand $C_K = \oplus_{I_n} \mathbf{Z}(p^n)$ which is pure in K and so $C_K \sqsubset K$, $K = K_1 \oplus C_K$, say. Therefore, $G = K \oplus A = K_1 \oplus C_K \oplus A \cong K_1 \oplus C_K = K$. Similarly, we get that $H \cong K$ and so $H \cong G$. Therefore, G is minimal, by Theorem 2.4.

We finish this section with some consequences of Theorem 3.2.

Corollary 3.3. *A direct sum of cyclic p-groups is minimal if and only if it is of the form* $\bigoplus_{I_1} \mathbf{Z}(p) \oplus \bigoplus_{I_2} \mathbf{Z}(p^2) \oplus \cdots \oplus \bigoplus_{I_n} \mathbf{Z}(p^n) \oplus \cdots$ *where* I_n *is infinite for all* $n < \omega$ *or there exists* n *such that* I_j *is infinite for all* $j < n$ *and* I_j *is empty for all* $j \geq n$.

Proof. If G is a direct sum of cyclic p-groups, then $f_n(G)$ is the number of copies of $\mathbf{Z}(p^{n+1})$ for each $n \geq 0$ and now the result is immediate from Theorem 3.2.

Corollary 3.4. *A direct sum of minimal p-groups is minimal.*

Proof. If $G = \oplus_{i \in I} G_i$ where each G_i is a minimal p-group, then $f_n(G)$ is infinite if some $f_n(G_i)$ is infinite since Ulm invariants are additive.

Note that, in contrast to Corollary 3.4, direct sums of minimal groups are, in general, not necessarily minimal. A simple torsion-free example is given by $G = G_1 \oplus G_2$, any completely decomposable group of rank 2 where the types of G_1 and G_2 are incomparable with a common finite entry. Note also that a summand of a minimal p-group need not be minimal since, if A is minimal, then $f_n(A \oplus B)$ is infinite if $f_n(A)$ is, for all p-groups B. Also, the only Ulm invariants involved in the characterization are the finite ones. This is reflected in the following corollary.

Corollary 3.5. *A reduced p-group* G *is minimal if and only if* $G/p^\omega G$ *is minimal.*

Proof. Lemma 2.6 shows that $f_n(G) = f_n(G/p^\omega G)$ for all $n < \omega$ and now the result follows trivially from Theorem 3.2.

4. Purely minimal p-groups. In keeping with the general approach to minimality outlined in the introduction, we now proceed to consider the larger class of purely minimal p-groups.

Definition 4.1. A group G is *purely minimal* if G is isomorphic to all its pure subgroups of finite index.

Note that, if $H \leq_* G$ and G/H is finite, then $H \sqsubset G$. Therefore, G is purely minimal if and only if G is isomorphic to all its direct summands of finite index. We do not, however, have a corresponding reduction to subgroups of index p in the case of purely minimal groups, as the following example shows:

Example. $G = \bigoplus_{\aleph_0} \mathbf{Z}(p) \oplus \mathbf{Z}(p^2)$ is obviously not purely minimal; however, if $H \leq_* G$ of index p, then $H = \bigoplus_I \mathbf{Z}(p) \oplus \bigoplus_J \mathbf{Z}(p^2)$ where $|I| = \aleph_0$ and $|J| \leq 1$. Now $G \cong H \oplus \mathbf{Z}(p)$ and so $|J| = 1$ and thus $G \cong H$.

Lemma 4.2. *If a reduced p-group G is purely minimal and $f_n(G) \neq 0$ for some $n < \omega$, then $f_n(G)$ is infinite.*

Proof. The arguments are the same as in the proof of Proposition 2.8.

