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STRICTLY NONZERO CHARGES

RÜDIGER GÖBEL AND K.P.S. BHASKARA RAO

This paper is dedicated to the memory of our dear friend, Rae Michael Shortt.

Kelley in [**4**] discovered necessary and sufficient conditions on a Boolean algebra to admit a strictly positive bounded real-valued charge. As was noted in [**5**], the same conditions also characterize Boolean algebras that admit strictly nonzero bounded real-valued charges.

If G is a group and $\mathcal{A}$ is a Boolean algebra when would there exist a charge

$$\mu : \mathcal{A} \to G$$

which is strictly nonzero in the sense that $\mu(A) \neq 0$ whenever $A \in \mathcal{A}$ and $A \neq \emptyset$? The present paper is devoted to a study of this problem and its ramifications. We shall start with a result which says that to study group valued charges one has to look at only commutative groups. $\mathcal{A}, \mathcal{B}, \ldots$ stand for Boolean algebras or fields of sets and G stands for a group written additively.

Proposition 1. *If $\mu : \mathcal{A} \to G$ is a charge, there exists an abelian subgroup H of G such that $\mu(A) \in H$ whenever $A \in \mathcal{A}$.*

Proof. Let $D = \{\mu(A) : A \in \mathcal{A}\}$, the range of μ. If A and $B \in \mathcal{A}$, then $\mu(A)+\mu(B)-\mu(A\cap B) = \mu(A\cup B) = \mu(B\cup A) = \mu(B)+\mu(A)-\mu(A\cap B)$. Thus, if $x, y \in D$, then $x+y = y+x$. This implies that $\langle D\rangle$, the group generated by D, is commutative. □

Thus we assume that *all groups are abelian.* The next results says

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that there are plenty of nonzero G-valued charges on any given Boolean algebra provided G is nontrivial.

Proposition 2. *Given any field $\mathcal{A}$ of subsets of a given set and any nontrivial group G, there are many nonzero charges $\mu : \mathcal{A} \to G$. In fact, the number of charges defined on $\mathcal{A}$ taking values in G is $|G|^{|\mathcal{A}|}$ if $\mathcal{A}$ is infinite, and $|G|^{\log_2 |\mathcal{A}|}$ if $\mathcal{A}$ is finite.*

Proof. First let $\mathcal{A}$ be infinite. Let $S(X, \mathcal{A})$ be the set of all functions $f : X \to \mathbf{Z}$ such that range of f is finite and such that $f^{-1}(n) \in \mathcal{A}$ for all $n \in \mathbf{Z}$. This ring $S(X, \mathcal{A})$ is a Specker subgroup of the Nöbeling group $B(X) = S(X, \mathcal{P}(X))$, hence free, see [**1**]. Any map from a set of free generators of $S(X, \mathcal{A})$ to G extends naturally to a group homomorphism from $S(X, \mathcal{A})$ to G which in turn will induce a charge $\mu : \mathcal{A} \to G$ in a natural way. The number of charges in the proposition is now obvious. In case $\mathcal{A}$ is finite, observe that any charge on $\mathcal{A}$ is determined by its values on the atoms of $\mathcal{A}$. This gives the proof of the second part regarding the number of charges. $\square$

Note that ultrafilters in $\mathcal{A}$ can also be used to construct some (two-valued) charges. Having established that we have to look at only commutative groups and that there are many group valued charges let us look at strictly nonzero group valued charges.

Definition 3. We shall say that a charge $\mu : \mathcal{A} \to G$ is a *strictly nonzero G-valued charge* (snz *G-valued charge*) if $\mu(A) \neq 0$ whenever $A \in \mathcal{A}$ and $A \neq \emptyset$.

For the group $\mathbf{R}$ of real numbers, it is possible that on a field $\mathcal{A}$ of sets there is an snz $\mathbf{R}$-valued charge but there is no bounded snz $\mathbf{R}$-valued charge. Let us first see that, on a field $\mathcal{A}$ of sets, there is an snz bounded $\mathbf{R}$-valued charge if and only if there is a strictly positive ($\mu(A) > 0$ whenever $A \in \mathcal{A}$ and $A \neq \emptyset$) bounded $\mathbf{R}$-valued charge on $\mathcal{A}$.

