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A GEOMETRIC SETTING FOR SOME PROPERTIES OF TORSION-FREE MODULES

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Dedicated to Jim Reid

1. Introduction. In this note we give examples of how certain properties of torsion-free modules that have been of recent interest arise in algebro-geometric settings. In particular, we find subrings of function fields whose torsion-free modules behave in ways similar to that of torsion-free abelian groups, and we indicate how these module-theoretic properties distinguish the associated rings and their geometry.

Some of our results are expository, the necessary commutative algebra having been developed elsewhere. In Section 2, relying on the work in [**7**], we interpret some Krull-Schmidt properties for modules geometrically. Our focus in this section is the Noetherian case, so it is natural that we find geometric analogues for the algebraic characterizations in [**7**].

In Section 3, we report on some recent results on the existence of finite character Prüfer domains in function fields. Here the "geometry" is more abstract in that our Prüfer rings do not arise as coordinate rings of varieties. Instead, they are intersections of valuation overrings of coordinate rings. The class of Noetherian Prüfer domains is precisely the class of Dedekind domains, so it is not surprising that one must search beyond coordinate rings of varieties to find Prüfer examples in function fields of transcendence degree greater than one.

We also push this construction a bit farther by indicating how h-local Prüfer domains can be found in function fields. An integral domain R is *h-local* if R has finite character and each nonzero prime ideal of R is contained in a unique maximal ideal of R. The classes of finite character and h-local Prüfer domains are playing an increasingly central role in a number of aspects of module theory, as is evidenced by the recent text [**5**].

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Our construction of h-local Prüfer domains leads to issues involving Warfield duality of torsion-free modules, and in Section 4 we explore this topic further. We indicate briefly why for rings of Krull dimension greater than one, the full strength of Warfield duality pushes our construction in Section 3 beyond the geometric setting of function fields. Then we show that, although we have left our original setting, one can still construct Warfield domains with diverse prime spectra.

Notation and terminology. Let R be an integral domain with quotient field F. Then R has *finite character* if each nonzero element $r \in R$ is contained in at most finitely many maximal ideals of R. The domain R is a *Prüfer domain* if every finitely generated ideal of R is invertible; equivalently, R_M is a valuation domain for all maximal ideals M of R. If G is a torsion-free R-module, then $R(G)$ is the *ring of coefficients* of G, that is, $R(G) := \{f \in F : fG \subseteq G\}$. Note that if G has rank one as a torsion-free R-module, then $R(G)$ can be identified with $\mathrm{End}_R(G)$. In Section 2 we will be particularly interested in the class of *torsionless* modules, those R-modules that are isomorphic to a submodule of a finitely generated free R-module.

2. The Krull-Schmidt property. Let R be a domain and $\mathbf{C}$ a class of R-modules. The *Krull-Schmidt* property holds for $\mathbf{C}$ if whenever

$$G_1 \oplus G_2 \oplus \cdots \oplus G_n \cong H_1 \oplus H_2 \oplus \cdots \oplus H_m$$

for G_i, $H_j \in \mathbf{C}$, then $n = m$ and after reindexing, $G_i \cong H_i$ for all $i \leq n$. If, instead of $G_i \cong H_i$, we require only that there exists $k > 0$ such that $G_i^{(k)} \cong H_i^{(k)}$ for all i, then we say the *weak Krull-Schmidt property* holds for $\mathbf{C}$. (We write $G^{(k)}$ for a direct sum of k copies of a module G.)

Although variations of the Krull-Schmidt property have been studied by many authors, we are mainly concerned here with those properties studied in [**7**], and it may be convenient for the reader to have a copy of this article at hand since it is quoted often in this section. We are interested in how the following properties are manifested geometrically.

- **TFKS**: the class of indecomposable torsionless modules has the Krull-Schmidt property

- **weak TFKS**: the class of indecomposable torsionless modules has the weak Krull-Schmidt property
- **UDI**: the class of ideals has the Krull-Schmidt property
- **weak UDI**: the class of ideals has the weak Krull-Schmidt property

The classification of the Noetherian cases of these properties is reviewed in [**7**]. In this section we focus exclusively on the Noetherian case and give geometrical interpretations of these properties.