Theorem 4.3. *A reduced p-group G is purely minimal if and only if whenever $f_n(G) \neq 0$ then $f_n(G)$ is infinite.*

Proof. The necessity is given by the previous lemma. For sufficiency, let $G = H \oplus A$, where A is finite. If $f_n(G)$ is infinite, then so is $f_n(H)$ and if $f_n(G) = 0$, then so are $f_n(H)$ and $f_n(A)$. Let B_H be a basic subgroup of H. For any n such that $f_n(A) \neq 0$ we have that $f_n(G)$ is infinite and so $f_n(B_H) = f_n(H)$ is also infinite. If $A = \bigoplus_{I_1} \mathbf{Z}(p^{n_1}) \oplus \bigoplus_{I_2} \mathbf{Z}(p^{n_2}) \oplus \cdots \oplus \bigoplus_{I_r} \mathbf{Z}(p^{n_r})$ where $I_1, I_2, \dots, I_r$ are finite, then B_H, and hence H, has a summand $C_H = \bigoplus_{J_1} \mathbf{Z}(p^{n_1}) \oplus \bigoplus_{J_2} \mathbf{Z}(p^{n_2}) \oplus \cdots \oplus \bigoplus_{J_r} \mathbf{Z}(p^{n_r})$ where $J_1, J_2, \dots, J_r$ are infinite, $H = H_1 \oplus C_H$, say. Thus $G = H \oplus A = H_1 \oplus C_H \oplus A \cong H_1 \oplus C_H = H$, and hence G is purely minimal.

Note that a minimal p-group is obviously purely minimal but the converse is not necessarily true as we see in the following example.

Example. The group $G = \bigoplus_{\aleph_0} \mathbf{Z}(p) \oplus \bigoplus_{\aleph_0} \mathbf{Z}(p^3)$ is purely minimal but not minimal by Theorems 3.2 and 4.3.

5. Mixed minimal groups. We now consider mixed minimal groups. First we have some results on mixed minimal groups in general and then we concentrate on mixed groups of torsion-free rank 1.

Theorem 5.1. *Let G be a mixed group. If G is minimal, then both $t(G)$ and $G/t(G)$ are minimal where $t(G)$ is the torsion subgroup of G.*

Proof. First we show that $t(G)$ is minimal. Let S be of finite index in $t(G)$. Then $G/t(G) \cong (G/S)/(t(G)/S)$ so $(G/S)/(t(G)/S)$ is torsion-free and hence $t(G)/S \leq_* G/S$. But $t(G)/S$ is finite, especially it is bounded, and hence $tG/S \sqsubset G/S$, $G/S = t(G)/S \oplus K/S$, say. Now $G/K \cong (G/S)(K/S) \cong t(G)/S$ is finite and so $K \cong G$ since G is minimal. Therefore, $t(G) \cong t(K)$. Also $K/S \cong (G/S)/(t(G)/S) \cong G/t(G)$ is torsion-free, so $S = t(K)$, i.e., $S \cong t(G)$ as required.

It remains to show that $G/t(G)$ is minimal. Let $H/t(G)$ be of finite index in $G/t(G)$. Then $G/H \cong (G/t(G))/(H/t(G))$ is finite and so $G \cong H$ since G is minimal. Therefore, $G/t(G) \cong H/t(H)$. But $t(H) = H \cap t(G) = t(G)$ since $t(G) \leq H$ and so $G/t(G) \cong H/t(G)$.

The next theorem shows that, if G splits, then the converse is true. First we prove a lemma, due to Procházka [**10**], but the proof given here is based on Proposition 3.1.

Lemma 5.2. *Let G be a mixed group and H a finite index subgroup of G. Then, if G splits, so does H.*

Proof. If G/H is finite, then $t(G)/t(H) = t(G)/(H \cap t(G)) \cong (t(G)+H)/H \leq G/H$, so $t(G)/t(H)$ is finite. Now $t(G) = \bigoplus_{p\in\Pi} G_p$, $t(H) = \bigoplus_{p\in\Pi} H_p$ and $t(G)/t(H) \cong \bigoplus_{p\in\Pi}(G_p/H_p)$, so $H_p = G_p$ for almost all p. Suppose $H_p \neq G_p$ for $p = p_1, p_2, \dots, p_n$ and we have equality for all other p. For each p_i, $i = 1, 2, \dots, n$, by Theorem 3.1 there exists $K_{p_i} \leq H_{p_i}$ such that $G_{p_i} = K_{p_i} \oplus A_{p_i}$ and $H_{p_i} = K_{p_i} \oplus B_{p_i}$ where $B_{p_i} \leq A_{p_i}$ and A_{p_i} is finite. Therefore,

