ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 4, Winter 2002

M-FREE ABELIAN GROUPS

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ABSTRACT. We study M-free abelian groups with M-basis X, i.e., each map $f : X \to M$ extends uniquely to a homomorphism $\varphi : A \to M$. We will find conditions under which X generates a direct summand of A.

If F is an object in a concrete category, X a nonempty set and $i : X \to F$ a map, then F is free on the set X if for each M in the category and for each map $f : X \to M$ there is a morphism $\varphi : F \to M$ such that $\varphi \circ i = f$, cf. [**7**]. We will investigate, in the category of abelian groups only, which objects are "free" if, in the above definition, "each M" is replaced by "some fixed M." The answer, of course, depends on what kind of abelian group M actually is.

Definition. Let A, M be abelian groups and X a subset of A. Then A is *M-free with M-basis* X if, for each map $f : X \to M$, there is a *unique* $\varphi \in \operatorname{Hom}(A, M)$ such that $\varphi \restriction_X = f$ where $\varphi \restriction_X$ is the restriction of φ to X.

We say that A is *split-M-free* if $A = H \oplus \langle X \rangle$ such that $\langle X \rangle$ is free abelian with basis X and $\operatorname{Hom}(H, M) = 0$.

Of course, split-M-free implies M-free, and the main purpose of this paper is to investigate for which abelian groups M we have that all M-free groups A are actually split-M-free.

Let $\operatorname{Cent}(R)$ denote the center of a ring R. We will prove:

Main theorem. *Let A be M-free with M-basis X and M slender. If either*

(a) *M is countable and* $\operatorname{End}(M)^+$, *the additive group of the endomorphism ring of M, is free abelian, or*

1991 AMS *Mathematics Subject Classification.* Primary 20K30.
Research of the first author partially supported by Baylor University's Summer Sabbatical program.
Received by the editors on August 13, 2001, and in revised form on October 29, 2001.

(b) $|X| < \aleph_m$, *the first measurable cardinal and* $[Cent(\text{End}(M))]^+$ *is free abelian, then* A *is split-*M*-free with* M*-basis* X.

Our notations are standard as in [**5**]. We define $R_M(A) = \cap\{\ker(\varphi) : \varphi \in \text{Hom}(A, M)\}$ and call A *M-torsionless* if $R_M(A) = 0$. The subgroup $R_M(A)$ is sometimes referred to as the M-radical of A. Note that $A = R_M(A) + \langle X\rangle$ if A is split-M-free with M-basis X. We will see that $A = R_M(A) \oplus \langle X\rangle$ if M is not bounded. We will call A trivially M-free if $A = R_M(A)$, i.e., $X = \varnothing$. To contrast our result above, we will show that *any* abelian group A is M-free if $M = \mathbf{Q}$ or $M = J_p$, the additive group of p-adic integers.

We say that an abelian group A has *invariant M-rank* if A is M-free with M-basis X and whenever A is M-free with M-basis Y, then X and Y have the same cardinality. We will show that each M-free A has invariant M-rank if M is *not* torsion-free. We also have partial results for torsion-free M.

Throughout the paper we will use the following fact derived from elementary homological algebra, cf. [**5**]: An abelian group A is M-free with M-basis X if and only if $\text{Hom}(A/\langle X\rangle, M) = 0$ (this gives "uniqueness" in the definition) and the natural sequence $0 \to \text{Ext}(A/\langle X\rangle, M) \to \text{Ext}(A, M) \to \text{Ext}(\langle X\rangle, M) \to 0$ is short-exact. This implies that $\text{Ext}(A/\langle X\rangle, M) \cong \text{Ext}(A, M)$ via the natural map, if $\langle X\rangle$ is free abelian.

We begin by collecting some elementary facts about M-free groups.

Lemma 1. *Let A be M-free with M-basis $X \neq \varnothing$.*

(a) *If M is not bounded, then $\langle X\rangle$ is a free subgroup of A with basis X.*

(b) *If $pM \neq M$, p a prime, then $pA \cap \langle X\rangle = p\langle X\rangle$ and, if A is p-torsion-free, then $\langle X\rangle$ is p-pure in A.*

(c) *If N is a subgroup of M such that* (i) *M/N is isomorphic to a subgroup of M, or* (ii) *M/N is M-torsionless, or* (iii) $\text{Hom}(A/\langle X\rangle, M/N) = 0$, *then A is N-free with N-basis X.*

Proof. To show (a), let $a = \sum_{i=1}^{k} r_i x_i \in \langle X\rangle$ be an element of finite order n such that $r_1 \neq 0$. Define $f : X \to M$ by $f(x_1) = m \in M$ and

$f(x) = 0$ for all $x_1 \neq x \in X$. Let $\varphi \in \operatorname{Hom}(A, M)$ with $\varphi \restriction_X = f$. Then $0 = \varphi(na) = \sum_{i=1}^{k} \varphi(nr_i x_i) = r_1 nm$ and, since $m \in M$ was arbitrary, we have $r_1 nM = 0$, a contradiction to M not bounded. If $a = 0$, i.e., $n = 1$, we get the same contradiction, which implies that X is a basis for the free abelian group $\langle X \rangle$.

To show (b), let $m \in M \setminus pM$ and assume $py = \sum_{i=1}^{t} r_i x_i$, $x_i \in X$, $y \in A$ such that p doesn't divide r_1. Define a map $f : X \to M$ as above and obtain $\bar{y} = r_1 m \in M$ for some $\bar{y} \in M$. Since $\gcd(p, r_1) = 1$, we infer $m \in pM$, a contradiction. Thus, $r_i \in p\mathbf{Z}$ for all i.