Two key features of our Krull-Schmidt properties help determine their geometric interpretations. First, domains possessing weak UDI must have a *complemented* maximal ideal, that is, a maximal ideal M such that every ideal not contained in M is invertible [**7**, Lemma 2.4]. Since all our Krull-Schmidt properties imply weak UDI, this phenomenon occurs for all of them. It is quite natural from the geometric standpoint: a noninvertible complemented maximal ideal corresponds to a unique singular point.

Second, domains with UDI have trivial Picard group, while domains with weak UDI must have torsion Picard group [**7**, Lemma 2.4]. Since TFKS domains possess UDI and weak TFKS domains possess weak UDI, one may deduce similar assertions for weak UDI and weak TFKS. Theorem 2.1 shows that UDI and TFKS are very strong geometrically. We denote the affine coordinate ring of a variety X by $A(X)$. Recall that a *rational* curve is a curve of genus 0, and that a nonsingular curve C is rational if and only if $\operatorname{Pic}(C) = 0$.

Theorem 2.1. *Let X be an irreducible variety over an algebraically closed field. Then X is a nonsingular rational curve if and only if $A(X)$ has* UDI, TFKS *or the torsion-free rank one modules of $A(X)$ satisfy the Krull-Schmidt property.*

Proof. Suppose first that $A(X)$ has UDI. Every UDI Noetherian domain is h-local [**6**, Lemma 2.2], so if $A(X)$ has UDI, $A(X)$ is an h-local Hilbert ring. Hence $A(X)$ has Krull dimension one and X is a curve. Let $\tilde{X}$ be the normalization of X. If X is singular, then the kernel of the group homomorphism, $\operatorname{Pic}(X) \to \operatorname{Pic}(\tilde{X})$ is never finitely generated [**22**, Theorem 3.2]. But since $A(X)$ has UDI,

$\operatorname{Pic}(X) = 0$; hence, X is nonsingular. The converse follows from the fact that the affine coordinate ring of an irreducible nonsingular curve with $\operatorname{Pic}(X) = 0$ is a PID. □

It is a classical theorem of Baer that the ring of integers has the Krull-Schmidt property for rank one modules (i.e., torsion-free rank one abelian groups). A consequence of Theorem 2.1 is that if R is a domain that is finitely generated over an algebraically closed field, then the Krull-Schmidt property holds for rank one R-modules if and only if R is a PID.

Examination of the proof of 2.1 shows that the fact that UDI forces the Picard group to be trivial bears sharply on the strength of UDI and related properties in the geometric setting. The search for versions of UDI and TFKS that would "register" geometrically led to the introduction of weak UDI and weak TFKS in [**7**]. Curves with torsion Picard group arise naturally in arithmetic geometry, as is indicated by the examples that close the section.

Let C be an irreducible curve over an algebraically closed field k with singularity at some point P. Let $\nu : \tilde{C} \to C$ denote the normalization of C. Then a *branch* of C at P is an element of $\nu^{-1}(P)$. A branch Q of C at P is *linear* if $M \not\subseteq N^2$ where M is the maximal ideal of the local ring of C at P and N is the maximal ideal of the local ring of $\tilde{C}$ at Q.

Theorem 2.2. *Let X be an irreducible variety over an algebraically closed field. Then $A(X)$ has weak UDI if and only if* $\operatorname{Pic}(X)$ *is torsion and X is a nonsingular curve or X is a curve with unique singularity p such that if $\nu : \tilde{X} \to X$ is the normalization of X, then p satisfies one of the following conditions*:

(i) *p is unibranched*,

(ii) *p has two branches, one of which is linear, or*

(iii) *p has three branches, all of which are linear.*

Proof. Suppose $A := A(X)$ has weak UDI. As in Theorem 2.1, X is a curve. If X is singular, then by [**7**, Theorem 3.4], it has at most one singularity, say p. Thus if M is the maximal ideal of A corresponding to p, A_M has UDI [**7**, Theorem 3.4]. Now apply Theorem 4.1 of [**7**] to A_M to obtain the geometrical analogues of (i)–(iii) of the referenced theorem. Conversely, suppose X is an irreducible curve, $\mathrm{Pic}\,(X)$ is torsion and p satisfies (i), (ii) or (iii). To show A has weak UDI, it suffices to show A_p has UDI [**7**, Theorem 3.4]. We do this by direct appeal to Theorem 4.1 of [**7**]. Since the ground field k of A is algebraically closed, the residue fields of the normalization of X remain the same as the residue fields of X, namely, they are all isomorphic to k. Thus, since the integral closure of A is module-finite, A_p, and hence A, have UDI. □