$t(G) = \bigoplus_{p \neq p_1, \dots, p_n} G_{p_i} \oplus \bigoplus_{i=1}^n (K_{p_i} \oplus A_{p_i}) = K \oplus \bigoplus_{i=1}^n A_{p_i}$ and $t(H) = \bigoplus_{p \neq p_1, \dots, p_n} G_{p_i} \oplus \bigoplus_{i=1}^n (K_{p_i} \oplus B_{p_i}) = K \oplus \bigoplus_{i=1}^n B_{p_i}$ where $K = \bigoplus_{p \neq p_1, \dots, p_n} G_{p_i} \oplus \bigoplus_{i=1}^n K_{p_i}$. Now G splits, by assumption, and so $G = t(G) \oplus G'$ where G' is torsion-free, i.e., $G = K \oplus A \oplus G'$ where $A = \bigoplus_{i=1}^n A_{p_i}$. Hence $H = H \cap G = K \oplus H \cap (A \oplus G') = K \oplus H'$ where $H' = H \cap (A \oplus G')$. But $t(H')$ is finite and so it is a summand of H'. Hence $H = K \oplus t(H') \oplus \overline{H}$ where $\overline{H}$ is torsion-free and so H also splits.

Using Lemma 5.2 we can now establish the following partial converse to Theorem 5.1.

Theorem 5.3. *If G splits and tG and $G/t(G)$ are both minimal, then G is minimal.*

Proof. Let H be of finite index in G. Then, as in the previous lemma, $t(H)$ is of finite index in $t(G)$, so $t(G) \cong t(H)$. Also $(G/t(G))/((t(G)+H)/t(G)) \cong G/(t(G)+H) \cong (G/H)/((t(G)+H)/H)$ again is finite, so $G/t(G) \cong (t(G)+H)/t(G) \cong H/(H \cap t(G)) = H/t(H)$. Now if G splits, then so does every finite index subgroup of G, by Lemma 5.2. Hence, if $G = t(G) \oplus K$ and $H = t(H) \oplus L$, say, then $K \cong G/t(G) \cong H/t(H) \cong L$ and $t(G) \cong t(H)$. Therefore $G \cong H$.

The second part of Theorem 5.1 and the necessary part of Proposition 2.2 are examples of a more general result concerning preradicals and socles; see, e.g., [**1**] for definitions.

Theorem 5.4. *If a group G is minimal and R is a socle preradical, then G/RG is also minimal.*

Proof. Let H/RG be of finite index in G/RG. Then H is of finite index in G and so $G \overset{\phi}{\cong} H$. We have $(RG)\phi \leq RH$ and $(RH)\phi^{-1} \leq RG$. Therefore $(RH)\phi^{-1}\phi \leq (RG)\phi$, i.e., $RH \leq (RG)\phi$ and so $(RG)\phi = RH$. Hence $G/RG \cong H/RH$. But $i : H \to G$ is a homomorphism, so $(RH)i \leq RG$, i.e., $RH \leq RG$. Also $RG \leq H$, by assumption, so $R(RG) \leq RH$ (consider $i : RG \to H$) and $R(RG) = RG$ so $RG \leq RH$.

We have $RG = RH$ and hence $G/RG \cong H/RG$.

6. Mixed groups of torsion-free rank 1. For the remainder of this paper we concentrate on mixed groups of torsion-free rank 1. First recall the definition of the generalized p-height of an element in a group.

Definition 6.1. If $a \in A$, the *generalized p-height* of a in A, $h_p^{*A}(a)$ or just $h_p^*(a)$, if A is understood, is defined by $h_p^*(a) = \sigma$ if $a \in p^\sigma A \setminus p^{\sigma+1}A$ where $p^\sigma A$ is defined inductively. If $p^\tau A = p^{\tau+1}A$ (i.e., $p^\tau A$ is p-divisible and $a \in p^\tau A$), we set $h_p^*(a) = \infty$ and we consider ∞ larger than every ordinal.

Lemma 6.2. *If $a \in A$ and $n \in \mathbf{Z}$ and if p is a prime such that p does not divide n, then $h_p^*(a) = h_p^*(na)$.*

Proof. Straightforward.