In fact, if $\mu : \mathcal{A} \to \mathbf{R}$ is an snz bounded $\mathbf{R}$-valued charge by taking the positive and negative variations μ^+ and μ^- on $\mathcal{A}$ (as in [**5**]), one sees that $\mu^+(A) + \mu^-(A) > 0$ whenever $\mu^+(A) - \mu^-(A) \neq 0$. Thus

$|\mu| = \mu^+ + \mu^-$ is a strictly positive bounded charge on $\mathcal{A}$. The converse is trivial.

Example 4. Let $\mathcal{A}$ be the finite co-finite field on an uncountable set X. Since X is uncountable there are uncountably many pairwise disjoint sets in $\mathcal{A}$. Hence $\mathcal{A}$ has no bounded snz charge. Let us define an snz $\mathbf{R}$-valued (in fact $\mathbf{Z}$-valued) charge μ on $\mathcal{A}$. We define

$$\mu(A) = \begin{cases} 2 \cdot |A| & \text{if } A \text{ is finite} \\ 1 - 2 \cdot |A^c| & \text{if } A \text{ is co-finite.} \end{cases}$$

Then μ is an snz $\mathbf{R}$-valued charge on $\mathcal{A}$. This μ is also an snz $\mathbf{Z}$-valued charge.

This example makes the following problem interesting.

Problem 1. *Find necessary and sufficient conditions on a field $\mathcal{A}$ so that it admits an snz $\mathbf{R}$-valued charge.*

Later we shall also investigate the problem of finding necessary and sufficient conditions on a field $\mathcal{A}$ so that it admits an snz $\mathbf{Z}$-valued charge.

As in Proposition 6, *infra* we can see that if there is an snz $\mathbf{R}$-valued charge on a field of sets $\mathcal{A}$, then every linearly ordered (under inclusion) collection of sets in $\mathcal{A}$ must be of power c (the power of the continuum) at most. Let us now look at snz G-valued charges for other groups. Finite groups are easily dealt with.

Proposition 5. *Let G be a finite group and $\mathcal{A}$ a field of subsets of a set X.*

a) *If there is an* snz *G-valued charge then $\mathcal{A}$ is finite.*

b) *If $\mathcal{A}$ is finite, say it has 2^n sets, then there is an* snz *G-valued charge on $\mathcal{A}$ if and only if $*(G) \geq n$ where $*(G)$ is defined as* $\max\{k :$ *there exists $g_1, g_2, \dots, g_k \in G$ such that $\sum_{i \in I} g_i \neq 0$ for every nonempty set $I \subset \{1, 2, \dots, k\}$*$\}$.

Next we consider chain conditions on $\mathcal{A}$ caused by the existence of G-valued charges.

Proposition 6. *In any field $\mathcal{A}$ of sets which admits an* snz *G-valued charge, every linearly ordered subset of $\mathcal{A}$ has cardinality at most $|G|$.*

Note that any linearly ordered set as in the above proposition must be at most countable if $G = \mathbf{Z}$.

Proof. Let $\mathcal{L} \subset A$ be a linearly ordered subset of $\mathcal{A}$ of size $> |G|$. If $\mu : \mathcal{A} \to G$ is a G-valued snz charge, then by the pigeon hole principle there are $A \neq B \in \mathcal{L}$ such that $\mu(A) = \mu(B)$. We may assume $A \subset B$ by linearity. Hence $B - A \neq \emptyset$ and $\mu(B - A) = \mu(B) - \mu(A) = 0$. This contradicts that μ is snz. So $|\mathcal{L}| \leq |G|$. □

We apply Proposition 6 to a classical case.

If we consider $\mathcal{A} = \mathcal{B}/\mathcal{I}$ where $\mathcal{B}$ is the Borel σ-field of $[0, 1]$ and $\mathcal{I}$ is an ideal of Lebesgue null sets, then $\mathcal{A}$ has an snz $\mathbf{R}$-valued charge but $\mathcal{A}$ does not admit an snz $\mathbf{Z}$-valued charge.

