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COMPLETELY DECOMPOSABLE ABELIAN GROUPS WITH A DISTINGUISHED CD SUBGROUP

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ABSTRACT. We define a category $CD^1(T, p)$, p a prime and T a set of types, consisting of all pairs $V = (C, D)$ where C is a completely decomposable group with critical type set T and D a completely decomposable subgroup with $p^e C \subseteq D \subseteq C$ for some $e \geq 1$. We show that while indecomposables in this category have rank at most one if T is an antichain, we observe "wild" behavior if T contains comparable elements.

I. Introduction. One of the true chestnuts of the theory of abelian groups is the stacked basis theorem that goes back to work of C.F. Gauss [**4**] which can be stated as follows: Let F be a free module of finite rank over a principal ideal domain S and X a submodule of F. Then F has a basis $B = \{b_1, \dots, b_n\}$ such that $X = \oplus_{i=1}^n b_i s_i S$ for some $s_i \in S$ with $s_i \in s_{i-1}S$, $1 \leq i \leq n$. This implies that F/X is a direct sum of cyclic S-modules. We refer to [**3**] for a discussion of the history and references for this result. More recently, Benabdallah and Ould-Beddi [**2**] proved a version of the stacked basis theorem for homogeneous completely decomposable (hcd) groups of finite rank, which was extended by Ould-Beddi and Strüngmann [**5**] to the infinite rank case: Let C be an hcd group and X a (hcd) subgroup of C of bounded index. Then C has a stacked basis B for X, i.e., $C = \oplus_{b \in B} \langle b \rangle_*^C$ and $X = \oplus_{b \in B} \langle s_b b \rangle_*^X$ for some $s_b \in \mathbf{Z}$, where $\langle b \rangle_*^X$ denotes the purification of the subgroup generated by b inside X. This makes it natural to ask about the case where C is not hcd but completely decomposable (cd) of finite rank with critical type set T.

It is easy to see that the same result holds in this case if T is an antichain but, as we shall see, things are dramatically different if the critical type set T contains comparable elements. This note attempts

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to look at the stacked basis theorem for cd groups from the point of view of representations of partially ordered sets. In order to employ the language of representations (see Arnold's book [**1**] for notations), we define a category $\mathbf{CD}^1(\mathbf{T}, \mathbf{p})$ to consist of all pairs $V = (C, D)$ such that C is a cd group of finite rank with critical type set T, p a prime and D is a cd subgroup of C such that $p^e C \subseteq D$ for some $e \geq 1$. The stacked basis theorem due to Benabdallah and Ould-Beddi stated above (in the p-local case) can now be rephrased as: If T is a singleton, then indecomposable objects in $CD^1(T, p)$ have rank at most 1. The same holds if T is an antichain. On the other hand, if T is not an antichain, i.e., if it contains comparable elements, we will show that $CD^1(T, p)$ is "*wild*" in the sense that each finite dimensional $\mathbf{Z}/p\mathbf{Z}$-algebra can be obtained as an epimorphic image of $\mathrm{End}\,(V)$ for some object V in $CD^1(T, p)$. As a biproduct, we are led to the construction of a number of large indecomposable groups with a single distinguished subgroup.

II. The categories $\mathbf{Rep}_S(n\frac{1}{2}, p, e)$.

Definition. Let S be a PID and p a prime element of S. Moreover, let $n, e \in \mathbf{N}$. Then $V = (F, F_1, \ldots, F_n, F_{n+1})$ is in $\mathrm{Rep}_S(n\frac{1}{2}, p, e)$ if and only if F is a free S-module of finite rank, $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n$ is a chain of pure submodules of F (and thus direct summands of F) and F_{n+1} is a free submodule of F such that $p^e F \subseteq F_{n+1}$. A morphism in this category is any S-linear map $\varphi : F \to F'$ such that $(F_j)\varphi \subseteq F'_j$ for all $1 \leq j \leq n + 1$.

Similar categories have been studied in [**3**], and we refer to [**1**] as a guide to the literature. For example, if $n = 2$ and $F_3 = F$, then indecomposables have rank ≤ 2, cf. [**3**]. We will show that $\mathrm{Re}\,p_S(1\frac{1}{2}, p, e)$ has infinite representation type already for small values of e. First we will construct some examples.

Proposition 1. *There are arbitrarily large indecomposables in* $\mathrm{Re}\,p_S(1\frac{1}{2}, p, 6)$.