Let N be given as in (c). Then (i) implies (ii), which implies (iii). Let $f : X \to N$ be a map. Then there is a $\varphi \in \operatorname{Hom}(A, M)$ with $\varphi \restriction_X = f$. Now φ induces a homomorphism $\bar{\varphi} : A/\langle X \rangle \to M/N$, which must be the 0-map by (iii). Thus $\varphi(A) \subseteq N$ and $\varphi \in \operatorname{Hom}(A, N)$. Note that $\operatorname{Hom}(A/\langle X \rangle, N) = 0$ since N is a subgroup of M and $\operatorname{Hom}(A/\langle X \rangle, M) = 0$. $\square$

Lemma 2. *Let A, M be abelian groups and $X \subseteq M$.*

(a) *If A is M-free with M-basis X and N a direct summand of M, then A is N-free with N-basis X.*

(b) *A is $\Pi_{i \in I} M_i$-free with $\Pi_{i \in I} M_i$-basis X if and only if A is M_i-free with M_i-basis X for all $i \in I$.*

(c) *If $M = \oplus_{i \in I} M_i$ and A is M-free with M-basis X, then A is M_i-free with M_i-basis X. The converse holds if I is finite, but not if I is infinite.*

Proof. (a), (b) and the first part of (c) follow easily from the definition. To obtain a counterexample for the second part of (c), let $M = \oplus_{n=1}^{\infty} e_n J_p$ where J_p is the additive group of the ring of p-adic integers, $e_n = 1$ in the nth copy of J_p. Let $0 \neq a_i \in \mathbf{Z}$ and $A = \langle \oplus_{n=1}^{\infty} e_n \mathbf{Z}, (e_n p^n a_n)_{n \in \mathbf{N}} \rangle_*$, a pure subgroup of $\Pi_{n=1}^{\infty} e_n \mathbf{Z}$. Let $X = \{e_n \mid n \in \mathbf{N}\}$. Then $\langle X \rangle$ is pure in A with $A/\langle X \rangle$ p-divisible, J_p is pure injective, which shows that A is J_p-free with J_p-basis X. Now define $f : X \to M$ by $f(e_n) = e_n$. This function does not extend to a $\varphi \in \operatorname{Hom}(A, M)$. Thus A is *not* M-free. $\square$

The following result will allow us to assume that M-free groups are

M-torsionless.

Lemma 3. *Let $A, M \neq 0$ be abelian groups and $B \subseteq R_M(A)$. Then A is M-free with M-basis X if and only if A/B is M-free with M-basis $\overline{X} = \{x + B : x \in X\}$.*

Proof. Assume that A is M-free with M-basis X, and let $\bar{f} : \overline{X} \to M$ be a map. Let $^- : A \to A/B$ be the natural epimorphism, which induces a one-to-one map from X onto $\overline{X}$ because $M \neq 0$. define $f : X \to M$ by $f(x) = \bar{f}(\bar{x})$. Then there is $\varphi \in \mathrm{Hom}\,(A, M)$ with $\varphi \restriction_X = f$. Thus $\bar{\varphi} \restriction_{\overline{X}} = \bar{f}$ where $\bar{\varphi} \in \mathrm{Hom}\,(A/B, M)$ is the map induced by φ. (Note that $B \subseteq R_M(A) \subseteq \ker(\varphi)$). Moreover, $\mathrm{Hom}\,(A/\langle X \rangle, M) = 0$ implies $\mathrm{Hom}\,(\bar{A}/\langle \overline{X} \rangle, M) = 0$, and we have that $\bar{A}$ is M-free with M-basis $\overline{X}$. The converse is proved in a similar fashion. □

Lemma 4. *If A is M-free with M-basis $X = \{x\} \cup Y$, $x \notin Y$. Then $A/\langle Y \rangle$ is M-free with M-basis $\{x + \langle Y \rangle\}$.*

The easy proof is left to the reader.

Since the class of free abelian groups is closed with respect to direct summands, one might expect the same to be true for M-free groups. The following is a counterexample. Let p, q be two distinct primes and $\mathbf{1} = (1, 1) \in S = \mathbf{Z}_p \oplus \mathbf{Z}_q$. Then S is S-free with S-basis $\{\mathbf{1}\}$. But $\mathbf{Z}_p$ is not S-free for any X since $\mathrm{Hom}\,(\mathbf{Z}_p, \mathbf{Z}_q) = 0$ but $\mathrm{Hom}\,(\mathbf{Z}_p, S) \neq 0$.

If A and B are free abelian groups with bases $\{a_i : i \in I\}$ and $\{b_j : j \in J\}$, respectively, then the tensor product $A \otimes_{\mathbf{Z}} B$ is again a free group with basis $\{a_i \otimes b_j : i \in I, j \in J\}$. Thus it is natural to ask if tensor products of M-free abelian groups behave in the same way. We can prove that tensor products of M-free groups are M-free again only if we assume some additional conditions. We do not know if these conditions are actually necessary.

Proposition 5. *Let M be unbounded, A M-free with M-basis X and B M-free with M-basis Y such that A is torsion-free or B is torsion-*

free. Then $A \otimes B$ is M-free with M-basis $X \otimes Y = \{x \otimes y : x \in X, y \in Y\}$.

Proof. Since M is unbounded, $\langle X\rangle$ and $\langle Y\rangle$ are free by Lemma 1. Assume that A is torsion-free. The natural map $\langle X\rangle \otimes \langle Y\rangle \to A \otimes \langle Y\rangle$ is an embedding since $\langle Y\rangle$ is torsion-free by Lemma 1 and the map $A \otimes \langle Y\rangle \to A \otimes B$ is a natural embedding, since A is torsion-free. Thus $\langle X\rangle \otimes \langle Y\rangle = \oplus_{x\in X, y\in Y}(x \otimes y)\mathbf{Z}$ is free and naturally embedded in $A \otimes B$. Let $X \otimes Y = \{x \otimes y \mid x \in X, y \in Y\}$ and $f : X \otimes Y \to M$ be any map. For $y \in Y$ define $f_y : X \to M$ by $f_y(x) = f(x \otimes y)$. This map extends to $\varphi_y \in \mathrm{Hom}\,(A, M)$. Since $A \otimes \langle Y\rangle = \oplus_{y\in Y} A \otimes y$ and $A \otimes y \cong A$, we can define $\varphi : A \otimes \langle Y\rangle \to M$ by $\varphi \restriction_{A\otimes y} = \varphi_y$. Let $0 \neq a \in A$. Since $\langle Y\rangle$ is torsion-free, $\langle a\rangle \otimes \langle Y\rangle$ embeds into $A \otimes \langle Y\rangle$. Define $g_a : Y \to M$ by $g_a(y) = \varphi(a \otimes y)$. Since B is M-free with M-basis Y, g_a extends to $\gamma_a : B \to M$. Now define $\sigma : A \times B \to M$ by $\sigma(a, b) = \gamma_a(b)$. Since $\gamma_a \in \mathrm{Hom}\,(B, M)$, σ is linear in the second variable. Moreover, define for $y \in Y$ and $a_1, a_2 \in A \theta(y) = \sigma(a_1 + a_2, y) - \sigma(a_1, y) - \sigma(a_2, y) = \gamma_{a_1+a_2}(y) - \gamma_{a_1}(y) - \gamma_{a_2}(y) = g_{a_1+a_2}(y) - g_{a_1}(y) - g_{a_2}(y) = \varphi((a_1+a_2)\otimes y) - \varphi(a_1 \otimes y) - \varphi(a_2 \otimes y) = \varphi(0) = 0$, since $\varphi \in \mathrm{Hom}\,(A \otimes \langle Y\rangle, M)$. Thus θ extends to a unique $\bar{\theta} \in \mathrm{Hom}\,(B, M)$, defined in terms of σ. Now $\hat{\theta} = 0$ since $\theta = 0$ and σ is linear in the first variable as well. This implies that $\sigma(az, b) = \sigma(a, zb)$ for all $z \in \mathbf{Z}$, and therefore σ is a middle linear map, cf. [**7**]. Thus there is $\psi : A \otimes B \to M$ with $\psi(x \otimes y) = \sigma(x, y) = \gamma_x(y) = g_x(y) = \varphi(x \otimes y) = \varphi_y(x) = f_y(x) = f(x \otimes y)$, i.e., ψ is extending f.