Another example would be:

Example 7. $\mathcal{A} = \{$the field generated by sets of the form $[a, b)$ where $0 \leq a < b \leq 1\}$ on the set $X = [0, 1)$ on which there is an snz $\mathbf{R}$-valued change (the Lebesgue measure) and $\mathcal{A}$ does not admit an snz $\mathbf{Z}$-valued charge.

We shall now see that any field $\mathcal{A}$ which admits an snz $\mathbf{Z}$-valued charge should also satisfy a version of the countable chain condition *ccc* (i.e., every family of pairwise disjoint sets is countable). We first need a simple result.

Proposition 8. *If $\mathcal{A}$ is an infinite field of sets with an* snz *$\mathbf{Z}$-valued charge μ, then μ is unbounded both in the positive and negative directions, i.e., for any integer k, there exist $A, B \in \mathcal{A}$ such that $\mu(A) > k$ and $\mu(B) < -k$.*

Proof. Since $\mathcal{A}$ is infinite there are sets $A_1 \subset A_2 \subset A_3 \ldots$, all distinct, in $\mathcal{A}$. Since $\{\mu(A_i) : i \geq 1\}$ is a set of integers, distinct, $\{\mu(A_i) : i \geq 1\}$ is unbounded either in the positive direction or in the negative direction. Then $\{\mu(X - A_i) : i \geq 1\}$ is unbounded in the negative direction or in the positive direction correspondingly. In any case μ is unbounded both in the positive direction and in the negative direction. $\square$

Given a field of sets $\mathcal{A}$ let us say that a set $A \in \mathcal{A}$ is $*$-infinite if $|A \cap \mathcal{A}|$ is infinite, where $A \cap \mathcal{A} = \{A \cap B : B \in \mathcal{A}\}$.

Proposition 9. (a) *If $\mathcal{A}$ is any field of subsets of a set X with an* snz **Z**-*valued charge, then any family of pairwise disjoint $*$-infinite sets is countable.*

(b) *If $\mathcal{A}$ is atomless* (*i.e., $\mathcal{A}$ is nonatomic*) *with an* snz **Z**-*valued charge, then $\mathcal{A}$ satisfies ccc.*

Proof. (a) Let μ be an snz **Z**-valued charge on $\mathcal{A}$ and $\{A_i\}_{i\in I}$ a family of pairwise disjoint $*$-infinite sets from $\mathcal{A}$, and let I be uncountable. Clearly $\mu(A_i) \neq 0$ for all $i \in I$. Hence there is an integer $a \neq 0$ and an uncountable set $J \subset I$ such that $\mu(A_i) = a$ for all $i \in J$. Let us assume that $\mu(A_i) = a$ for all $i \in I$. Since each A_i is $*$-infinite, $A_i \cap \mathcal{A}$, as a field of subsets of A_i is infinite, and by Proposition 8, there exist $B_i \subset A_i$, $B_i \in \mathcal{A}$, for all $i \in I$ such that $\mu(B_i) > a$ for all $i \in I$. Again, without loss of generality, let us assume that $\mu(B_i) = b$ for all $i \in I$ and $b > a$. Then $a - b < 0$. Take sets $I_0 \subset I$ and $I_1 \subset I$ such that $|I_0| = a$ and $|I_1| = b - a$. Then $\mu\big(\bigcup_{i\in I_0}(A_i - B_i) \cup (\bigcup_{i\in I_1} A_i)\big) = a(a-b) + a(b-a) = 0$. Thus μ is not an snz charge.

(b) Follows from (a) because in an atomless field $\mathcal{A}$ every nonempty set $A \in \mathcal{A}$ is $*$-infinite. $\square$

We can use the above proposition to construct an example of a field of sets in which every linearly ordered set (under inclusion) is countable, but the field does not admit an snz **Z**-valued charge.