Proof. We will prove this proposition in some detail since the proofs of subsequent propositions will be modeled after this. Details of similar approaches are found in [**1**, Section 4]. Let $F = \oplus_{i=1}^4 e_i X$, where

$X \cong S^{(k)}$, a free S-module of rank k. Note that F has rank $4k$. Let $F_1 = e_3X \oplus e_4X$ and $F_2 = (e_1p^2 + e_3 + e_4p)X \oplus (e_2p^4 + e_3p)X \oplus (e_3p^2A + e_4p^3)X \oplus e_3p^3X$ where A is some $k \times k$-matrix over S. It is easy to verify that this sum is direct and that $p^6F \subseteq F_2$, but $p^5F \not\subseteq F_2$. The module F_2 can be represented by the matrix

$$M = \begin{bmatrix} p^2 & 0 & : & 1 & p \\ 0 & p^4 & : & p & 0 \\ 0 & 0 & : & p^2A & p^3 \end{bmatrix},$$

where we suppress the summand e_3p^3X, but keep in mind that we read entries in the third column modulo p^3. Now consider $V = (F, F_1, F_2) \in \mathrm{Re}\,p_S(1\frac{1}{2}, p, 6)$. Let $\varphi \in \mathrm{End}\,(V)$. Then φ has the form

$$\varphi = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \gamma_{11} & \gamma_{12} \\ \alpha_{21} & \alpha_{22} & \gamma_{21} & \gamma_{22} \\ 0 & 0 & \beta_{11} & \beta_{12} \\ 0 & 0 & \beta_{21} & \beta_{22} \end{bmatrix},$$

an element in the matrix ring $Mat_{4k\times 4k}(S)$ of $4k \times 4k$-matrices over S, since $(F_1)\varphi \subseteq F_1$. Note that maps operate on the right. Since $(F_2)\varphi \subseteq F_2$, there is a matrix $U = (u_{ij}) \in \mathrm{Mat}_{4k\times 4k}(S)$ such that $M\varphi = UM$ where we read the third column modulo p^3. We compute

$$M\varphi = \begin{bmatrix} p^2\alpha_{11} & p^2\alpha_{12} & p^2\gamma_{11} + \beta_{11} + p\beta_{21} & p^2\gamma_{12} + \beta_{12} + p\beta_{22} \\ p^4\alpha_{21} & p^4\alpha_{22} & p^4\gamma_{21} + p\beta_{11} & p^4\gamma_{22} + p\beta_{12} \\ 0 & 0 & p^2A\beta_{11} + p^3\beta_{21} & p^2A\beta_{12} + p^3\beta_{22} \end{bmatrix}$$

and

$$UM = \begin{bmatrix} p^2u_{11} & p^4u_{12} & u_{11} + pu_{12} + p^2u_{13}A & pu_{11} + p^3u_{13} \\ p^2u_{21} & p^4u_{22} & u_{21} + pu_{22} + p^2u_{23}A & pu_{21} + p^3u_{23} \\ p^2u_{31} & p^4u_{32} & u_{31} + pu_{32} + p^2u_{33}A & pu_{31} + p^3u_{33} \end{bmatrix}.$$

From $M\varphi = UM$, we derive congruences mod p^i where we will read the first column mod p^5, the second mod p^6, the third mod p^3 and the fourth mod p^4. We obtain:

$\alpha_{11} \equiv u_{11} \bmod (p^3)$, $\alpha_{12} \equiv 0 \bmod (p^2)$, $u_{21} \equiv 0 \bmod (p^2)$, $\alpha_{22} \equiv u_{22} \bmod (p^2)$, $u_{31} \equiv 0 \bmod (p^3)$ and $u_{32} \equiv 0 \bmod (p^2)$. Moreover,

$\beta_{11} \equiv u_{11} \bmod (p)$, $\beta_{12} + p\beta_{22} \equiv pu_{11} \bmod (p^2)$, $p\beta_{11} \equiv pu_{22} \bmod (p^2)$, $p\beta_{12} \equiv pu_{21} \bmod (p^3)$ and $p^2 A\beta_{11} \equiv p^2 u_{33} A \bmod (p^3)$, $p^2 A\beta_{12} + p^3\beta_{22} \equiv p^3 u_{33} \bmod (p^4)$. We infer $\beta_{12} \equiv 0 \bmod (p^2)$ and $\beta_{22} \equiv u_{33} \bmod (p)$. Thus

$$\varphi \equiv \begin{bmatrix} \alpha & 0 & \gamma_{11} & \gamma_{12} \\ \alpha_{21} & \alpha & \gamma_{21} & \gamma_{22} \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & \beta_{21} & \alpha \end{bmatrix} \bmod p$$

with $\alpha \bmod p \in C(A \bmod p)$, the centralizer of $A \bmod p$.