For the uniqueness of this extension, consider $0 \to \langle Y\rangle \to B \to B/\langle Y\rangle \to 0$, which gives rise to $0 \to A \otimes \langle Y\rangle \to A \otimes B \to A \otimes (B/\langle Y\rangle) \to 0$ and $\mathrm{Hom}\,(A \otimes (B/\langle Y\rangle), M) \cong \mathrm{Hom}\,(A, \mathrm{Hom}\,(B/\langle Y\rangle, M)) \cong \mathrm{Hom}\,(A, 0) = 0$. Thus, each $\varphi \in \mathrm{Hom}\,(A \otimes B, M)$ with $\varphi \restriction_{A\otimes\langle Y\rangle} = 0$ is the 0-map. Similarly, let $0 \to \langle X\rangle \to A \to A/\langle X\rangle \to 0$ induce $0 \to \langle X\rangle \otimes \langle Y\rangle \to A \otimes \langle Y\rangle \to (A/\langle X\rangle) \otimes \langle Y\rangle \to 0$ with $\mathrm{Hom}\,((A/\langle X\rangle) \otimes \langle Y\rangle, M) \cong \mathrm{Hom}\,(A/\langle X\rangle, \mathrm{Hom}\,(\langle Y\rangle, M)) \cong \mathrm{Hom}\,(A/\langle X\rangle, M^Y) \cong (\mathrm{Hom}\,(A/\langle X\rangle, M))^Y = 0^Y = 0$. Therefore, each $\varphi \in \mathrm{Hom}\,(A \otimes \langle Y\rangle, M)$ with $\varphi \restriction_{\langle X\rangle\otimes\langle Y\rangle} = 0$ is the 0-map. Thus $\mathrm{Hom}\,((A \otimes B)/(\langle X\rangle \otimes \langle Y\rangle), M) = 0$ and $A \otimes B$ is M-free with M-basis $X \otimes Y$. $\square$

We obtain:

Corollary 6. *If M is a torsion-free $\mathbf{Z}_p$-module and A is M-free with M-basis X and torsion-free, then $A \otimes \mathbf{Z}_p$ is M-free with M-basis $\{x \otimes 1 \mid x \in X\}$.*

Proof. $\mathbf{Z}_p$ is M-free with M-basis $\{1\}$. □

The following is easy to show.

Observation 7. *Let A be M-free with basis X and B M-free with M-basis Y; then $A \oplus B$ is M-free with M-basis $X \cup Y$.*

For a few groups M, M-free groups are easy to describe.

Proposition 8. *Let A and M be abelian groups.*

(a) *If M contains $\mathbf{Q}/\mathbf{Z}$, then A is M-free with basis X if and only if $A = \langle X \rangle$ is free abelian with M-basis X.*

(b) *There is a subset X of A such that A is $\mathbf{Q}$-free with $\mathbf{Q}$-basis X.*

(c) *Let p be a prime. Then there is a subset X of A such that A is J_p-free with J_p-basis X.*

Proof. $\mathbf{Q}/\mathbf{Z}$ is a cogenerator [**1**] in the category of abelian groups and $\mathrm{Hom}\,(A/\langle X \rangle, \mathbf{Q}/\mathbf{Z}) = 0$. Thus $A/\langle X \rangle = 0$ and $A = \langle X \rangle$ is free by Lemma 1. Let $t(A)$ be the torsion subgroup of A and $X \subseteq A$ induce a basis of the $\mathbf{Q}$-vector space $(A/t(A)) \otimes \mathbf{Q}$. Then $A/t(A)$ is $\mathbf{Q}$-free with $\mathbf{Q}$-basis $\{x + t(A) \mid x \in X\}$. By Lemma 3, A is $\mathbf{Q}$-free with $\mathbf{Q}$-basis X since $R_{\mathbf{Q}}(A) = t(A)$. To show (c), let X be a subset of A such that $\overline{X} = \{x + R_{J_p}(A) \mid x \in X\}$ is a p-basis of $\bar{A} = A/R_{J_p}(A)$. Since J_p is p-pure-injective and $\bar{A}/\langle \overline{X} \rangle$ is p-divisible, $\bar{A}$ is J_p-free with J_p-basis $\overline{X}$. Now apply Lemma 3. □

A well-known notation that resembles "M-free" is "M-projective." Recall [**1**] that, for R-modules A and M, the module A is M-projective if, for each epimorphism $\psi : M \to B$ and for each homomorphism $\varphi : A \to M$ there is some $\gamma \in \mathrm{Hom}\,(A, M)$ such that $\varphi = \psi \circ \gamma$, i.e., the natural map $\mathrm{Hom}\,(A, M) \to \mathrm{Hom}\,(A, B)$ is surjective. The following example shows that M-free does not imply M-projective, even for abelian groups:

Let J_p be the group of p-adic integers. Then J_p is also a ring with $1 \in J_p$ and J_p is J_p-free with J_p-basis $\{1\}$ since $\mathrm{End}\,(J_p) = J_p$. Moreover, $|J_p| = c = 2^{\aleph_0}$, the cardinality of the continuum. There exists a short exact sequence $0 \to K \to J_p \xrightarrow{\psi} \mathbf{Z}(p^\infty) \to 0$ and J_p has $\oplus_c \mathbf{Z}(p^\infty)$ as an epimorphic image, which implies that $|\mathrm{Hom}\,(J_p, \mathbf{Z}(p^\infty))| = 2^c > c = |\mathrm{Hom}\,(J_p, J_p)|$. This shows that J_p is not J_p-projective but is J_p-free with J_p-basis $\{1\}$. Thus, "M-free" does not imply "M-projective." However we can modify the definition of M-projective to obtain a positive result:

Definition. An abelian group A is M-torsionless-projective if, for each epimorphism $\psi : M \to B$ with $R_M(B) = 0$ and homomorphism $\varphi : A \to B$ there is a $\beta \in \mathrm{Hom}\,(A, M)$ with $\psi \circ \beta = \varphi$.

We have:

Proposition 9. *If A is M-free with M-basis X, then A is M-torsionless-projective.*

Proof. With notations as in the definition, pick $m_x \in M$ such that $\varphi(x) = \psi(m_x)$. Then there is $\beta \in \mathrm{Hom}\,(A, M)$ with $\beta(x) = m_x$. Let $\eta = \psi \circ \beta - \varphi$. Then $\eta \in \mathrm{Hom}\,(A, B)$ with $X \subseteq \ker(\eta)$. Assume that there is an $a \in A$ with $\eta(a) = b \neq 0$. Since B is M-torsionless, there is $\theta \in \mathrm{Hom}\,(B, M)$ with $\theta(b) \neq 0$. Thus $\theta \circ \eta : A \to M$ with $\theta \circ \eta \restriction_X = 0$. Thus $\theta \circ \eta = 0$, a contradiction to $(\theta \circ \eta)(a) = \theta(b) \neq 0$. □

Observation. *Let H, M be groups such that* $\mathrm{Hom}\,(H, M) = 0$. *Then* $\mathrm{Hom}\,(H, (\mathrm{End}\,(M))^+) = 0$.

Proof. Let $E = \mathrm{End}\,(M)$ and $m \in M$. Then $Em = \{\varphi(m) : \varphi \in E\} \cong E/\mathrm{ann}_E(m)$ is a subgroup of M, which implies $\mathrm{Hom}\,(H, Em) = 0$. Thus $\mathrm{Hom}\,(H, \prod_{m \in M} Em) = 0$ and there is an embedding $\varphi \mapsto (\varphi(m))_{m \in M}$ of E^+ into $\Pi_{m \in M} E_m$. This shows $\mathrm{Hom}\,(H, E^+) = 0$. □

We will use this observation in the proof of the following essential result.

Lemma 10. *Let A, M be abelian groups. The following are equivalent*:

(a) *A is M-torsionless and M-free with M-basis $X = \{x\}$, a singleton.*

(b) *There is a subgroup B of* $[\text{Cent}\,(\text{End}\,(M))]^+$ *such that* $\text{id}_M \in B$ *and an isomorphism $\gamma : A \to B$ with $\gamma(x) = id_M$ and* $\text{Hom}\,(B/\langle id_M\rangle, M) = 0$.

Proof. First we show that (a) implies (b). For $a \in A$ define a map $\gamma(a) : M \to M$ by $[\gamma(a)](m) = \varphi_m(a)$, where φ_m is the unique homomorphism $\varphi_m : A \to M$ with $\varphi_m(x) = m$. It is easy to see that $\gamma(a) \in \text{End}\,(M)$ and $\gamma \in \text{Hom}\,(A, (\text{End}\,(m))^+)$. If $\gamma(a) = 0$, then $\varphi_m(a) = 0$ for all $m \in M$. Since $\text{Hom}\,(A, M) = \{\varphi_m : m \in M\}$, we infer $a \in R_M(A) = 0$ since A is M-torsionless. Thus $\gamma : A \to \text{End}\,(M)$ is an imbedding. Note that $[\gamma(x)](m) = \varphi_m(x) = m$ for all $m \in M$. Thus $\gamma(x) = \text{id}_M$. Let $B = \gamma(A) \subseteq \text{End}\,(M)$. Since A and B are isomorphic via γ and $\gamma(x) = \text{id}_M$, we have that B is M-free with M-basis $\{\text{id}_M\}$. Thus $\text{Hom}\,(B/\langle \text{id}_M\rangle, M) = 0$, which, by the above observation, implies that $\text{Hom}\,(B/\langle \text{id}_M\rangle, \text{End}\,(M)) = 0$. For $\alpha \in \text{End}\,(M)$ one can define maps α_r and α_l from B into $\text{End}\,(M)$ with $\alpha_r(b) = b\alpha$ and $\alpha_l(b) = \alpha b$ for all $b \in B$. Since $\alpha_r(\text{id}_M) = \alpha_l(\text{id}_M)$, we have that $\alpha_r - \alpha_l$ induces an element of $\text{Hom}\,(B/\langle \text{id}_M\rangle, \text{End}\,(M)) = 0$. Thus, $\alpha_r = \alpha_l$ for all $\alpha \in \text{End}\,(M)$ and therefore $B \subseteq \text{Cent}\,(\text{End}\,(M))$.