Lemma 6.3. *For all $a \in A$ and for all $n \in \mathbf{N}$, $h_p^{*A}(a) \le h_p^{*nA}(na)$.*

Proof. If $a \in p^\sigma A$, then $na \in np^\sigma A$, so it suffices to show $np^\sigma A \le p^\sigma(nA)$ for all σ. We use transfinite induction on σ. It is obviously true for all $\sigma < \omega$. Suppose that $np^\sigma A \le p^\sigma(nA)$ for all $\sigma < \rho$ and first consider $\rho = \tau + 1$. Then $np^\rho A = npp^\tau A = pnp^\tau A \le pp^\tau(nA) = p^{\tau+1}(nA)$, by the induction hypothesis. If ρ is a limit ordinal, then $np^\rho A = n \cap_{\sigma<\rho} p^\sigma A \le \cap_{\sigma<\rho} np^\sigma A \le \cap_{\sigma<\rho} p^\sigma(nA) = p^\rho(nA)$, also by the induction hypothesis.

Again let A be an arbitrary group. Each element $a \in A$ has an associated matrix, called the height-matrix of a, defined in the following way.

Definition 6.4. The height-matrix of a is the $\omega \times \omega$ matrix (whose entries are ordinals or ∞) $H(a) = (\sigma_{nk})$, where $\sigma_{nk} = h_{p_n}^*(p_n^k a)$ and $n = 1, 2, \dots, k = 0, 1, \dots$ and the primes are arranged in order of magnitude.

Definition 6.5. Two $\omega \times \omega$ matrices, σ_{nk} and ρ_{nk} whose entries are ordinals or ∞, are equivalent if almost all their rows are identical and for each of the other rows there exist integers $l, m \geq 0$ such that $\sigma_{n,k+l} = \rho_{n,k+m}$ for all $k \geq 0$.

It is easily seen that this is an equivalence relation on the class of such matrices. Now if A is a mixed group of torsion-free rank 1 and $a, b \in A$ are any two torsion-free elements, then there exist some $r, s \in \mathbf{Z}$ such that $ra = sb$. If p_n is any prime such that p_n does not divide rs, then the nth rows of $H(a)$ and $H(b)$ are the same, by Lemma 6.2. If p_n divides rs (i.e., $r = p_n^l r_1$, $s = p_n^m s_1$) where $(p_n, r_1) = 1 = (p_n, s_1)$, then $h^*_{p_n}(ra) = h^*_{p_n}(sb)$, so $h^*_{p_n}(p_n^l a) = h^*_{p_n}(p_n^m b)$, again by Lemma 6.2, i.e., $\sigma_{nl} = \rho_{nm}$ where $H(a) = (\sigma_{nk})$ and $H(b) = (\rho_{nk})$. Similarly, for any $k > 0$, $r_1 p_n^{l+k} a = s_1 p_n^{m+k} b$ and so $h^*_{p_n}(p_n^{k+l} a) = h^*_{p_n}(p_n^{k+m} b)$, i.e., $\sigma_{n,k+l} = \rho_{n,k+m}$. Therefore, $H(a)$ and $H(b)$ are equivalent and we denote this equivalence class simply as $H(A)$.

We now state a theorem which gives a condition, in terms of height matrices, for the isomorphism of groups in large classes of mixed groups of torsion-free rank 1, and we then apply this theorem to investigate minimality in these classes of groups.

Theorem 6.6. *Let A and C be countable mixed groups of torsion-free rank* 1. *Then $A \cong C$ if and only if*

(i) $t(A) \cong t(C)$ *and*

(ii) $H(A) = H(C)$.

Proof. See [**3**, Theorem 104.3].

This theorem can be extended [**13**], [**8**] to the cases where

(a) $t(A)$ and $t(C)$ are totally projective.

(b) $t(A)$ and $t(C)$ are torsion-complete.

We first apply this result to prove a theorem concerning the minimality of local mixed groups of torsion-free rank 1 with divisible torsion-free quotient and then we use it to consider minimality in classes of mixed groups of torsion-free rank 1 whose height matrices have rows which are eventually gap-free.

From now on we will assume that G is a reduced mixed group of torsion-free rank 1 such that $t(G)$ belongs to one of the classes described in (a) or (b) above.