Define $C_p(A) = \{\psi \in \operatorname{End}_S(X) \mid \psi A \equiv A\psi \bmod p\}$, i.e., for $\psi \in C_p(X)$ we have $\psi A - A\psi : X \to pX$.

Then $\operatorname{End}(V) = C_p(A) I_{4\times 4} + J$ where $I_{4\times 4}$ is the 4×4 identity matrix and J is an ideal of $\operatorname{End}(V)$ with $J^4 \subseteq \operatorname{End}(V) \cap pMat_{4k\times 4k}(S)$. Thus if $0, 1$ are the only idempotents in $C_p(A)$, the same holds in $\operatorname{End}(V)$. Now let A be a $k \times k$ Jordan block with eigenvalue $\lambda = 0$ so that

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

and notice that $C(A) = S[A]$, $C(A \bmod p) = (S/pS)[A \bmod p] = C(A) \bmod p = C_p(A) \bmod p$ and $S[A]$, being a local ring, has only 0 and 1 as idempotents.

Remark. It might be of interest to note that if one changes the p's into I's inside the matrix M, one obtains a matrix

$$M^{\#} = \begin{bmatrix} I & 0 & : & I & I \\ 0 & I & : & I & 0 \\ 0 & 0 & : & A & I \end{bmatrix},$$

which is the "matrix problem" version [**1**] of a representation of the critical, tame poset (2, 2, 2) as follows: Let F be a field, X a vector space of dimension k over F and $U = \oplus_{i=1}^{4} e_i X$. Define $U_1 = e_2 X \subset U_1 \oplus e_1 X = U_2$, $U_3 = e_4 X \subset U_3 \oplus e_3 X = U_4$, $U_5 =$

$(e_2+e_3)X \subset U_5 \oplus (e_3A+e_4)X = U_6$. Finally, let $U^\# = (e_1+e_3+e_4)X$. Then $W^\# = (U, U_1 \subset U_2, U_3 \subset U_4, U_5 \subset U_6, U^\#)$ with $U = U_2 \oplus U_4$ a representation of the poset (2, 2, 2, 1) and $W = (U/U^\#, (U_i+U^\#)/U^\#, 1 \leq i \leq 6)$ a representation of (2, 2, 2) with $\mathrm{End}\,(W) \approx C(A)$. (Actually, the matrix $M^\#$ was our motivation for the matrix M).

Definition. Let $\mathbf{C}$ be a category of S-modules (with distinguished submodules).

(a) The category $\mathbf{C}$ has finite representation type if there is a finite upper bound for the rank of indecomposable objects in $\mathbf{C}$. Otherwise the representation type is called infinite.

(b) The category $\mathbf{C}$ has *wild modulo p representation type*, or "*wild mod p*" for short, cf. [**1**, p. 135] if, for each A-algebra Γ that is finitely generated and free as an S-module there is $V \in \mathbf{C}$ such that there is a ring epimorphism $\phi : \mathrm{End}\,(V) \to \Gamma/p\Gamma$.

(c) We call the category $\mathbf{C}$ *p-endo-wild* if, for *each* finite dimensional S/pS-algebra Δ, there is a $V \in \mathbf{C}$ and a ring epimorphism $\phi : \mathrm{End}\,(V) \to \Delta \to 0$.

Notice that if each S/pS-algebra Δ is the epimorphic image of a Γ as in (a), i.e., $\Delta \approx \Gamma/p\Gamma$, then the definitions in (b) and (c) are equivalent. It is an open problem (cf. [**1**, p. 139, Open Question 2]), whether (b) and (c) are equivalent in general.

We are now ready for the main proposition of this section:

Proposition 2. *The categories* $\mathrm{Re}\,p_S(n\frac{1}{2}, p, e)$ *have the following representation type*:

(a) $\mathrm{Re}\,p_S(1\frac{1}{2}, p, 6)$ *has infinite type.*

(b) $\mathrm{Re}\,p_S(1\frac{1}{2}, p, 7)$ *is wild modulo p.*

(c) $\mathrm{Re}\,p_S(1\frac{1}{2}, p, e)$ *is p-endo-wild for* $e \geq 9$.