Theorem 2.3. *Let X be an irreducible variety over an algebraically closed field. Then $A(X)$ has weak* TFKS *if and only if* $\mathrm{Pic}\,(X)$ *is torsion, and X is a nonsingular curve or X is a curve with unique singularity p such that if $\nu : \tilde{X} \to X$ is the normalization of X, then p is unibranched or p has two branches, both of which are linear.*

Proof. The proof is similar to that of Theorem 2.2, only we appeal to [**7**, Corollary 4.3] instead of [**7**, Theorem 4.1]. □

Remark. If K is the algebraic closure of a finite field and D is a one-dimensional domain that is finitely generated as a K-algebra, then $\mathrm{Pic}\,(D)$ is torsion [**21**, Lemma 2]. Thus it follows that, over such a field K, the requirement in Theorem 2.2 and Theorem 2.3 that $\mathrm{Pic}\,(X)$ be torsion is superfluous.

In light of 2.2, 2.3 and the remark, it is not difficult to give examples of curves with unique singularities whose coordinate rings are examples of weak UDI and weak TFKS. Let K be the algebraic closure of a finite field of characteristic $p > 5$. Then the following list shows each type of singularity can occur on a (rational) curve over K.

- The coordinate ring of the cusp $C : y^3 - x^2 = 0$ has weak TFKS, since the singularity $(0, 0)$ is unibranched.
- The coordinate ring of the node $C : y^2 - x^2 - x^3 = 0$ has weak TFKS, since the singularity $(0, 0)$ has two branches, both of which are linear.
- The coordinate ring of the curve $C : y^3 - x^2y - x^4 = 0$ has weak

UDI but not weak TFKS, since the singularity $(0,0)$ has three branches, all of which are linear.

3. Warfield's category equivalence. R.B. Warfield, Jr., in his 1968 paper on duality, introduced a category equivalence for torsion-free abelian groups that has recently been shown to hold in much more general settings. In this section we review the formulation of this category equivalence for torsion-free modules over domains and summarize its ring-theoretic characterization. Using this characterization, we associate these domains to a geometric notion.

Let R be an integral domain with quotient field F. If X is a rank one module and G is a torsion-free R module, then G *dominates* X if the canonical homomorphism

$$\mathrm{Hom}_R(X,G)\otimes_R F \longrightarrow \mathrm{Hom}_R(X, G\otimes_R F)$$

is an isomorphism. In case G has finite rank, G dominates X if and only if $\mathrm{Type}\, X \leq \mathrm{IT}(G)$. See [**18**] for a treatment of types in the setting of integral domains.

Let X be a torsion-free rank one R-module. Define $\mathcal{E}_X$ to be the category of torsion-free $R(X)$-modules, and let $\mathcal{D}_X$ be the category of torsion-free $R(X)$-modules that dominate X. Define a pair of functors

$$H_X : \mathcal{D}_X \longrightarrow \mathcal{E}_X \quad \text{and} \quad T_X : \mathcal{E}_X \longrightarrow \mathcal{D}_X$$

by $H_X(G) = \mathrm{Hom}_{R(X)}(X,G)$ for all $G \in \mathcal{D}_X$ and $T_X(H) = X \otimes_{R(X)} H$ for all $H \in \mathcal{E}_X$.

In [**13**], the following terminology was provisionally introduced. We recall it here in order to give a module-theoretic background for the notion of "stability."

(HT): For all torsion-free rank one modules X, $H_X \circ T_X \cong 1_{\mathcal{E}_X}$.

(TH): For all torsion-free rank one modules X, $T_X \circ H_X \cong 1_{\mathcal{D}_X}$.

In [**13**, Theorem 2.3], it is shown that an integral domain R satisfies (TH) if and only if R is *stable*, that is, every nonzero ideal of R is projective over its ring of endomorphisms. The desire for a classification of rings satisfying (TH) (as well as Warfield duality) has motivated much of the recent work on the structure and classification of stable domains. Moreover, (TH) implies (HT), so the integral domains that support Warfield's category equivalence are precisely the stable domains [**13**, Corollary 2.4]. (For Noetherian domains, (HT) implies (TH), but this is not always true for non-Noetherian domains [**13**, Proposition 4.6]. An ideal-theoretic characterization of (HT) domains is given in [**13**, Section 4].