Theorem 6.7. *Let $t(G)$ be a p-group for some prime p, and let $G/t(G) \cong \mathbf{Q}$. Then G is minimal if and only if $t(G)$ is minimal.*

Proof. If G is minimal, then $t(G)$ is minimal by Theorem 5.1. Conversely, suppose that $t(G)$ is minimal. By Theorem 2.4 it suffices to consider subgroups of G of prime index. First if A is a subgroup of G of some prime index $q \neq p$, then $t(G)/t(A) = t(G)/A \cap t(G) \cong (t(G)+A)/A \leq G/A$. Therefore, $t(G) = t(A)$ since $t(G)/t(A)$ is a p-group and $G/A \cong \mathbf{Z}(q)$. Now $G/A \cong [G/t(G)]/[A/t(G)]$ is both divisible and finite and so must be zero, a contradiction. Hence we need only consider subgroups A of G of index p. Then $pG \leq A$ and $t(G) \cong t(A)$ since $t(G)$ is minimal. Next note that if $t(G) \leq A$ then, as above, G/A is both divisible and finite and so must be zero. Therefore, $t(G) \not\leq A$ and $(A + t(G))/A$ is a nonzero subgroup of G/A and so $(A+t(G))/A = G/A$ since G/A is a simple group. Hence there exists some $x \in t(G) \setminus A$ such that $G = \langle A, x\rangle$. If $g \in G$ is any element in G, then $g = a + rx$ for some $a \in A$ and some $0 \leq r < p$ and, since $t(G)$ is a p-group, there exists some $k \in \mathbf{N}$ such that $p^k x = 0$ and so $p^k g = p^k a \in A$ and therefore $p^k G = p^k A$. Now let $g \in G$ be any torsion-free element in G. Then $g_1 = p^k g \in p^k G = p^k A$. For all $n \geq 0$ we have $h_p^{*G}(p^n g_1) \geq h_p^{*A}(p^n g_1)$ and if $h_p^{*G}(p^n g_1) > h_p^{*A}(p^n g_1) = \sigma$, say, where $\sigma \geq k$, then $p^n g_1 = p g_2$ for some $g_2 \in p^\sigma G = p^\sigma A$ so $p^n g_1 = p a_1$ for some $a_1 \in p^\sigma A$, a contradiction to $h_p^{*A}(p^n g_1) = \sigma$. Therefore, $h_p^{*G}(p^n g_1) = h_p^{*A}(p^n g_1)$ for all $n \geq 0$. If $q \neq p$, then $h_q^{*G}(q^n g_1) \leq h_q^{*pG}(pq^n g_1) \leq h_q^{*A}(pq^n g_1) = h_q^{*A}(q^n g_1)$ for all $n \geq 0$ by Lemmas 6.2 and 6.3, and the converse inequality is true since $A \leq G$. We conclude that $H^G(g_1)$ is equivalent to $H^A(g_1)$ and so $H(A) = H(G)$. Hence $A \cong G$, by Theorem 6.6, and it follows that G is minimal by Theorem 2.4.

In our next result we do not need to assume that the torsion subgroup is a p-group.

Theorem 6.8. *Suppose that each row of $H(G)$ is eventually gap-free, i.e., given any torsion-free element a in G, for each prime p, there exists some n_p such that $h_p^{*G}(p^{k+1}a) = h_p^{*G}(p^k a) + 1$ for all $k \geq n_p$. Then G is minimal if and only if tG is minimal.*

