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THE DEVELOPMENT OF THE THEORY OF p-GROUPS

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1. Introduction. As usual, all groups considered here are abelian. The request initially presented to me by the kind invitation of the conference organizers was to give an account of the development, especially in the United States, of the theory of torsion and mixed groups. My interpretation of this was that I should include everything except torsion-free groups. A rough draft of the first part of this project dealing with torsion groups convinced me that it would be an enormous challenge, for even a more talented writer, to accomplish this within the pages allocated without diluting both parts beyond tasteful limits. Therefore, I decided to restrict this paper to torsion groups and consequently to p-groups, with the hope that the mixed case could be reserved for a later time.

In a further attempt to rein in the scope to manageable limits, I have essentially restricted the account given here of the development of p-groups to the period beginning with the appearance of László Fuchs' famous book [**9**], which was a catalyst for the tremendous advancement in abelian groups that followed. Although this survey begins after the time that Reinhold Baer was a U.S. resident, I would be remiss if I did not mention Baer's contributions to the subject including his academic prodigies (of which I am one of a large group; my thesis advisor was Baer's student at the University of Illinois). Likewise, I am compelled to mention at the outset the tremendous influence that the approach used by Leon Zippin [**64**] in his proof of Ulm's theorem ultimately had on my own techniques. Indeed, virtually all who have joined the quest for the classification of p-groups have to some extent followed Zippin's approach.

The organization of what is to follow is intended to be basically chronological. However, the formal organization is by topic, so the chronology often overlaps and occasionally may be completely reversed. Unless otherwise stated, all groups henceforth are p-groups, although

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we repeat this hypothesis from time to time for emphasis. It is my hope that the references are adequate in number, but they are not intended to be exhaustive.

2. Subsocles and pure subgroups. In or around 1960, some attention was being given to the notion of pure subgroups having prescribed socles. The significance of purity had already been recognized for a long time, and it was becoming clear that socles and subsocles were going to play an increasing role in the theory of p-groups. In this connection, there were two basic questions open at the time.

Uniqueness. If two pure subgroups H and K are supported by the same subsocle S of the p-group G, must H and K be isomorphic?

Existence. When does a given subsocle S of G support a pure subgroup?

A casual acquaintance with the preceding problems soon introduces one to the notion of a pure-complete group, where *each* subsocle supports a pure subgroup.

It was Charles Megibben, as I recall, who first brought to my attention the fact that if $A \oplus B$ is pure complete, then A and B are isomorphic if (and only if) there is a height-preserving isomorphism between their socles. This provided additional motivation for studying what already seemed to be a fundamental issue: pure subgroups with prescribed socles.

The above uniqueness question was clearly related to another question open at the time that had been posed by John Irwin and Elbert Walker [**45**]: are all high subgroups of a given group necessarily isomorphic? Incidentally, my recollection is that I first met Elbert Walker at a meeting of the American Mathematical Society in Stillwater, Oklahoma, in the summer of 1961. He was kind enough to invite me subsequently to what would become known as the First New Mexico State Conference, held in June of 1962. I came to the conference empty handed and did not give a talk, but I was able to solve the preceding uniqueness question, as well as the companion high-subgroup problem, in time to be included in the proceedings of the conference [**15**].

A few years later Megibben and I would prove [**38**] that it is not unusual for a group of cardinality $c = 2^{\aleph_0}$ to have fixed subsocles supporting as many as 2^c isomorphically distinct subgroups!

While the uniqueness question was quickly and emphatically settled, the existence question, stated in much more general terms, was a different matter. Megibben and I also considered this question in the early to mid 1960's. A noteworthy result in [**36**] is that a dense subsocle always supports a pure subgroup. In fact, it was shown that if S is a dense subsocle of a p-group G, any subgroup H of G which is maximal with respect to $H[p] = S$ must be pure. This generalized the result that N-high subgroups are pure whenever $N \subseteq p^\omega G$ [**44**], [**45**].

It was shown in [**37**] that the existence question is related to a closure property in the p-adic topology. Calling a group G (without elements of infinite height) *quasi-closed* if the closure of each of its pure subgroups is again pure, Megibben and I proved in [**37**] that a quasi-closed group is pure complete. Naturally, a closed (= torsion complete) group is quasi-closed. We went on in [**37**] to find a useful characterization of quasi-closed groups, and in so doing we established the fact that the pure-complete groups form a fairly large class but one which is not so well behaved; for example, they are not closed with respect to direct sums.