(d) $\mathrm{Re}\,p_S(2\frac{1}{2}, p, 4)$ *has infinite type.*

(e) $\mathrm{Re}\,p_S(n\frac{1}{2}, p, e)$ *is p-endo-wild for* $n \geq 2$ *and* $e \geq 6$; *or* $n = 3$ *and* $e \geq 5$.

(f) $\mathrm{Re}\,p_S(1\frac{1}{2}, p, 2)$ *contains indecomposables of rank* 2.

Proof. (a) was proved in Proposition 1. (b) We refer to [**1**] for the notation and the approach that we shall follow below. Let, as before, X be a free S-module of rank k and let $\alpha, \beta \in \operatorname{End}_S(X)$ be represented by $k \times k$-matrices over S. Let $F = \oplus_{i=1}^5 e_i X$, $F_1 = e_4 X \oplus e_5 X$ and F_2 the row space of the matrix

$$M = \begin{bmatrix} p^2 & 0 & 0 & A & pB \\ 0 & p^4 & 0 & p & 0 \\ 0 & 0 & p^6 & p^2 & p^3 \\ 0 & 0 & 0 & p^3 & p^4 \\ 0 & 0 & 0 & p^4 & 0 \end{bmatrix}$$

where A and B are some $k \times k$-matrices to be specified later. Then

$$M^{-1} = \begin{bmatrix} \frac{1}{p^2} & 0 & 0 & -\frac{B}{p^5} & -\frac{-B+A}{p^6} \\ 0 & \frac{1}{p^4} & 0 & 0 & -\frac{1}{p^7} \\ 0 & 0 & \frac{1}{p^6} & -\frac{1}{p^7} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{p^4} \\ 0 & 0 & 0 & \frac{1}{p^4} & -\frac{1}{p^5} \end{bmatrix},$$

which implies that $e_1 p^6 X \oplus e_2 p^7 X \oplus e_3 p^7 X \oplus e_4 p^4 X \oplus e_5 p^5 X \subseteq F_2$, i.e., $V = (F, F_1, F_2) \in \operatorname{Rep}_S(1\frac{1}{2}, p, 7)$. We also see that $p^6 F \not\subseteq F_2$. Each $\varphi \in \operatorname{End}(V)$ has the form

$$\varphi = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \gamma_{11} & \gamma_{12} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \gamma_{21} & \gamma_{22} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \gamma_{31} & \gamma_{32} \\ 0 & 0 & 0 & \beta_{11} & \beta_{12} \\ 0 & 0 & 0 & \beta_{21} & \beta_{22} \end{bmatrix}.$$

Since $F_2\varphi \subseteq F_2$ there is a 5×5-matrix U over S,

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} \\ u_{21} & u_{22} & u_{23} & u_{24} & u_{25} \\ u_{31} & u_{32} & u_{33} & u_{34} & u_{35} \\ u_{41} & u_{42} & u_{43} & u_{44} & u_{45} \\ u_{51} & u_{52} & u_{53} & u_{54} & u_{55} \end{bmatrix}$$

such that $M\varphi = UM$. Now

$$M\varphi = \begin{bmatrix} p^2\alpha_{11} & p^2\alpha_{12} & p^2\alpha_{13} & p^2\gamma_{11} + A\beta_{11} + pB\beta_{21} & p^2\gamma_{12} + A\beta_{12} + pB\beta_{22} \\ p^4\alpha_{21} & p^4\alpha_{22} & p^4\alpha_{23} & p^4\gamma_{21} + p\beta_{11} & p^4\gamma_{22} + p\beta_{12} \\ p^6\alpha_{31} & p^6\alpha_{32} & p^6\alpha_{33} & p^6\gamma_{31} + p^2\beta_{11} + p^3\beta_{21} & p^6\gamma_{32} + p^2\beta_{12} + p^3\beta_{22} \\ 0 & 0 & 0 & p^3\beta_{11} + p^4\beta_{21} & p^3\beta_{12} + p^4\beta_{22} \\ 0 & 0 & 0 & p^4\beta_{11} & p^4\beta_{12} \end{bmatrix}$$