The stability property has an interesting history dating back at least to a 1963 paper of Bass [**2**]. For surveys of the recent work on stability and the uses to which stability has been put, see [**12**] and [**13**]. Noetherian stable domains are well-understood, and it is known that geometrically, stable Noetherian rings arise as the coordinate rings of irreducible curves whose singularities are double points [**12**].

From a structural point of view, the Prüfer case is also well-understood: *An integrally closed domain R is a stable domain if and only if R is a strongly discrete Prüfer domain of finite character* [**14**, Theorem 4.6], but see [**11**, Theorem 3.3] for a more general result. A Prüfer domain R is *strongly discrete* if $P \neq P^2$ for all nonzero prime ideals P of R. Valuation domains having free value groups that are ordered anti-lexicographically are strongly discrete [**5**]. Valuation domains with such value groups arise naturally in the study of function fields, and we exploit this observation in the construction of stable Prüfer domains in function fields.

Rather comprehensive existence results for stable Prüfer domains already appear in the literature [**14**, Proposition 5.4]. These results, however, rely ultimately on the Kaplansky-Jaffard-Ohm construction [**5**, Theorem III.5.3], so that the stable domains there constructed are certain overrings of the group ring $K[G]$, where K is a field and G is a partially ordered abelian group. If a stable Prüfer domain R having infinitely many maximal ideals is constructed in such a manner, then the construction forces $K[G]$ to have infinite transcendence degree over K.

By contrast, the function fields that arise in classical algebraic geometry have finite transcendence degree over K. Thus the finite character Prüfer domains arising in the Kaplansky-Jaffard-Ohm construction are often outside the scope of the classical geometric setting.

One of the intents of the article [**17**] is to remedy this problem by finding examples of stable, hence finite, character, Prüfer domains in function fields. We first state a realization theorem for stable Prüfer domains, then we outline the construction of such rings.

If $(X, \leq)$ is a partially ordered set, then X is a *tree* if for all $x, y, z \in X$ with $x \leq z$ and $y \leq z$, it is the case that $x \leq y$ or $y \leq x$. The *dimension* of X is n if there is a chain of elements $x_0 < x_1 < \cdots < x_n$ in X but no longer chain. If X has a least element x_0, then X has *finite character* if, given any $x \in X$ such that $x \neq x_0$, there are only finitely many maximal elements y of X such that $x \leq y$.

Theorem 3.1 [**17**]. *Let D be a domain of Krull dimension n that is a finitely generated K-algebra, where K is a field or $K = \mathbf{Z}$, and let $(X, \leq)$ be a countable partially ordered set. The following are equivalent for X.*

(1) *There exists a stable Prüfer overring H of D such that* Spec(H) *is order isomorphic to X.*

(2) *X has a least element and X is a finite character tree of dimension $\leq n$.*

Note that Theorem 3.1 also provides a source of h-local strongly discrete Prüfer domains. This class of domains admits several nice descriptions. In [**15**], a commutative ring R is defined to be a *Zerlegung in Prim- und Umkehrbaridealen (ZPUI)* ring if every ideal I of R that contains a nonzero divisor can be written in the form $I = UP_1^{e_1} \cdots P_n^{e_n}$, where U is an invertible ideal and the P_i are prime ideals of R. A commutative ring R is a ZPUI ring if and only if R is a finite direct product of ZPUI integral domains and special primary rings [**15**, Theorem 3.3]. Thus the study of ZPUI rings reduces to the domain case, and it is shown in [**15**, Theorem 2.3] that a domain R is a ZPUI domain if and only if R is an h-local strongly discrete Prüfer domain. Hence, by Theorem 3.1, one may find nontrivial examples of ZPUI domains H in function fields.

There is another interpretation of h-local Prüfer domains that is interesting in the present context and anticipates issues treated in the next section. Bazzoni and Salce show in [**3**] (see [**10**, Theorem 3.1] for further details) that a domain R is an h-local strongly discrete Prüfer domain if and only if R is an integrally closed domain such that, for all rank one R-modules X and Y with $X \subseteq Y$ and $R(Y) \subseteq R(X)$, the canonical homomorphism

$$X \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(X, Y), Y)$$

is an isomorphism. Thus, in the terminology of the next section, there exist nontrivial examples in function fields of Prüfer domains for which "Warfield duality" holds for rank one modules.