It is interesting to note that a corresponding study of isotype subgroups with prescribed socles was not investigated until many years later [**41**].

3. Closed groups and their direct sums. In retrospect, with the advantage of hindsight, I would say that one of the most misguided efforts of the 60's in p-group research was that of attempting to classify the pure subgroups between B and $\overline{B}$, that is, between a direct sum of cyclic groups B and its torsion completion $\overline{B}$. At least I can speak for myself. The closed group $\overline{B}$ has a number of interesting properties most of which are not inherited by pure subgroups. Not the least of these is the property of being pure complete.

Since closed groups are pure complete, Meggiben's observation mentioned in the previous section implies that closed groups are determined by their socles in the sense that two closed groups are isomorphic if and only if there is a height-preserving isomorphism between their socles. By exactly the same reason, a direct sum of cyclic groups is determined by its socle. Since a direct sum of cyclic groups is a prototype for a direct sum of closed groups, the question naturally arose as to whether or not a direct sum of closed groups is determined by its socle. More precisely, if A and B are both direct sums of closed groups, must they be isomorphic if there is a height-preserving isomorphism between their socles? An affirmative answer was provided in [**17**]. We mention that the proof used in the preceding special cases was not available since it was not known at the time that a direct sum of closed groups is pure complete. The latter fact first appeared in [**38**].

To avoid possible confusion, we should mention that Doyle Cutler [**3**] has used the terminology of a group being *determined by its socle* to mean something a little stronger than our usage here; namely, he allows for the groups to range over all p-groups rather than a designated subclass.

At this point in the development of abelian p-groups, say by the mid 1960's, it was becoming clear that if a given class of p-groups has a propensity for subsocles to support pure subgroups, then the structure theory for that class of groups tends to be greatly enhanced. As for direct sums of closed groups, the structure theory was further advanced when it was shown in [**19**] that a direct summand of a direct sum of closed groups is again a direct sum of closed groups. This result, together with a theorem of Edgar Enochs [**8**] provided the isomorphic refinement theorem for direct sums of closed groups. This capped a long list of partial results.

4. The advancement of homological methods. Not only were great strides being made in the 1960's in the structural theory of abelian groups and especially p-groups, but the homological theory was rapidly advancing as well. And the two arms of abelian group theory were beginning to interlock firmly adding to the strength of the whole. Among those who initially had a strong influence on the applications of homological methods to abelian groups were David Harrison [**14**], Saunders MacLane [**51**] and Ronald Nunke [**57**].

The functors Hom, Ext, $\otimes$ and Tor quickly became commonplace in abelian group papers, both as tools for the structure theory and as objects of study in their own right. Reflecting the significance of purity, the functor Pext became especially useful to p-group theory. Using the concepts of large subgroups and small homomorphisms, Richard Pierce [**59**] introduced another important functor, Shom (= small homomorphisms). Some applications of small homomorphisms and large subgroups are given in [**59**] and [**52**]. In regard to p-group development, Nunke's paper *Homology and direct sums of countable abelian groups* [**57**] certainly deserves a special commendation.

The programs of the New Mexico State Conference in 1962 and the Tihany Conference in 1963 seem to reflect that there was at that time fully as much interest in the homological aspects of abelian groups as there was in the structure theory, but by the time of the Montpellier conference in 1967, structure theory appeared to have regained its dominant role.

Even in this brief account, a few additional words about Tor are in order since it is basically a functor for p-groups and has been an important part of their development. In fact, Tor was a fundamental ingredient of the paper by Nunke that we have already acclaimed, and he and others used it as the main tool or the object of study in several other papers. There are interesting open questions that still remain concerning Tor. For example, when is $\operatorname{Tor}(A, B)$ an axiom 3 group? We refer to a very good survey paper by Patrick Keef [**49**] for a fairly current account of this and other open questions concerning Tor as well as for additional references.