Example 10. Let $\mathcal{B}$ be the finite co-finite field on an uncountable set X. Let $(A_i)_{i\in I}$ be an uncountable partition of X such that $|A_i|$ is infinite for each $i \in I$. Let $\mathcal{A}$ be the field generated by $\{\mathcal{B}, (A_i)_{i\in I}\}$.

Then every linearly ordered collection of sets from $\mathcal{A}$ is countable.

To see this, define $\mathcal{I} = \{\bigcup_{i\in I_1} B_i : I_1 \subset I, I_1$ is finite and $B_i \in A_i \cap \mathcal{B}\}$ and observe that, for every $A \in \mathcal{A}$ either $A \in \mathcal{I}$ or $X - A \in \mathcal{I}$. But, by Proposition 8, there is no snz $\mathbf{Z}$-valued charge on $\mathcal{A}$.

Thus, for an atomless field of sets $\mathcal{A}$, if there is an snz $\mathbf{Z}$-valued charge on $\mathcal{A}$, then in $\mathcal{A}$ every linearly ordered collection (under inclusion) is countable and also $\mathcal{A}$ satisfies ccc. The field of clopen sets of the product topological space $\{0,1\}^Y$ for any set Y satisfies both these conditions. We shall now study the existence of snz $\mathbf{Z}$-valued charges on these fields of sets. We shall denote the Cantor set by $C = 2^{\aleph_0} = \{0,1\}^{\aleph_0} = \{0,1\}^\omega$. We shall write $\mathcal{B}(2^Y)$ for the clopen sets of $\{0,1\}^Y$. In particular, $\mathcal{B}(2^{\aleph_0})$ stands for the clopen sets of C.

Let Y be an infinite set. For $A, B \subset Y$, where $|A|$ and $|B|$ are finite and $A \cap B = \emptyset$, we shall write

$$H(A,B) = \{f \in 2^Y : f(y) = 0 \text{ for } y \in A \text{ and } f(y) = 1 \text{ for } y \in B\}.$$

Every $H(A,B)$ is clopen in the product topology on 2^Y and every set in $\mathcal{B}(2^Y)$ is a finite disjoint union of sets of the type $H(A,B)$. For example, $H(\emptyset,\emptyset) = 2^Y$. If $Y = \omega = \{0,1,2,\dots\}$ then $H(\{1\},\emptyset) = \{0\} \times \{0,1\} \times \{0,1\} \times \dots$. If $A, B \subset Y$, where $|A|$ and $|B|$ are finite, $A \cap B = \emptyset$ and $y_0 \notin A \cup B$, then an innocent but important formula that holds is

$$H(A \cup \{y_0\}, B) \uplus H(A, B \cup \{y_0\}) = H(A,B), \tag{1}$$

where $\uplus$ stands for disjoint union. For $Y = \omega = \{0,1,2,\dots\}$

$$\big\{H(A,B) : A \cap B = \emptyset, A \cup B = \{0,1,2,\dots,n\}\big\}$$

form a partition of 2^ω. We shall denote all possible unions of this collection of sets for a given n by $\mathcal{B}_n$. Then $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \dots$ and $\bigcup_{n=0}^\infty \mathcal{B}_n = \mathcal{B}(2^\omega)$.

Definition 11. For a group G we shall call an snz G-valued charge a *good charge* and we shall call an snz G-valued charge μ on $\mathcal{B}(2^Y)$ a *very good charge* if $\mu(H(A,B))$ is a function of $|A|$ and $|B|$ only.

If μ is a very good G-valued charge on $\mathcal{B}(2^Y)$, let us write

$$p_n = \mu(H(n,\ \emptyset)) = \mu(H(A,\ \emptyset)) \quad \text{where } A \subset Y \text{ such that } |A| = n. \tag{2}$$

Thus $\mu(2^Y) = p_0$. Then $\mu(H(m,k))$ is uniquely defined and induction using (1) shows

$$\mu(H(m,k)) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \, p_{m+i}. \tag{3}$$

If $D \in \mathcal{B}(2^Y)$, then there exists a finite set $\{y_1, y_2, \dots, y_n\} \subset Y$ such that D is a disjoint union of sets of the type $H(A,B)$ where $A \cap B = \emptyset$ and $A \cup B = \{y_1, y_2, \dots, y_n\}$. Now $\mu(D)$ can be calculated from $\{\mu(H(m,k)) : m+k=n\}$. Also

$$\mu(D) = \sum_{\substack{m=0 \\ m+k=n}}^{n} i_m \, \mu(H(m,k))$$

for some $0 \le i_m \le \binom{n}{m}$ for $m = 0, 1, \dots, n$. For any group G, let the torsion-free rank be denoted by $r_0 G$ (see [**1**]).