and

$$UM = \begin{bmatrix} u_{11}p^2 & u_{12}p^4 & u_{13}p^6 & u_{11}A + u_{12}p + u_{13}p^2 + u_{14}p^3 + u_{15}p^4 & u_{11}pB + u_{13}p^3 + u_{14}p^4 \\ u_{21}p^2 & u_{22}p^4 & u_{23}p^6 & u_{21}A + u_{22}p + u_{23}p^2 + u_{24}p^3 + u_{25}p^4 & u_{21}pB + u_{23}p^3 + u_{24}p^4 \\ u_{31}p^2 & u_{32}p^4 & u_{33}p^6 & u_{31}A + u_{32}p + u_{33}p^2 + u_{34}p^3 + u_{35}p^4 & u_{31}pB + u_{33}p^3 + u_{34}p^4 \\ y_{41}p^2 & u_{42}p^4 & u_{43}p^6 & u_{41}A + u_{42}p + u_{43}p^2 + u_{44}p^3 + u_{45}p^4 & u_{41}pB + u_{43}p^3 + u_{44}p^4 \\ u_{51}p^2 & u_{52}p^4 & u_{53}p^6 & u_{51}A + u_{52}p + u_{53}p^2 + u_{54}p^3 + u_{55}p^4 & u_{51}pB + u_{53}p^3 + u_{54}p^4 \end{bmatrix}$$

We now exploit the equation $M\varphi = UM$ to derive that

$$\varphi \bmod p = \begin{bmatrix} \alpha & 0 & 0 & \gamma_{11} & \gamma_{12} \\ \alpha_{21} & \alpha & 0 & \gamma_{21} & \gamma_{22} \\ \alpha_{31} & \alpha_{32} & \alpha & \gamma_{31} & \gamma_{32} \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta_{21} & \alpha \end{bmatrix}$$

with $\alpha A \equiv A\alpha \bmod p$ and $\alpha B \equiv B\alpha \bmod p$. Note that, for $\alpha = 0$, $(\varphi \bmod p)^7 = 0$.

Given an S-algebra Γ, free of rank k as an S-module, we may choose S-matrices A, B such that $\Gamma \approx C(A, B)$, the centralizer of A and B, such that $\Gamma/p\Gamma \approx C(A \bmod p, B \bmod p)$ (cf. [**1**, p. 136, Lemma 4.2.1; and Example 1.1.7]). Since $C(A, B)I_{5\times 5} \approx \Gamma$ is contained in $\operatorname{End}(V)$ and $\operatorname{End}(V) = C(A, B)I_{5\times 5} + J$, J an ideal of $\operatorname{End}(V)$ with $J \cap C(A, B)I_{5\times 5} = pC(A, B)I_{5\times 5}$, the S/pS-algebra $\Gamma/p\Gamma$ is an epimorphic image of $\operatorname{End}(V)$.

To prove (c), we us a similar construction but with $F = \oplus_{i=1}^6 e_i X$, $F_1 = \oplus_{i=4}^6 e_i X$ and F_2 the row space of the 6×6-matrix

$$M = \begin{bmatrix} p^3 & 0 & 0 & 1 & 0 & 0 \\ 0 & p^5 & 0 & p & p^2 & 0 \\ 0 & 0 & p^7 & p^2 & 0 & p^4 \\ 0 & 0 & 0 & p^3 & p^4 A & p^5 B \\ 0 & 0 & 0 & 0 & p^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & p^6 \end{bmatrix}.$$

Since

$$M^{-1} = \begin{bmatrix} \frac{1}{p^3} & 0 & 0 & -\frac{1}{p^6} & \frac{A}{p^7} & \frac{B}{p^7} \\ 0 & \frac{1}{p^5} & 0 & -\frac{1}{p^7} & \frac{-1+A}{p^8} & \frac{B}{p^8} \\ 0 & 0 & \frac{1}{p^7} & -\frac{1}{p^8} & \frac{A}{p^9} & \frac{-1+B}{p^9} \\ 0 & 0 & 0 & \frac{1}{p^3} & -\frac{A}{p^4} & -\frac{B}{p^4} \\ 0 & 0 & 0 & 0 & \frac{1}{p^5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{p^6} \end{bmatrix},$$

we have that $e_1 p^7 X \oplus e_2 p^8 X \oplus e_3 p^9 X \oplus e_4 p^4 X \oplus e_5 p^5 X \oplus e_6 p^6 X \subseteq F_2$, and it is readily seen that $C_p(A,B) I_{6\times 6}$ is contained in End (V), which will imply that we can obtain any S/pS-algebra as epimorphic image of End (V). If $\varphi \in$ End (V), it can be shown that $\varphi \bmod p = \alpha I_{6\times 6} + N$ where

$$N = \begin{bmatrix} 0 & 0 & 0 & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ \alpha_{21} & 0 & 0 & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ \alpha_{31} & \alpha_{32} & 0 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{54} & 0 & 0 \\ 0 & 0 & 0 & \alpha_{64} & \alpha_{65} & 0 \end{bmatrix}.$$

It is easy to verify that $N^7 = 0$.