Proof. If G is minimal, then $t(G)$ is minimal, again by Theorem 5.1. Conversely, suppose that $t(G)$ is minimal. As in Theorem 6.7 it suffices to consider a subgroup A of G of prime index p in G for any prime p. Then $pG \leq A$ and $t(A) \cong t(G)$ since $t(G)$ is minimal. The group A contains torsion-free elements since $pG \leq A$ and so A is also mixed of torsion-free rank 1. Let $a \in A$ be torsion-free and first consider $q \neq p$. Then, for any $k, h_q^{*G}(q^k a) \leq h_q^{*pG}(pq^k a) \leq h_q^{*A}(pq^k a) = h_q^{*A}(q^k a)$, using Lemmas 6.2 and 6.3, and, since $A \leq G$, we have $h_q^{*A}(q^k a) \leq h_q^{*G}(q^k a)$. We conclude that $h_p^{*A}(q^k a) = h_p^{*G}(q^k a)$. Now suppose that $H^G(a)$ has no gaps after the kth entry in the row corresponding to p. Then $h_p^{*G}(p^k a) \leq h_p^{*pG}(p^{k+1}a) \leq h_p^{*A}(p^{k+1}a) \leq h_p^{*G}(p^{k+1}a) \leq h_p^{*pG}(p^{k+2}a) \leq h_p^{*A}(p^{k+2}a) \leq h_p^{*G}(p^{k+2}a) \leq \dots$. So, if $H^G(a) = (\sigma_{nm})$ and $H^A(a) = (\rho_{nm})$, then we get that $\sigma_{pk} \leq \rho_{p,k+1} \leq \sigma_{p,k+1} \leq \rho_{p,k+2} \leq \sigma_{p,k+2} \leq \dots$; and, since both sequences are strictly increasing and $\sigma_{pk}, \sigma_{p,k+1}, \sigma_{p,k+2} \dots$ has no gaps, we get that $H^G(a)$ and $H^A(a)$ are equivalent and so $H(A) = H(G)$ and hence $A \cong G$ by Theorem 6.6. Therefore, G is minimal.

If G is any mixed group of torsion-free rank 1, Megibben [**8**] has observed that the rank 1 torsion-free group $G/t(G)$ can be recovered from $H(G)$. If (σ_{ij}) is any matrix in $H(G)$, then we define a sequence $(k_1, k_2, \dots, k_n, \dots)$ as follows: $k_n = \infty$ if the nth row of (σ_{ij}) contains an infinite ordinal or the symbol ∞ or has infinitely many gaps and $k_n = \sigma_{nj} - j$ if the nth row of (σ_{ij}) contains only integers and has no gaps after $\sigma_{n,j-1}$. Then it is not difficult to see that $(k_1, k_2, \dots, k_n, \dots)$ is the characteristic of some element in $G/t(G)$ and hence determines $G/t(G)$. If $t(G)$ is a p-group for some prime p, then let $p_n \neq p$ be any other prime, and let $0 \neq g + t(G) = \bar{g} \in \overline{G} = G/t(G)$ have characteristic $(k_1, k_2, \dots, k_n, \dots)$. We have $h_{p_n}^{*G}(g) \leq h_{p_n}^{*\overline{G}}(\bar{g}) = k_n$. If k_n is finite, then $\bar{g} = p_n^{k_n}\overline{g_1}$ and $g = p_n^{k_n} g_1 + t$ where $g_1 \in G$ and t is torsion; so $p^s g = p^s p_n^{k_n} g_1$ for some s, and hence $h_{p_n}^*(g) = h_{p_n}^*(p_n^{k_n} g_1) \geq k_n$. Therefore $h_{p_n}^*(g) = k_n$. Similarly, $h_{p_n}^{*G}(p_n^k g) = h_{p_n}^{*\overline{G}}(p_n^k)\bar{g} = k_n + k$ for all $k \in \mathbf{N}$. Now consider $k_n = \infty$. First note that $p_n^{\omega}\overline{G} = p_n^{\infty}\overline{G}$ since $\overline{G}$ is torsion-free. We show by transfinite induction that whenever $\bar{g} \in p_n^{\sigma}\overline{G}$ then $g \in p_n^{\sigma}G$. If $\sigma < \omega$, then it has been proved above. Assume that it is true for all ordinals $< \sigma$ and first consider $\sigma = \rho + 1$. Then if $\bar{g} \in p_n^{\sigma}\overline{G}$, we have $\bar{g} = p_n\overline{g_1}$ where $\overline{g_1} \in p_n^{\rho}\overline{G}$. If $\overline{g_1} = g_1 + t(G)$, then the induction hypothesis tells us that $g_1 \in p_n^{\rho}G$. But $g = p_n g_1 + t$ where $p^s t = 0$ for some $s \in \mathbf{N}$, so $h_{p_n}^*(g) \geq h_{p_n}^*(g_1) + 1 \geq \rho + 1 = \sigma$. If σ is a limit ordinal, then $\bar{g} \in p_n^{\rho}\overline{G}$ for all $\rho < \sigma$; so, again appealing to the induction hypothesis, we get that $g \in p_n^{\rho}G$ for all $\rho < \sigma$ and so $g \in p_n^{\sigma}G$. Therefore, if $k_n = \infty$, we must have that $h_{p_n}(g) = \infty$.