Now we show that (b) implies (a). Let $m \in M$. Define a map $\psi_m : B \to M$ by $\psi_m(b) = b(m)$ for all $b \in B$. Then $\psi_m \in \text{Hom}\,(B, M)$ with $\psi_m(\text{id}_M) = \text{id}_M(m) = m$. This, together with the hypothesis $\text{Hom}\,(B/\langle \text{id}_M\rangle, M) = 0$ implies that B is M-free with M-basis $\{\text{id}_M\}$. Moreover, if $b \in B$ such that $\psi_m(b) = 0$ for all $m \in M$, then $b(m) = 0$ for all $m \in M$, i.e., $b = 0$ and we infer $R_M(B) = 0$ and B is M-torsionless. Now use the isomorphism γ^{-1} from (b) to conclude that A is M-free with M-basis $\{x\}$ and also M-torsionless. □

Remark. Assume that the group B is as in part (b) of Lemma 10. Let $\varphi \in \text{End}\,(B) \subseteq \text{Hom}\,(B, \text{Cent}\,(\text{End}\,(M)))$. Since $\text{Hom}\,(B/\langle \text{id})M\rangle, M) = 0$ and $\text{Hom}\,(B/\langle \text{id}_M\rangle, \text{End}\,(M) = 0$ by the above observation, we infer that there is an element $\alpha_\varphi \in \mathrm{Cent}\,(\mathrm{End}\,(M))$ such that $\varphi(b) = b\alpha_\varphi$ for all $b \in B$ and $\varphi(\mathrm{id}_M) = \mathrm{id}_M\alpha_\varphi = \alpha_\varphi \in B$. Thus $\mathrm{End}\,(B) \cong \{b \in B : Bb \subseteq B\}$, a subring of $\mathrm{Cent}\,(\mathrm{End}\,(M))$. Now we apply Lemma 10 to obtain:

If A is M-free with M-basis $X = \{x\}$ a singleton and A is also M-torsionless, then $\mathrm{End}\,(A)$ is commutative.

We want to extend Lemma 10 to the case where the M-basis has arbitrary size. This will involve an embedding of A in the double-dual $\mathrm{Hom}\,(\mathrm{Hom}\,(A, M), M)$. We know that (provided that M is unbounded) $\mathrm{Hom}\,(A, M)$ is isomorphic to $\prod_{x\in X} M$, a Cartesian product of $|X|$ copies of M. Thus $\mathrm{Hom}\,(\mathrm{Hom}\,(A, M), M) \cong \mathrm{Hom}\,(\prod_{x\in X} M, M)$. Nothing much can be said about the latter, unless M is a slender group. Recall [**5**] that M is slender if and only if M is torsion-free, cotorsion-free and contains no copy of the Baer-Specker group.

Theorem 11. *Let M be slender and $\aleph_m$ the first measurable cardinal* (*if it exists*). *The following are equivalent*:

(a) *A is M-torsionless and M-free with M-basis X and $|X| < \aleph_m$.*

(b) *There is a family of M-torsionless groups A_x, $x \in X$, such that $x \in A_x$, $|X| < \aleph_m$, each A_x is M-free with M-basis $\{x\}$ and $\oplus_{x\in X}\langle x\rangle \subseteq A \subseteq_{x\in X} A_x$, the natural projection of A into A_x is surjective for each $x \in X$ and* $\mathrm{Hom}\,(A/\langle X\rangle, M) = 0$.

(Note: If X is finite, then (a) and (b) are equivalent assuming M unbounded instead of slender.)

Proof. Obviously, (b) implies (a). The converse requires more work. Since A is M-free with M-basis X, $\mathrm{Hom}\,(A, M) = \{(m_x)_{x\in X} \mid m_x \in M\} \cong M^X$, where $[(m_x)_{x\in X}](y) = m_y$ for all $y \in X$. By [**6**, p. 204], which is a generalization of Łos's result on $\mathrm{Hom}\,(\Pi\mathbf{Z}, \mathbf{Z})$, we have $\mathrm{Hom}\,(\mathrm{Hom}\,(A, M), M) = \oplus_{x\in X}\mathrm{End}\,(M) \circ \pi_x$ where π_x is the coordinate projection which can be identified with id_M in the xth coordinate. Consider $\# : A \to \mathrm{Hom}\,(\mathrm{Hom}\,(A, M), M)$ defined by $a^\#(\varphi) = \varphi(a)$ for all $\varphi \in \mathrm{Hom}\,(A, M)$. Note that $\ker(\#) = R_M(A) = 0$ and $A \cong A^\#$. If $\varphi = (m_y)_{y\in X}$, then $x^\#(\varphi) = \varphi(x) = m_x$ and $x^\# = \pi_x$ follows. Thus $\oplus_{x\in X}\pi_x\mathbf{Z} \subseteq A^\# \subseteq \oplus_{x\in X}\mathrm{End}\,(M) \circ \pi_x$. Let

$E = \oplus_{x\in X}\mathrm{End}\,(M)\circ\pi_x$ and A_x the image of the projection η_x of $A^{\#}$ into the xth coordinate, i.e., $\oplus_{x\in X}\pi_x\mathbf{Z}\subseteq A^{\#}\subseteq\oplus_{x\in X}A_x\subseteq E$. Since $\pi_x\in A_x\subseteq\mathrm{End}\,(M)$, each map $f:\{\pi_x\}\to M$ extends to a $\varphi\in\mathrm{Hom}\,(A_x,M)$. Let $\psi\in\mathrm{Hom}\,(A_x,M)$ with $\psi(\pi_x)=0$. Extend ψ to $\tilde{\psi}\in\mathrm{Hom}\,(\oplus_{y\in X}A_y,M)$ by setting it 0 on the A_y with $y\neq x$. Then $\tilde{\psi}\restriction_{A^{\#}}(X^{\#})=\{0\}$ which implies $\tilde{x}\restriction_{A^{\#}}=0$. Since the projection η_x of $A^{\#}$ into A_x is onto and $\tilde{\psi}\restriction_{A^{\#}}=\psi\circ\eta_x=0$ we conclude that $\psi=0$. Thus A_x is M-free with a one element M-basis. □

Lemma 12. *Let $0\neq M$ be a slender group and I,J sets with $M^I\cong M^J$. If at least one of I,J is infinite, then $|I|=|J|$.*

Proof. Since $M\neq 0$ and M^I is slender if and only if I is finite, both I,J are infinite. Pick $0\neq m\in M$ and define $m_i\in M^I$ by $m_i(j)=m$ if $i=j$ and $m_i(j)=0$ if $i\neq j$. Then $\prod_{i\in I}m_iZ$ is a subgroup of M^I. We may assume that J is well-ordered by some order relation. Let $\theta:M^I\to M^J$ be some isomorphism and, for $i\in I$, define $E_i=\{j\in J\mid[\theta(m_i)](j)\neq 0\}$. Note that each $E_i\neq\varnothing$ and, for each $j\in J$, $\{i\in I\mid j\in E_i\}$ is finite since M is slender. Define $\sigma:I\to J$ by $\sigma(i)=$ least element in E_i and call $i_1,i_2\in I$ equivalent if $\sigma(i_1)=\sigma(i_2)$. Of course, this is an equivalence relation such that all equivalence classes are finite, i.e., there are as many classes as there are elements in the infinite set I, and there is a one-to-one function from the set of classes into J. Thus $|I|\leq|J|$. □

Theorem 13. *Let M be slender and A be M-free with M-bases X and Y. If X or Y is infinite, then $|X|=|Y|$ and $\langle X\rangle\cong\langle Y\rangle$.*

Proof. $M^X\cong\mathrm{Hom}\,(A,M)\cong M^Y$ and Lemma 12.