We now outline the construction of stable Prüfer domains in Theorem 3.1. Let K be an algebraically closed field and V an irreducible projective variety over K. We say an infinite set of points $\{p_\alpha\}$ on V *determines* V if, whenever W is a subvariety of V that contains infinitely many of the p_α, then $W = V$.

Generalizing this notion to schemes, we say an infinite set of closed points $\{p_\alpha\}$ of a projective scheme X over a domain D determines X if the only closed subscheme of X that contains infinitely many of the $\{p_\alpha\}$ is X. We say the set $\{p_\alpha\}$ is a *determinative set* of closed points of X.

For affine integral schemes, we can rephrase the definition: An infinite set of maximal ideals $\{M_\alpha\}$ of an integral domain D determines Spec$\,(D)$ if, whenever P is a prime ideal of D such that P is contained in infinitely many of the maximal ideals M_α, then $P = 0$. We say $\{M_\alpha\}$ is a determinative set of maximal ideals of D.

To relate determinative sets of points to sets of valuations, we adopt a more sophisticated way to view points on varieties or schemes, namely as centers of valuations. To simplify the discussion, we explain this switch to valuations for affine integral K-schemes X, where K is a field or $K = \mathbf{Z}$. In case K is an algebraically closed field, one may simply treat X as an irreducible K-variety. Whether or not we make this restriction on K, we may view X as Spec$\,(D)$ for some finitely generated K-algebra D such that D is a domain. If V is a valuation overring of D with maximal ideal M, then the *center* of V on D is the prime ideal $M \cap D$ of D. Thus, if Δ is a finite character set of infinitely many valuation overrings of D (meaning every nonzero $d \in D$ is a unit in all but finitely many of the elements $V \in \Delta$), and each valuation overring $V \in \Delta$ is centered on a maximal ideal of D, then it follows that the set of all such centers forms a determinative set of maximal ideals of D.

Moreover, if the corresponding valuations are chosen to be zero-dimensional, then each valuation remains centered on a closed point of a projective variety Y over D, and the set of centers of the valuations on Y determines Y. In [**17**] it is shown that for any projective D-scheme, where D is a finitely generated algebra over a field or $\mathbf{Z}$, one can always find such an infinite set of zero-dimensional valuations whose centers on every projective scheme Y over D determines Y, and such that the intersection of these valuation overrings is a finite character Prüfer domain. (Stronger versions of this statement can be found in [**17**].)

Motivating example. Let I be a compact subset of $\mathbf{C}$. Let $\Gamma = \{(x, f(x)) : x \in I\}$ be the graph of $y = f(x)$ over I for some transcendental function f, e.g., $f(x) = e^x$. Then for each $p \in \Gamma$, there exists a rank two valuation V_p centered on p. For example, for each $p := (t, f(t)) \in \Gamma$, one may view elements of $\mathbf{C}[x, y]$ as elements of $\mathbf{C}[x - t, y - f(t)]$. Then a valuation ring V_p can be defined as the ring corresponding to the valuation that maps an element $g = \sum_{i,j} c_{ij}(x - t)^i(y - f(t))^j$ to $\min\{(i, j) : c_{ij} \neq 0\} \in \mathbf{Z} \oplus \mathbf{Z}$, where $\mathbf{Z} \oplus \mathbf{Z}$ is ordered lexicographically. Moreover, no matter how the V_p are defined, if they are of rank two, then the set of centers of the V_p's forms a determinative set of maximal ideals of $\mathbf{C}[x, y]$. This is because no algebraic curve in $\mathbf{C}^2$ can intersect a compact subset of the graph of f infinitely many times. (Details can be found in [**17**].) The corresponding holomorphy ring $H := \cap_{p \in \Gamma} V_p$ is a stable Prüfer domain [**17**]. Consequently, H supports Warfield's category equivalence for torsion-free modules.