Proposition 12. *Let* $|Y| \ge \aleph_0$.

(a) $\mathcal{B}(2^Y)$ *has a very good* G*-valued charge if and only if* $\mathcal{B}(2^\omega)$ *has a very good* G*-valued charge.*

(b) *If* $r_0 G \ge \aleph_0$, *then* $\mathcal{B}(2^\omega)$ *has a very good* G*-valued charge. Hence* $\mathcal{B}(2^Y)$ *has a very good* G*-valued charge for all* Y *if* $r_0(G) \ge \aleph_0$.

(c) *If* $r_0 G \ge 2^{|Y|}$, *then on* $\mathcal{P}(Y)$ *there is an* snz G*-valued charge.*

Proof. (a) If $|Y| \ge \aleph_0$, let us assume that $\omega \subset Y$. Also $\{B \times 2^{Y \setminus \omega} : B \in \mathcal{B}(2^\omega)\}$ is a subfield of $\mathcal{B}(2^Y)$. If μ is a very good charge on $\mathcal{B}(2^Y)$, then its restriction to $\{B \times 2^{Y \setminus \omega} : B \in \mathcal{B}(2^\omega)\}$ is also a very good charge. This can be identified as a very good charge on $\mathcal{B}(2^\omega)$.

Conversely, if there is a very good charge μ on $\mathcal{B}(2^\omega)$, then by defining τ on $\mathcal{B}(2^Y)$ by $\tau(H(A,B)) = \mu(H(|A|,|B|))$, one sees that τ is a very good charge on $\mathcal{B}(2^\omega)$.

(b) Let $\{p_0, p_1, p_2, \dots\}$ be torsion-free independent elements of G. Define μ on $\mathcal{B}(2^\omega)$ by

$$\mu(H(n,0)) = p_n \quad \text{for all } n$$

$$\mu(H(A,B)) = \mu(H(|A|,|B|)) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \, p_{m+i}$$

for $A \cap B = \emptyset$, $|A| = m$ and $|B| = k$, and also $0 \leq i_m \leq \binom{n}{m}$ for $m = 0, 1, \ldots, n$, and we can write

$$\mu(D) = \sum_{\substack{m=0 \\ m+k=n}}^{n} i_m \, \mu(H(m,k))$$

for $D \in \mathcal{B}(2^\omega)$ where D is written as a disjoint union of sets of the type $H(A,B)$ where $A \cap B = \emptyset$ and $A \cup B$ is some fixed finite set of cardinality n.

Now $\mu(D)$ can be rewritten as $\sum_{l=0}^{n} k_l p_l$ for some k_l's where

$$k_l = i_0 \binom{n}{l} (-1)^l + i_1 \binom{n-1}{l-1} (-1)^{l-1} + \cdots + i_l \binom{n-l}{0} (-1)^0.$$

Since $p_0, p_1, \ldots$ are torsion-free and independent in G, if $\mu(B) = 0$ we get that $k_l = 0$ for $l = 0, 1, \ldots, n$. But $k_0 = 0$ gives us that $i_0 = 0$. Then $k_1 = 0$ gives us that $i_1 = 0 \ldots$ and so on. Thus $i_0, i_1, \ldots, i_n$ are all equal to 0. Hence D is the empty set. Thus μ is a very good charge on $\mathcal{B}(2^\omega)$.

The second part of (b) follows from (a).

(c) can be proved in an easy way by defining an injective homomorphism from the Nöbeling group $S(Y, \mathcal{P}(Y))$ to G. □

Let us now turn to **Z**-valued charges.