To show (d), let $F = \oplus_{i=1}^3 e_i X$ with $F_1 = e_3 X$, $F_2 = e_2 X \oplus e_3 X$ and F_3 the row space of the matrix

$$M = \begin{bmatrix} p^2 & p & 1 \\ 0 & p^2 & pA \\ 0 & 0 & p \end{bmatrix}.$$

For $V = (F, F_1, F_2, F_3)$ and $\varphi \in$ End (V) we compute

$$\varphi \bmod p = \begin{bmatrix} \alpha & \alpha_{12} & \alpha_{13} \\ 0 & \alpha & \alpha_{23} \\ 0 & 0 & \alpha \end{bmatrix}$$

with $\alpha A \equiv A\alpha \bmod p$. Now use a Jordan block matrix A to make V indecomposable.

(e) Here we use $F = \oplus_{i=1}^{6} e_i X$ with $F_1 = e_5 X \oplus e_6 X$, $F_2 = \oplus_{i=3}^{6} e_i X$, F_3 the row space of the matrix

$$M = \begin{bmatrix} p^2 & 0 & p & 0 & 1 & 0 \\ 0 & p^2 & 0 & p & 0 & 1 \\ 0 & 0 & 0 & p^2 & p & 0 \\ 0 & 0 & p^4 & 0 & p^2 A & p^2 B \\ 0 & 0 & 0 & 0 & p^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & p^3 \end{bmatrix}.$$

Let $V = (F, F_1, F_2, F_3)$. Each $\varphi \in \mathrm{End}\,(V)$ has the form

$$\varphi = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ 0 & 0 & \alpha_{33} & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ 0 & 0 & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} \\ 0 & 0 & 0 & 0 & \alpha_{55} & \alpha_{56} \\ 0 & 0 & 0 & 0 & \alpha_{65} & \alpha_{66} \end{bmatrix}.$$

Notice that $C_p(A, B) I_{6\times 6} \subseteq \mathrm{End}\,(V)$ and $e_1 p^6 X \oplus e_2 p^5 X \oplus e_3 p^5 X \oplus e_4 p^4 \oplus e_5 p^3 X \oplus e_6 p^3 X \subseteq F_3$. We use again the equation $M\varphi = UM$ to compute that

$$\varphi \bmod p = \begin{bmatrix} \alpha & 0 & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ 0 & \alpha & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ 0 & 0 & \alpha & 0 & \alpha_{35} & \alpha_{36} \\ 0 & 0 & 0 & \alpha & \alpha_{45} & \alpha_{46} \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha \end{bmatrix}.$$

(The computation is a little involved, but straightforward). Moreover, if $\beta \in \mathrm{End}\,(X)$ such that $\beta A \equiv A\beta \pmod p$ and $\beta B \equiv B\beta \bmod p$, then $I_{6\times 6}\beta \in \mathrm{End}\,(V)$. This shows that $\mathrm{End}\,(V)$ contains a copy of $C_p(A, B)$ and thus has $C(A \bmod p, B \bmod p)$ as a ring epimorphic image with, modulo p, a nilpotent kernel. Thus $\mathrm{Re}\,p_S(2\frac{1}{2}, p, 6)$ is p-endowild.

For the second part, let $F = \oplus_{i=1}^{4} e_i X$ and $F_j = \oplus_{i=4}^{5-j} e_i X$ for $1 \le j \le 3$ and F_4 the row space of the matrix

$$M = \begin{bmatrix} p^3 & p^2 & p & 1 \\ 0 & p^3 & p^2 A & pB \\ 0 & 0 & p^3 & 0 \\ 0 & 0 & 0 & p^2 \end{bmatrix}.$$

Then $p^5e_1X \oplus p^4e_2X \oplus p^3e_3X \oplus p^2e_4X \subseteq F_4$ and $C_p(A,B)I_{4\times 4}$ is contained in End (V), $V = (F, F_j, 1 \le j \le 4)$. Note that this matrix M is derived by the "matrix problem" associated with the standard wild five subspace representation.