We will employ some results by Eda [**3**] that can be found in [**4**] to get rid of the cardinality restriction on $|X|$ in Theorem 11.

Theorem 14. *Let M be a countable, slender group with* $(\mathrm{End}\,(M))^+$ *free abelian. If A is M-free with M-basis X, then A is split-M-free with M-basis X.*

Proof. We have $\mathrm{Hom}\,(A,M)\cong M^X$. Let $\mathbf{D}$ be the set of all $\aleph_1$-complete ultrafilters on X, cf. [EM]. Let $\bar{A}=A/R_M(A)$. Then $\bar{A}$ embeds into $\mathrm{Hom}\,(A^X,M)=\oplus_{D\in\mathbf{D}}\mathrm{Hom}\,(M^X/D,M)$ where $M^X/D=M^X/\{(m_x)_{x\in X}\mid\{x\in X\mid m_x=0\}\in D\}$. While—in general—there may be nonprincipal ultrafilters $D\in\mathbf{D}$, the fact that M is countable implies that $M^X/D\cong M$ and $\mathrm{Hom}\,(M^X,M)\cong\oplus_{D\in\mathbf{D}}\mathrm{End}\,(M)$, a free abelian gorup. Thus $\bar{A}$ is free and $A=R_M(A)\oplus F$, $\bar{A}\cong F$ is free and $\mathrm{Hom}\,(R_M(A),M)=0$. For $x\in X$ write $x=x_1+x_2$ with $x_1\in R_M(A)$ and $x_2\in F$. Then $x\mapsto x_2$ is a one-to-one map since $\langle X\rangle\cap R_M(A)=0$. It is easy to see that F is M-free with M-basis $X_2=\{x_2\mid x\in X\}$. Let $G=F/\langle X_2\rangle$. Then $\mathrm{Hom}\,(G,M)=0$ and $0\to\mathrm{Hom}\,(G,M)=0\to\mathrm{Hom}\,(F,M)\to\mathrm{Hom}\,(\langle X_2\rangle,M)\to\mathrm{Ext}\,(G,M)\to\mathrm{Ext}\,(F,M)=0$. Since F is M-free with M-basis X_2, we conclude $\mathrm{Ext}\,(G,M)=0=\mathrm{Hom}\,(G,M)$. Let p be a prime. Then $pM\neq M$ since $\mathrm{End}\,(M)^+$ is free abelian. Now $0\to pM\to M\to M/pM\to 0$ gives rise to $\mathrm{Hom}\,(G,M)=0\to\mathrm{Hom}\,(G,M/pM)\to\mathrm{Ext}\,(G,pM)=0$ since the slender group M is torsion-free and thus $M\cong pM$. We infer $\mathrm{Hom}\,(G,M/pM)=0$ and G is divisible and torsion-free by Lemma 1(b).

Assume that $G\neq 0$. Then G contains a summand isomorphic to $\mathbf{Q}$ and $\mathrm{Ext}\,(\mathbf{Q},M)=0$ follows. This means that M is cotorsion, a contradiction to M slender. Thus $G=0$ and $F=\langle X_2\rangle$, We may change complements and obtain $A=R_M(A)\oplus\langle X\rangle$, i.e., A is split-M-free. $\square$

The main result is now a consequence of Theorem 14. By Lemma 1(a) we obtain:

Corollary 15. (a) *Let A be M-free with M-basis X such that M has a summand isomorphic to $\mathbf{Z}$. Then $A\cong H\oplus\langle X\rangle$ with* $\mathrm{Hom}\,(H,M)=0$.

(b) *Let M be slender such that* $[\mathrm{Cent}\,(\mathrm{End}\,(M))]^+$ *is free abelian, and let A be M-free with M-basis X and $|X|<\aleph_m$. Then A is split-M-free.*

Proof. A is $\mathbf{Z}$-free with $\mathbf{Z}$-basis X and thus $A\cong H\oplus\langle X\rangle$, such that $\mathrm{Hom}\,(A/\langle X\rangle,M)=0=\mathrm{Hom}\,(H,M)$.

Item (b) follows from Theorem 11 and Lemma 10. $\square$

We will now study the case of nontorsion-free groups M.

Lemma 16. *If the p-primary torsion part M_p of M is not* 0 *and A is M-free with M-basis X, then $A/\langle X\rangle$ is p-divisible.*

Proof. Let $m \in M_p$ have order p. If $\bar{A} = A/\langle X\rangle$ is not p-divisible, then $p\bar{A} \neq \bar{A}$. We can find maps $\bar{A} \to \bar{A}/p\bar{A} \to \mathbf{Z}/p\mathbf{Z} \to m\mathbf{Z} \subseteq M$ whose composition is $\neq 0$, a contradiction to $\operatorname{Hom}(\bar{A}, M) = 0$. □

Suppose that $\langle X\rangle$ is not p-pure in A. Then there are $x_i \in X$, $r_i \in \mathbf{Z}$, and $y \in A$ such that $p^k y = \sum_{i=1}^m r_i x_i$ and, we may assume, p^e is the highest power of p that divides r_1 and $e < k$. Mapping x_1 to $m \in M$ and the other x's to 0, we get $p^k y_m = msp^e$ where $\gcd(s,p) = 1$. Let $1 = s\alpha + p^{k-e}\beta$ with $a, \beta \in \mathbf{Z}$. Then $p^k y_m \alpha = m(s\alpha)p^e = m(1 - p^{k-e}\beta)p^e$ and $mp^e = p^k(y_m\alpha + m\beta)$ and thus $p^e(m - p^{k-e}(y_m\alpha + m\beta)) = 0$. This shows that $M = M[p^e] + p^{k-e}M$. Since $k - e \geq 1$, we infer $M = M[p^e] + p^n M$ for all natural numbers n. If M_p contains a copy of $\mathbf{Z}(p^\infty)$, then $A/\langle X\rangle$ is p-torsion-free and, by Lemma 1(b), $\langle X\rangle$ is p-pure in A, a contradiction to our assumption. Thus M_p is bounded and, with Lemma 16, we have:

Theorem 17. *Let A be M-free with M-basis X such that, for some prime p, the p-primary part M_p of M is not bounded. Then $\langle X\rangle$ is a p-pure free subgroup of A with basis X such that $A/\langle X\rangle$ is p-divisible, i.e., $\langle X\rangle$ is a p-basic subgroup of A.* (*All such subgroups have the same rank.*)

The following follows from Lemma 1(a) and the fact that $\mathbf{Z}(p^\infty)$ is injective in the category of abelian groups.

Lemma 18. *An abelian group A is $\mathbf{Z}(p^\infty)$-free with $\mathbf{Z}(p^\infty)$-basis X if and only if $\langle X\rangle$ is a free subgroup of A such that $A/\langle X\rangle$ is torsion with $(A/\langle X\rangle)_p = 0$, i.e., $A/\langle X\rangle$ is a torsion p'-group.*

We now look at the case where M_p is bounded and thus has a summand isomorphic to $\mathbf{Z}(p^e)$, $e \geq 1$.

Lemma 19. *A is $\mathbf{Z}(p^e)$-free with $\mathbf{Z}(p^e)$-basis X if and only if $A = \langle X\rangle + p^eA$ and $p^eA \cap \langle X\rangle = p^e\langle X\rangle$ and $\langle X\rangle/p^e\langle X\rangle$ is a free $\mathbf{Z}(p^e)$-module.*

Proof. The "if" part is easy and left to the reader. Assume that A is $\mathbf{Z}(p^e)$-free with $\mathbf{Z}(p^e)$-basis X. Let $x_i \in X$ and $z_i \in \mathbf{Z}$ such that $w = \sum_{i=1}^k z_i x_i \in p^eA$. Fix $1 \le j \le k$ and define $f : X \to \mathbf{Z}(p^e)$ by $f(x_j) = 1$ and $f(x_i) = 0$ for $i \ne j$. Let $\varphi \in \operatorname{Hom}(A, Z(p^\infty))$ extend the map f. Then $0 = \varphi(w) = \varphi(x_j z_j) = z_j \in \mathbf{Z}(p^\infty)$, i.e., p^e divides z_j. This shows that $\langle X\rangle/p^e\langle X\rangle$ is a free $\mathbf{Z}(p^e)$-module. Moreover, $p^eA\cap\langle X\rangle = p^e\langle X\rangle$ follows as well. Note that $p^eA \subseteq R_{\mathbf{Z}(p^e)}(A)$ and, by Lemma 3, $\bar{A} = A/p^eA$ is $\mathbf{Z}(p^e)$-free with $\mathbf{Z}(p^e)$-basis $\overline{X} = \{x + p^eA : x \in X\}$. Now $\bar{A}$ is p^e-bounded and $\operatorname{Hom}(\bar{A}/\langle\overline{X}\rangle, Z(p^e)) = 0$. Thus $\bar{A} = \langle\overline{X}\rangle$ and we have $A = \langle X\rangle + p^eA$. $\square$

Theorem 20. (a) *If A is $\mathbf{Z}(p^\infty)$-free with $\mathbf{Z}(p^\infty)$-basis X, then A is $\mathbf{Z}(p^e)$-free for each $e \ge 1$.*

(b) *If A is $\mathbf{Z}(p^e)$-free with $\mathbf{Z}(p^e)$-basis X for each $e \ge 1$ and $A/\langle X\rangle$ is torsion with $(A/\langle X\rangle)_p = 0$, then A is $\mathbf{Z}(p^\infty)$-free with $\mathbf{Z}(p^\infty)$-basis X.*

Proof. (a) follows from Lemma 1(c). To show (b), note that Lemma 19 implies that $\langle X\rangle/p^e\langle X\rangle$ is a free $\mathbf{Z}(p^e)$-module with basis $\{x + p^e\langle X\rangle \mid x \in X\}$. This implies that $\langle X\rangle$ is free abelian with basis X. Now apply Lemma 18. $\square$

The hypothesis in (b) that $A/\langle X\rangle$ is p'-torsion is actually needed as the following example shows.

Let $0 \to K \to F \to \mathbf{Z}[1/p] \to 0$ be a free resolution of the p-divisible torsion-free rank 1 subgroup $\mathbf{Z}[1/p]$ of $\mathbf{Q}$ and K is free with basis X. For all $e \ge 1$ we have $F = K + p^eF$. Then F is $Z(p^e)$-free with $\mathbf{Z}(p^e)$-basis X, but F is not $\mathbf{Z}(p^\infty)$-free with $\mathbf{Z}(p^\infty)$-basis X. $\square$

Lemma 21. *Let A be split-M-free with M-basis X and M unbounded. If A is M-free with M-basis Y, then $\langle X\rangle \cong \langle Y\rangle$ and A has* *invariant M-rank.*

Proof. $A = H \oplus \langle X \rangle$ with $\langle X \rangle$ free and $\mathrm{Hom}\,(H, M) = 0$. Also $\langle Y \rangle$ is free with $H \cap \langle Y \rangle = 0$ and thus $\mathrm{rank}\,\langle Y \rangle \leq \mathrm{rank}\,\langle X \rangle$. If $\mathrm{rank}\,\langle Y \rangle < \mathrm{rank}\,\langle X \rangle$, then $A = B \oplus \langle x \rangle$ with $0 \neq x \in \langle X \rangle$ and $Y \subseteq B$. Thus $\mathrm{Hom}\,(A/\langle Y \rangle, M) \neq 0$, a contradiction. □

We leave the proof of the following result to the reader.