We can conclude that if $t(G)$ is a p-group then $G/t(G)$ determines all the rows of $H(G)$ except the row corresponding to p. We now use this fact to establish our final result.

Theorem 6.9. *Suppose that $t(G)$ is a p-group and the row corresponding to p in $H(G)$ is eventually gap-free. Then G is minimal if and only if both $t(G)$ and $G/t(G)$ are minimal.*

Proof. If G is minimal, then both $t(G)$ and $G/t(G)$ are minimal by Theorem 5.1. Conversely, suppose that $t(G)$ and $G/t(G)$ are minimal. Let A be a subgroup of G of prime index. Then, as in Theorem 5.3, $t(A) \cong t(G)$ and $A/t(A) \cong G/t(G)$. Let $a \in A$ be torsion-free. We wish to show that $H^A(a)$ is equivalent to $H^G(a)$. By what has been said above, we need only consider the row corresponding to p. If $|G/A| = q \neq p$, then, for any $k \in \mathbf{N}$, we have $h_p^{*G}(p^k a) \leq h_p^{*qG}(qp^k a) \leq h_p^{*A}(qp^k a) = h_p^{*A}(p^k a)$ and so $h_p^{*G}(p^k a) = h_p^{*A}(p^k a)$. Hence $H(A) = H(G)$ and $A \cong G$. If $|G/A| = p$, then, since the row corresponding to p has only a finite number of gaps, proceeding as in Theorem 6.8, we get that $H^A(a)$ is equivalent to $H^G(a)$, and again $A \cong G$. We conclude that G is minimal.

REFERENCES

1. B. Charles, *Sous-groupes fonctoriels et topologies*, Springer-Verlag, Dunod, 1967.

2. L. Fuchs, *Infinite abelian groups* I, Academic Press, New York, 1970.

3. ———, *Infinite abelian groups* II, Academic Press, New York, 1973.

4. B. Goldsmith, S. Ó hÓgáin and S. Wallutis, *Quasi-minimal abelian groups*, submitted.

5. J. Ginsburg and B. Sands, *Minimal infinite topological spaces*, Amer. Math. Monthly **86** (1979), 574–576.

6. J. Matier and T.B.M. McMaster, *Iteration of the "anti" operation in ordered topological spaces and in other categorical contexts*, Boll. Un. Mat. Ital. **7** (1993), 369–378.

7. P.T. Matthews and T.B.M. McMaster, *A viewpoint on minimality in topology*, Irish Math. Soc. Bull. **32** (1994), 7–16.

8. C. Megibben, *On mixed groups of torsion-free rank* 1, Illinois J. Math. **11** (1967), 134–144.

9. R.S. Pierce, *Homomorphisms of primary abelian groups*, *Topics in abelian groups*, Chicago, IL, 1963, 215–310.

10. L. Procházka, *Bemerkung über die Spaltbarkeit der gemischten abelschen Gruppen*, Czechoslovak. Math. J. **10** (1960), 479–492 (in Russian).

11. D.J.S. Robinson and M. Timm, *On groups that are isomorphic with every subgroup of finite index and their topology*, J. London Math. Soc. **57** (1998), 91–104.

12. L. Salce, *Struttura dei p-Gruppi Abeliani*, Quaderni dell'Unione Matematica Italiana, 1980.

13. K.D. Wallace, *On mixed groups of torsion-free rank* 1 *with totally projective primary components*, J. Algebra **17** (1971), 482–488.

School of Mathematical Sciences, Dublin Institute of Technology, Kevin Street, Dublin 8, Ireland
E-mail address: Brendan.Goldsmith@DIT.ie

School of Mathematical Sciences, Dublin Institute of Technology, Kevin Street, Dublin 8, Ireland
E-mail address: johog@maths.tcd.ie