(f) Consider $M = \left[\begin{smallmatrix} p & 1 \\ 0 & p \end{smallmatrix}\right]$ and proceed as before considering (F, F_1, F_2) with F a rank-2 S-module and F_2 the row space of M.

We will use invertible row/column operations to prove

Proposition 3. *The categories* $\mathrm{Re}\,p_S(n\frac{1}{2}, p, 3)$ *have finite representation type for all* $n \ge 1$.

Proof. Let $V = (F, F_1, \dots, F_n, F_{n+1}) \in CD^1(n\frac{1}{2}, p, 3)$ be indecomposable such that F_{n+1} is the row space of the matrix

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & \cdots & \cdots \\ 0 & M_{22} & M_{23} & \cdots & \cdots \\ 0 & 0 & M_{33} & \cdots & \cdots \\ 0 & 0 & 0 & M_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{bmatrix}.$$

Since $p^3F \subseteq F_{n+1}$, we may treat multiples of p^3 inside M as zeros. The columns of M correspond to the complements of F_{j-1} inside F_j. Each M_{jj} can be reduced to a diagonal matrix with entries of the main diagonal either 1, p or p^2. Inductively, we show that no p^2 actually occurs there: Assume a p^2 occurs as a diagonal (i.e., pivotal) entry in M_{11}. Then look up the first $\neq 0$ entry in that row of M. One of these entries will cancel the other, a contradiction, because we get a rank 1 summand of V. Thus we may assume that $M_{11} = pI$, I some $k \times k$-identity matrix. If there is a pivot p^2 in M_{22}, then on top of it are only zeros or ones. Any 1 can be used to cancel the p^2 pivot, and we are able to produce another rank 1 summand $\dots$. Thus, we may assume that each M_{jj} has the form pI and therefore all matrices M_{ij}, $i < j$, can be assumed to contain only entries that are 0 or units modulo p. Above or to the left of such a unit all entries can be cancelled and turned into 0's. Thus the only indecomposable module of rank ≥ 2 is represented by

$$M = \begin{bmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{bmatrix},$$

a representation of rank 3.

III. Categories of completely decomposable (cd) groups with a distinguished subgroup.

For this section we fix a prime integer p. All subgroups $A_i \subset \mathbf{Q}$ are assumed to be p-locally free, i.e., $1/p \notin A_i$ but $1 \in A_i$. Then $\tau(A_i) = \tau_i$ denotes the type of A_i and we call τ_i p-locally free as well. A cd group C with critical type set $T = \{\tau_1, \dots, \tau_k\}$ is of the form $C = \oplus_{i=1}^{k} A_i^{(m_i)}$ where the A_i are p-locally free subgroups of $\mathbf{Q}$ of type τ_i.

Definition. The category $\mathbf{CD}^1(\mathbf{T}, \mathbf{p})$ consists of all pairs $V = (C, D)$ such that:

(a) C is a finite rank cd group with critical type set T, a set of p-locally free types.

(b) D is a subgroup of C such that D is again a cd group and $p^j C \subseteq D$ for some $j \geq 1$.

Morphisms $\phi : (C, D) \to (C', D')$ in this category are the homomorphisms $\varphi : D \to D'$ such that $\varphi(D) \subseteq D'$. For $V = (C, D) \in CD^1(T, p)$, let $\mathrm{End}\,(V)$ denote the set of morphisms from V into V.

Using our terminology, we may state (a special case of) a result by Benabdallah and Ould-Beddi.

Theorem [2]. *If $|T| = 1$, then indecomposable objects in $CD^1(T, p)$ have rank 1, i.e., the stacked basis theorem holds in this case.*

Note that this result was generalized by Ould-Beddi and Strüngmann to the case of homogeneous cd groups of infinite rank [**5**]. We are now ready for our main result:

Theorem 4. *If the set T of critical types is an antichain, i.e., any two elements in T are incomparable, then indecomposable objects in $CD^1(T, p)$ have rank 1. If T is not an antichain, then $CD^1(T, p)$ is p-endowild, i.e., for each finite dimensional $\mathbf{Z}/p\mathbf{Z}$-algebra A there is an object $V \in CD^1(T, p)$ such that there exists a ring epimorphism $\varphi : \mathrm{End}\,(V) \to A$. Moreover, φ is the "modulo p" map followed by an* *epimorphisms with nilpotent kernel.*

Proof. If T is an antichain, then $C = \oplus_{i=1}^{k} C(\tau_i) = \oplus_{i=1}^{k} A_i^{(m_i)}$ and $D = \oplus_{i=1}^{k}(D \cap C(\tau_i))$ because the summands $C(\tau_i)$ of C are fully invariant in C. (The crucial fact here is that the $C(\tau_i)$'s are homogeneous of type τ_i as T is an antichain and D has finite index in C). Thus, $(C, D) = \oplus_{i=1}^{k}(C(\tau_i), D \cap C(\tau_i))$ and each of these summands is a direct sum of rank 1 summands by the stacked basis result in [**2**].