Lemma 22. *If A is J_p-free with J_p-basis X, then $\overline{X} = \{x + R_{J_p}(A) \mid x \in X\}$ is a p-basis of $\bar{A} = A/R_{J_p}(A)$ and $|X| = |\overline{X}|$. Thus A has invariant J_p-rank. Conversely, each p-basis of $\bar{A}$ gives rise to a J_p-basis of A.*

We now summarize our results on invariant M-rank as:

Theorem 23. *Let A be M-free with M-basis X and $M \neq 0$. Any of the following conditions implies that A has invariant M-rank*:

(a) *For some prime p, $pM \neq M$ and A is p-torsion-free.*

(b) *$M = J_p$ for some prime p.*

(c) *M is not torsion-free.*

(d) *M has a direct summand H that is countable, slender and* $[\mathrm{Cent}\,(\mathrm{End}\,(H))]^+$ *is free abelian.*

(e) *M has $\mathbf{Z}$ as a direct summand.*

(f) *M has a slender summand $S \neq 0$ and all M-bases of A have cardinality $< \aleph_m$.*

(g) *M is unbounded and A has a finite M-basis.*

Proof. We have to prove (g). Assume that X and Y are M-bases of A with X finite and $|X| \leq |Y|$. Let S be the center of $\mathrm{End}\,(M)$ and S^+ the additive group of S. By Theorem 11, using $X = \{x_i \mid i = 1, 2, \dots, n\}$, we have $A \subseteq \oplus_{i=1}^n A_i \subseteq \oplus_{i=1}^n S = S^{(n)}$ such that x_i is the 1 in the ith copy of S. Moreover, $\mathrm{Hom}\,(A, S^+) = \{\vec{b} = (b_1, \dots, b_n) \mid b_i \in S\}$ such that if $\vec{a} = (a_1, \dots, a_n) \in A$, then $\vec{b}(\vec{a}) = \vec{b} \cdot \vec{a} = \sum_{i=1}^n a_i b_i$, the natural dot product of the free S-module $S^{(n)}$. Note that $\mathrm{Hom}\,(A,S^+)$ is naturally isomorphic to $\mathrm{Hom}_S(S^{(n)},S)$. For $y\in Y$ consider the map $\theta_y : A\to S$ defined as the composition of the maps $A\to A/\langle Y-\{y\}\rangle\to S$ such that $\theta_y(y)=1\in S$. Here we use Lemma 10 and Lemma 4. Clearly, $\theta_y\in \mathrm{Hom}\,(A,S^+)$ with $\theta_y(x)=0$ for $y\neq x\in Y$. Note that all the maps in $\mathrm{Hom}\,(A,S^+)$ are S-linear. Let $y_1,y_2,\dots,y_k$ be elements of Y, a subset of $S^{(n)}$, and assume there are $s_i\in S$ such that $\sum_{i=1}^k y_is_i=0$. Then $0=\theta_{y_j}(\sum_{i=1}^k y_is_i)=\theta_{y_j}(s_j=s_j$, i.e., Y is a linearly independent subset of the S-module $S^{(n)}$. Let J be a maximal ideal of S. Then $F=S/J$ is a field and $\overline{S^{(n)}}=S^{(n)}/(JS^{(n)})$ is an F-vector space of dimension n. Since θ_y is S-linear, it includes $\bar\theta_y:\overline{S^{(n)}}\to F$ such that

$$\bar\theta_y(\bar x)=\begin{cases}1 & \text{if } x=y\\ 0 & \text{if } x\neq y\end{cases}.$$

As above, this shows that $\overline{Y}=\{\bar y\mid y\in Y\}$ is linearly independent over F.

Thus Y is finite and $|Y|\le n=|X|\le|Y|$, which implies $|X|=|Y|$.

We conclude this paper with some examples:

(1) Let p,q,r be three distinct primes and $M=(\mathbf{Z}[1/p]\oplus\mathbf{Z}[1/q])+[(1/r),(1/r)]\mathbf{Z}$. Then $\mathrm{End}\,(M)=\begin{pmatrix}1&0\\0&1\end{pmatrix}\mathbf{Z}\oplus\begin{pmatrix}0&0\\0&r\end{pmatrix}\mathbf{Z}$ is free and any A that is M-free with some M-basis X is split-M-free with M-basis X.

(2) A popular class of groups is obtained as follows: Let $\pi\in J_p$ be transcendental over Z_p, and let M_π be the p-pure subgroup of J_p generated by 1 and π such that M_π is homogeneous of type 0. Then $\mathrm{End}\,(M_\pi)\cong\mathbf{Z}$ and M_π-free implies split-M_π-free.

(3) If E is any E-ring, then E^+ is E^+-free with E^+-basis $\{1\}$.

(4) Let p,q be distinct primes and $A=(x\mathbf{Z}\oplus y\mathbf{Z}[1/q])+\sum_n\langle(x-y\pi_n)/p^n\rangle$ where $\pi_n=\sum_{i=0}^n p^ia_i$, $a_i\in\mathbf{Z}$. Then A is $\mathbf{Z}[1/p]$-free with basis $\{x\}$ and $R_{\mathbf{Z}[1/p]}(A)=y\mathbf{Z}[1/q]$ is *not* a summand of A. Note that $A/R_{\mathbf{Z}[1/p]}(A)\cong\mathbf{Z}[1/p]$.

(5) Let $\pi\in J_p$ be transcendental over $\mathbf{Z}_p$ and $E=Z_p[\pi]_*\subseteq J_p$ the pure subring of J_p generated by 1 and π. Then E is an E-ring. Let A be any pure subgroup of E with $1\in A$. Then A is E-free with E-basis $\{1\}$ and also E-torsionless. All of these As are indecomposable.

(6) Another example where $\langle X \rangle$ is not a direct summand: Let M be a torsion-free $\mathbf{Z}[1/p]$-module. Then $\operatorname{End}(M)$ is a $\mathbf{Z}[1/p]$-module and thus not free abelian. Let $0 \to \langle X \rangle \to F \to B \to 0$ be short exact with F free abelian, X a basis of $\langle X \rangle$ and B a torsion p-group. Then $\operatorname{Hom}(F/\langle X \rangle, M) = 0 = \operatorname{Ext}(B, M)$, since M is p-divisible and B is p-torsion. This shows that F is M-free with M-basis X, F is M-torsionless, but $\langle X \rangle$ is not a summand of F.

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