4. Warfield duality. Let R be an integral domain. Let G be a torsion-free finite rank R-module of rank n and Y be a rank one R-module. It was observed by Reid in [**18**] that the canonical map

$$G \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(G, Y), Y)$$

is an isomorphism only if (i) G is an $R(Y)$-module and (ii) $G \hookrightarrow Y \oplus Y \oplus \cdots \oplus Y$, n copies. An integral domain R is a *Warfield domain* if conditions (i) and (ii) are not only necessary but *sufficient* to guarantee the canonical map is an isomorphism for every torsion-free finite rank R-module G and rank one R-module Y.

Every Dedekind domain is a Warfield domain, but the converse fails roundly, since there exist Warfield domains that are not one-dimensional, Noetherian or integrally closed (see [**5**, Example XV.9.6], for example). Recent work of several authors has led to the classification of this class of domains; see [**5**, Chapter 15], [**10**] and [**19**] for different perspectives on this topic. In the present section we show how interesting examples of Warfield domains arise as overrings of complete regular local rings.

A Noetherian domain D is a Warfield domain precisely when every ideal of D can be generated by two elements [**5**, Theorem XV.9.3]. The class of Noetherian domains with 2-generated ideals has been well-studied and the geometric analogue of this property is known to be that singularities on irreducible curves are double points [**8**]. It is interesting to note that in case a Noetherian domain R has module-finite integral closure, then R is a stable domain if and only if R is a Warfield domain [**20**].

An integrally closed domain R is a Warfield domain if and only if R is a Prüfer domain that is strongly discrete and *almost maximal*, meaning R/I is a linearly compact R-module for all nonzero ideals I of R [**5**, Theorem XV.9.5]. This latter condition is absent from the version of Warfield duality for rank one modules mentioned in the last section, and it arises in this section from the introduction of modules of rank greater than one. We shall see that the property of almost maximality pushes us into an analytic setting, namely, our examples of Warfield duality are found in fields such as $K((x_1, \dots, x_n))$.

One reason for this is that if K is a field and L is a field of finite transcendence degree greater than one over K, then if W is a discrete valuation domain of Krull dimension greater than one, $K \subseteq W$ and W has quotient field L, then W cannot be almost maximal. Indeed, if W is almost maximal, then there exists a discrete valuation overring V of W of Krull dimension 2 that is almost maximal. If Q is the unique nonzero nonmaximal prime ideal of V, then V/Q is a maximal, hence complete, discrete valuation domain. By the Cohen structure theorem for complete rings, $V/Q \cong K[[w_1, \dots, w_m]]/I$ for some indeterminates $w_1, \dots, w_m$ and ideal I. The quotient field of V/Q thus has infinite transcendence degree over K. However, the quotient field of V/Q is isomorphic to V_Q/Q so it is a residue field of a valuation on L. As such, it has finite transcendence degree over K. (See [**1**, Theorem 1] for example.) This contradiction shows that V, hence W, cannot be almost maximal.

Warfield domains of Krull dimension one are Noetherian [**5**, Exercise XV.8.4], so we are interested here in examples of Warfield Prüfer domains of Krull dimension greater than one (since those of Krull dimension one are simply Dedekind domains). Theorem 4.2 shows Warfield Prüfer overrings can be found in a setting of central geometric importance.

Lemma 4.1. *Let D be a complete regular local ring of Krull dimension $n > 1$ and having residue field of cardinality α. Then, there are at least α many distinct Warfield valuation overrings of D of Krull dimension n.*

Proof. Let M be the maximal ideal of D. Since D is a regular local ring, M can be generated by n elements, say $x_1, x_2, \dots, x_n \in M$. For each $u \in D \setminus M$, note that since $M = (x_1 - ux_2, x_2, \dots, x_n)$, the sequence $(x_1 - ux_2, x_2, \dots, x_n)$ is a system of parameters of D; hence it is a regular sequence. Fix $u \in D \setminus M$ and define $P_1 := (x_1 - ux_2)$. For each i with $2 \le i \le n$, set $P_i := (x_1, x_2, \dots, x_i)$. Let i be such that $1 \le i < n$. Observe that the quotient field of $D_{P_{i+1}}/P_i D_{P_{i+1}}$ is isomorphic to $D_{P_i}/D_{P_i}P_i$. Now D/P_i is again a regular local ring since it has a regular system of parameters, so it is an integrally closed local domain. Thus $D_{P_{i+1}}/P_i D_{P_{i+1}}$ is a complete DVR. We now use induction to construct a valuation domain V with prime ideals $Q_1 \subset Q_2 \subset \cdots \subset Q_n$ such that $Q_i \cap D = P_i$ for all $i \le n$. Let V_1 be the complete DVR D_{P_1} and set $Q_1 = D_{P_1}P_1$. Let $k < n$ and suppose there exists a valuation domain V_k such that $D \subseteq V_k \subseteq V_1$ and V_k has prime ideals $Q_1 \subset \cdots \subset Q_k$ such that $(V_k)_{Q_{j+1}}/Q_j$ is a complete DVR for all j with $1 \le j < k$. Let V_{k+1} be defined as the pullback in the following commutative diagram:

$$\begin{array}{ccc} V_{k+1} & \longrightarrow & (D/P_k)_{(P_{k+1}/P_k)} \\ \big\downarrow & & \big\downarrow \\ V_k & \longrightarrow & V_k/Q_k \end{array}$$

The following argument justifies the existence of the right vertical mapping. Observe that, by the construction, V_k/Q_k is the residue field of $(D/P_{k-1})_{(P_k/P_{k-1})}$ so $V_k/Q_k \cong D_{P_k}/P_k D_{P_k}$. Hence V_k/Q_k

is isomorphic to the quotient field of $(D/P_k)_{(P_{k+1}/P_k)}$ and the right vertical mapping can be defined in the natural way. It follows that V_{k+1} is a Warfield valuation domain since it is a pullback of the complete DVR $(D/P_k)_{(P_{k+1}/P_k)}$ and the Warfield valuation domain V_k [**10**, Lemma 4.5]. If $V = V_n$, then V is the desired Warfield valuation domain. It is easy to see that each $u \in D \setminus M$ yields a different prime ideal $(x_1 - ux_2)$ of D so each u yields a different Warfield valuation domain V_u, and the claim follows. □

Theorem 4.2. *Let D be a complete regular local ring of Krull dimension n. If $k < \min\{\aleph_0, \alpha\}$ where α is the cardinality of the residue field of D, then D has an n-dimensional Warfield Prüfer overring with k many maximal ideals.*

Proof. In [**16**, Corollary 4.5], it is shown that the intersection of finitely many Warfield valuation domains sharing the same quotient field is a Warfield Prüfer domain. That such an intersection has Krull dimension n is a consequence of Nagata's theorem on the intersection of finitely many valuation domains [**5**, Theorem III.1.7]. □

Since Warfield domains are h-local, the argument in the proof of Theorem 4.2 shows any two distinct valuation overrings V and W constructed as in Lemma 4.1 are independent in the sense that they share no common prime ideals. This is somewhat striking because the prime ideals of V and W contract to the same prime ideals of the base ring D with the singular exception of the prime ideal P_1 in the proof of the lemma.

If D is a complete regular local ring with infinite residue field, then by Lemma 4.1 there are infinitely many Warfield valuation overrings of D having the same Krull dimension as D. In light of Theorem 4.2, the question arises as to whether an intersection of infinitely many of these Warfield valuation domains is a Warfield Prüfer domain. This is never the case, however, since each of these valuation domains is centered on the unique maximal ideal of D and a Warfield domain must have finite character [**5**, Corollary IV.5.5].

Thus, while yielding an interesting source of Warfield Prüfer domains, Theorem 4.2 does not provide examples of Warfield Prüfer domains that have Krull dimension greater than one and infinitely many maximal ideals. To my knowledge, examples of Warfield domains of dimension greater than one but *having infinitely many maximal ideals* have not been previously constructed. More generally, the literature does not seem to contain an example of an almost maximal Prüfer domain of Krull dimension greater than one but having infinitely many maximal ideals. See [**10**, pp. 35–36] for a discussion of this problem.

As a first example of such a phenomenon, we construct, as an overring of the ring of Puiseaux series, a two-dimensional Warfield Prüfer domain having infinitely many maximal ideals.

Example 4.3. Let K be a formally real field and x and y be indeterminates. Then there is a Warfield Prüfer overring of $K[[x]][y]$ of Krull dimension 2 with infinitely many maximal ideals.