If T is *not* an antichain, we may assume without loss of generality that T is a chain, $T = \{\tau_1 < \tau_2 < \cdots < \tau_{k+1}\}$ for some $k \geq 1$.

Let A be a finite dimensional $\mathbf{Z}/p\mathbf{Z}$-algebra. Then there exist a natural number m and $m \times m$-matrices A^*, B^* over $\mathbf{Z}/p\mathbf{Z}$ such that A is isomorphic to $C(A^*, B^*)$, the centralizer of A^*, B^* in the ring of all $m \times m$-matrices over $\mathbf{Z}/p\mathbf{Z}$, cf. Example 1.1.7 in [**1**]. Now let $A^{\#}, B^{\#}$ be matrices over $\mathbf{Z}$ such that $A^* = A^{\#} \bmod p$ and $B^* = B^{\#} \bmod p$.

For any $k \geq 2$ there is some e, depending on k, such that according to Proposition 2 in Section II there is some $V = (F_0, F_1, \ldots, F_k, F_{k+1}) \in \operatorname{Re} p_{\mathbf{Z}}(k\frac{1}{2}, e)$ with A an epimorphic image of $\operatorname{End}(V)$. Since $F_0 \sqsupseteq F_1 \sqsupseteq \cdots \sqsupseteq F_k$ is a descending chain of summands of F_0, we may write $F_{i-1} = S_i \oplus F_i$ for $1 \leq i \leq k$. For $A_i \subset \mathbf{Q}$ with $\tau(A_i) = \tau_i$, we define $C = S_1A_1 \oplus S_2A_2 \oplus \cdots \oplus S_kA_k \oplus F_kA_{k+1}$. The submodule F_{k+1} in the list V is defined as the row space of an upper triangular $k \times k$-matrix

$$M = \begin{bmatrix} R_1 \\ \cdots \\ \cdots \\ R_m \end{bmatrix}.$$

A typical such matrix used for $\operatorname{Re} p_S(2\frac{1}{2}, p, 6)$ is

$$M = \begin{bmatrix} p^2 & 0 & p & 0 & 1 & 0 \\ 0 & p^2 & 0 & p & 0 & 1 \\ 0 & 0 & 0 & p^2 & p & 0 \\ 0 & 0 & p^4 & 0 & p^2A^{\#} & p^2B^{\#} \\ 0 & 0 & 0 & 0 & p^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & p^3 \end{bmatrix}.$$

(In this particular case we need to switch the third and fourth row to make M upper triangular). Now define $D = (R_1A_1 \oplus R_2A_1) \oplus$

$(R_3A_2 \oplus R_4A_2) \oplus (R_5A_3 \oplus R_6A_3) \subset C$. (This sum is direct because M is upper triangular). Define $V^{\#} = (C, D)$. Let $\varphi \in \mathrm{End}\,(V^{\#})$. Then $\varphi \in \mathrm{End}\,(C)$ with $\varphi(D) \subseteq D$. Let $S = \mathbf{Z}_p$ be the ring of integers localized at p. Then φ induces $\tilde{\varphi} \in \mathrm{End}\,(VS)$ where $VS = (F_0S, F_1S, \dots, F_kS,) \in \mathrm{Re}\,p_S(k\frac{1}{2}, p, e)$. As the proof of Proposition 2 shows, both $\mathrm{End}\,(V)$ and $\mathrm{End}\,(VS)$ have A as epimorphic image. On the other hand, each $\psi \in \mathrm{End}\,(V)$ induces a $\hat{\psi} \in \mathrm{End}\,(V^{\#})$. Thus A is an epimorphic image of $\mathrm{End}\,(V^{\#})$ by the map modulo p followed by an epimorphism with nilpotent kernel.

Corollary. *The stacked basis theorem holds in $CD^1(T, p)$ if and only if T is an antichain.*

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