Proposition 13. (a) *On $\mathcal{B}(2^\omega)$ there is a good* **Z***-valued charge.*

(b) *On any countable field of sets there is a good* **Z***-valued charge.*

Proof. Suppose that $\mathcal{C}$ is a finite field of subsets of a set X given by a partition $\{C_1, C_2, \ldots C_n\}$ of nonempty sets from $\mathcal{C}$. Suppose that μ is an snz **Z**-valued charge on $\mathcal{C}$. Let $C_0 \subset C_1$ be such that $C_0 \neq \emptyset$ and $C_1 - C_0 \neq \emptyset$. Let $\mathcal{D}$ be the field generated by $\mathcal{C}$ and C_0. Let us show that μ can be extended as an snz **Z**-valued charge on $\mathcal{D}$.

To do this let $S = \{\mu(C) : C \in \mathcal{C}\}$. Let $t \in \mathbf{Z}$ be such that $t + s \neq 0$ for all $s \in S$ and $t - s \neq 0$ for all $s \in S$. Since S is finite such a t can always be found. Now define $\bar{\mu}$ on $\mathcal{D}$ by prescribing $\bar{\mu}(C_0) = t$. Then $\bar{\mu}$

is an snz $\mathbf{Z}$-valued charge on $\mathcal{D}$. But any countable Boolean algebra $\mathcal{A}$ can be written as $\mathcal{A} = \bigcup_{i\geq 1} \mathcal{A}_i$ where $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \dots$ and $\mathcal{A}_{n+1}$ is a finite subalgebra obtained from $\mathcal{A}_n$ by the above procedure. If on $\mathcal{A}_1$ we start with any snz $\mathbf{Z}$-valued charge, using the above procedure we obtain an snz $\mathbf{Z}$-valued charge on $\mathcal{A}$. □

We do not know if for every infinite Y the Boolean algebra $\mathcal{B}(2^Y)$ has a good $\mathbf{Z}$-valued charge and we also do not know if $\mathcal{B}(2^\omega)$ has a very good $\mathbf{Z}$-valued charge. An affirmative answer to the second problem gives an affirmative answer to the first. The converse conclusion might depend on set theory. In fact we will show below, assuming the existence of a large cardinal (measurable or even less, Ramsey or Erdös will do), that the existence of good $\mathbf{Z}$-valued charges on a large enough $\mathcal{B}(2^Y)$ implies the existence of very good $\mathbf{Z}$-valued charges on $\mathcal{B}(2^\omega)$, hence on any $\mathcal{B}(2^X)$ by Proposition 12(a). Note that Erdös cardinals seem to be designed for this problem. If Erdös cardinals are really needed, then extra set theoretic assumptions (besides ZFC) are necessary to refute the converse above.

First we want to see that the natural choice $\mu(H(n,0)) = p_n = q^n$ for some integer $q \neq 1$ and any $n \in \omega$ does not lead to snz $\mathbf{Z}$-valued charges on $\mathcal{B}(2^\omega)$. We have the following

Proposition 14. *Let μ be a $\mathbf{Z}$-valued charge on $\mathcal{B}(2^\omega)$ defined by $\mu(H(n,0)) = q^n$ and $\mu(H(A,B)) = \mu(H(|A|,|B|))$. Then μ is not an* snz *charge.*

Proof. Observe that, for $m + k = n$,

$$\mu(H(m,k)) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i q^{m+i} = q^m (1-q)^k. \tag{4}$$

If q is positive, then choose n and m so that $n = 2m$ and $q \leq \binom{n}{m}$. Then if we take q disjoint sets of the type $H(A,B)$ with $|A| = m$, $|B| = m$, and $q-1$ disjoint sets with $|A| = m+1$, $|B| = m-1$, and call the union of all these sets (which forms a family of pairwise disjoint sets) D, then

$$\begin{aligned}\mu(D) &= qH(m,m) + (q-1)H(m+1,m-1) \\ &= qq^m(1-q)^m + (q-1)q^{m+1}(1-q)^{m-1} = 0.\end{aligned}$$