Proof. Let $V = K[[x]]$ and $F = K((x))$, and let W be the valuation domain $V + yF[y]_{(y)}$. Then W is the pullback of the complete DVR V and the DVR $F[y]_{(y)}$, so W is the Warfield valuation domain [**10**, Lemma 4.5]. For each nonzero $f \in F$, define W_f to be the DVR, $F[y]_{(y-f)}$. Set $R := W \cap (\cap_{f \in F} W_f)$ and note that, since W and W_f each has formally real residue field, R is a Prüfer domain [**4**, Theorem 2.1.4]. Now the maximal ideal of each W_f lies over the prime ideal $(y - f)$ of $K[[x]][y]$ and the maximal ideal of W lies over the maximal ideal (x, y) of $K[[x]][y]$. Since every nonzero element of $F[y]$ is contained in at most finitely many of the prime ideals $(y - f)$, $f \in F$, it follows that the collection $\{(x, y)\} \cup \{(y - f) : 0 \neq f \in F\}$ of prime ideals of $K[[x][y]$ has finite character in the sense that every element of $K[[x]][y]$ is contained in at most finitely many of these prime ideals. Thus the collection $\{W_f : f \in F\} \cup \{W\}$ has finite character and, by [**17**], R is a finite character Prüfer domain and each localization of R at a maximal ideal is either W or W_f for some $f \in F$. Thus R is a finite character Prüfer domain that is locally a Warfield domain; hence R is a Warfield Prüfer domain [**10**, Lemma 4.2]. □

Examination of the justification of Example 4.3 shows that, while the Warfield Prüfer domain constructed in the proof has infinitely many maximal ideals, only one of these maximal ideals has height greater than one. This leaves the problem of constructing Warfield Prüfer domains with more diverse prime spectra, and to accomplish this we leave our geometric setting and work over large power series rings.

If D is a domain and X is a collection of indeterminates for F, then we adopt the following definition (from several possible) for a ring of power series over D:

$$D[[X]] := \left\{ \sum_{i=1}^{\infty} d_i x_1^{e_{1i}} x_2^{e_{2i}} \cdots x_n^{e_{ni}} : d_i \in D \text{ and } \{x_1, x_2, \dots, x_n\} \subseteq X \right\}.$$

Theorem 4.4. *Let α be a cardinal number and $\{n_\beta\}_{\beta<\alpha}$ be a sequence of natural numbers. Then there exists a Warfield Prüfer domain R with maximal ideals $\{M_\beta : \beta < \alpha\}$ such that for each β, M_β has height n_β.*

Proof. Let K be a formally real field and, for each $\beta < \alpha$, let $X_\beta := \{x_{\beta,1}, x_{\beta,2}, \dots, x_{\beta,n_\beta}\}$ be a set of indeterminates for K. Define $D := K[[X_\beta : \beta < \alpha]]$ and

$$D_\beta := K((X_\gamma : \gamma < \alpha \text{ and } \gamma \neq \beta))[[X_\beta]].$$

Then D_β is a complete regular local ring of Krull dimension n_β so, by Lemma 4.1, there exiats a Warfield valuation overring V_β of D_β of Krull dimension n_β. Define $R := \cap_{\beta<\alpha} V_\beta$. Then R is a Prüfer domain since each V_β has formally real residue field K [**4**, Theorem 2.1.4]. Moreover, $\{V_\beta : \beta < \alpha\}$ is a finite character collection of valuation overrings of D. This is because each element of D is contained in at most finitely many of the prime ideals P_β of D where P_β is generated by the set X_β, and each V_β is centered on P_β. Thus R is a finite character Prüfer domain and $\{V_\beta\}$ is precisely the set of localizations of R at maximal ideals [**17**]. It follows that R is locally a Warfield domain. Since R has finite character, R is a Warfield domain [**10**, Lemma 4.2], proving the claim. □

Motivated by Theorem 4.4, we pose the following question.

Suppose $(X, \leq)$ is a partially ordered set with least element x_0 such that X satisfies the ascending chain condition and, for all $x \in X$ with

$x \neq x_0$, the set $\{y \in X : y \geq x\}$ is linearly ordered. Does there exist a Warfield Prüfer domain R such that Spec (R) *is order isomorphic to X?*

Compare [**14**, Proposition 5.5] and note that by the referenced result the converse assertion is true; namely, the prime spectrum of a Warfield domain satisfies the properties in the supposition.

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