Thus μ is not an snz charge. If q is negative, then observe that $\mu(H(0,n)) = (1-q)^n$ and $1-q$ is positive. Hence we will be in a similar situation as above with $1-q$ replacing q. Thus, in any case, μ is not an snz charge. □

Now we want to show the connection to large cardinals mentioned above. We need some results from Erdös-Hajnal-Rado's partition calculus, see [**2**], [**3**]. Let us set up the notation. We shall write '$\simeq$' for two well-ordered sets to be of the same order type. For $\beta \in \mathrm{Ord}$ and $\gamma \in \mathrm{Ord}$ we write $[\beta]^\gamma = \{y \subset \beta : y \simeq \gamma\}$. For $\alpha, \beta, \gamma, \delta \in \mathrm{Ord}$ we write

$$\beta \longrightarrow (\alpha)^\gamma_\delta$$

if, for every function $F : [\beta]^\gamma \to \delta$ there is a set $H \subset \beta$ such that H has order type α and is homogeneous for F, i.e., F takes the same value for every point of $[H]^\gamma$.

In the same way we define for $\beta \in \mathrm{Ord}$ that $[\beta]^{<\omega} = \bigcup_{n\in\omega}[\beta]^n$ and, for $\alpha, \beta, \delta \in \mathrm{Ord}$ we write $\beta \to (\alpha)^{<\omega}_\delta$ if, for every function $F : [\beta]^{<\omega} \to \delta$ there is a set $H \subset \beta$ such that H has order type α and is homogeneous for F, i.e., F takes the same value (depending only on n) for every point of $[H]^n$ for all $n \in \omega$ uniformly.

The least cardinal λ such that $\lambda \to (\alpha)^{<\omega}_2$ is called the Erdös cardinal $\kappa(\alpha)$. We are mainly interested in $\kappa(\omega)$. A result of Silver (see [**3**, p. 82]) says that

(a) any $\kappa(\alpha)$ is inaccessible.

(b) If $\gamma < \kappa(\alpha)$, then also $\kappa(\alpha) \to (\alpha)^{<\omega}_\gamma$.

Hence $\kappa(\omega)$ may not exist in certain universes of set theory but surely any universe with a measurable cardinal (or just a Ramsey cardinal) has $\kappa(\omega)$, again see [**3**]. We want to use (b) for charges.

Proposition 15. *Let G be any torsion-free group. If there is a good G-valued charge on $\mathcal{B}(2^Y)$ for a set Y with $|Y| = \kappa(\omega)$, then there is a very good G-valued charge on $\mathcal{B}(2^\omega)$, hence on any $\mathcal{B}(2^X)$.*

Proof. By Proposition 12(a) it is enough to find a very good charge on $\mathcal{B}(2^\omega)$. By Proposition 12(b) we may also assume that G is countable.

Let $\mu : \mathcal{B}(2^Y) \to G$ be a good G-valued charge on a set of cardinality $\kappa(\omega)$. We may put $Y = \kappa(\omega)$ and define a function

$$F : [\kappa(\omega)]^{<\omega} \longrightarrow G$$

by $F(A) = \mu(H(A, \emptyset))$ (note that we can identify G and ω). From Silver's result (b) above we find a homogeneous subset $H \subset \kappa(\omega)$ such that $|H| = \aleph_0$ and F on $[H]^n$ takes only one value, say $0 \neq g_n \in G$ for each $n \in \omega$.

Now we define a charge $\tau : \mathcal{B}(2^\omega) \to G$ by $\tau(H(A,\ \emptyset)) = g_{|A|}$. Note that τ being the restriction of μ, is a good charge. Moreover, τ is very good because τ only depends on the size of A. □

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Rüdiger Göbel, Fachbereich 6, Mathematik, Universität Essen, 45117 Essen, Germany
E-mail address: R.Goebel@Uni-Essen.De

K.P.S. Bhaskara Rao, Southwestern College, Kansas, USA
E-mail address: brao@sckans.edu