5. Direct sums of countable groups and their subgroups. As we know, it took a quarter of a century to pass from the classification of countable p-groups [**62**], [**64**] to the same result for their direct sums due to George Kolettis [**50**]. The part of the story in between that is usually left out but which contributed significantly to this generalization was the introduction of the invariants $\dim(p^\alpha G[p]/p^{\alpha+1}G[p])$ by Irving Kaplansky and George Mackey ([**47**], [**46**]). Without (full) justification, these numbers have been referred to by me and many others as the Ulm invariants. They are more accurately called the Ulm-Kaplansky invariants by Fuchs in [**10**], so we shall (in spirit) follow Fuchs' example and refer to them here simply as the U-K invariants. It needs to be emphasized that the U-K invariants are assumed to include $\dim(p^\infty G[p])$ and therefore they also account for the divisible part of the group if it is not reduced.

A much shorter proof of Kolettis's uniqueness theorem was given in

[**18**] and then amazingly enough in 1969, Fred Richman and Elbert Walker observed in [**60**] that no group theory at all is needed for the generalization from countable groups to d.s.c.'s. There was a common feature of all three proofs (including the original) that facilitated, in progressive degree, going forward from countable groups to their direct sums but which in a way may have actually hindered progress toward further generalizations. For the latter purpose, it seems that all (including my own) strayed too far from the fundamentals of constructing isomorphisms by extending height-preserving isomorphisms on subgroups. This deficiency, if we can call it that, was first manifested only by the failure to achieve Zippin's result about extending automorphisms of $p^\alpha G$ to G. We shall hereinafter refer to this result as Zippin's theorem. Zippin's theorem for d.s.c.'s was established by Hill and Megibben in [**39**] along with some uniqueness theorems for certain generalizations of d.s.c.'s but, here too, there were limitations as to how far we could go without returning to the fundamentals which, of course, the present author did a little later when he introduced the third axiom of countability.

What about subgroups of d.s.c.'s? It may be hard now for some to imagine that it was unknown at the time when Kolettis classified d.s.c.'s whether or not a subgroup of a d.s.c. must again be a d.s.c.. Using Tor, Nunke [**56**] gave the first counterexample at the Tihany Conference in 1963. Counterexamples now abound to the point that one could almost say that it is the rule, not the exception, for subgroups of d.s.c.'s to lose this inheritance. Indeed, it was shown in [**22**] that the direct sum of uncountably many copies of the Prüfer group contains a subgroup without elements of infinite height that is not pure complete, so it cannot be a direct sum of countable groups. The same question, however, for isotype subgroups cannot be dismissed quite so easily. Using very ad hoc and laborious procedures, I first proved in [**23**] that the answer is in the affirmative for isotype subgroups of countable length. An example was given that showed that the condition on the length is necessary.

6. Axiom 3. In this section when we refer to the uniqueness theorem we always mean the theorem which states that two p-groups belonging to a given class are isomorphic whenever they have the same U-K invariants (frequently called Ulm invariants). Once Kolettis had classified d.s.c.'s, there was an ardent effort to push this result beyond d.s.c.'s to a larger class of groups. When Nunke introduced totally projective groups and characterized d.s.c.'s as being the totally projectives of length not exceeding Ω, the class of totally projective groups naturally became the target class for the uniqueness theorem. Let us hasten to add, however, that Peter Crawley and Alfred Hales set upon a different course. Their target was the class of simply presented groups [**1**], [**2**]. In essence, their target was the same, but their course was more scenic. Simple presentations are considered by many to be the most aesthetic approach.

In a paper [**58**] presented to the Montpellier conference in 1967, Larry Parker and Elbert Walker did a masterful job of applying all the latest known results, along with their own innovative ideas, to prove the uniqueness theorem for totally projective groups of length less than $\Omega\omega$. In fact, they proved Zippin's theorem for these groups.

Immediately after the Montpellier conference I decided to go back to the beginning, so to speak, or to the fundamentals as I called it in the previous section. I invented Axiom 3 groups (then called the third axiom of countability) for the express purpose of proving the uniqueness theorem by means of extending height-preserving isomorphisms on subgroups [**20**]. However, I certainly was mindful that these Axiom 3 groups would constitute a well-behaved class closed with respect to direct sums, elementary elongations and summands (once I forced the issue on summands by adjusting the definition of Axiom 3 from what today is called Griffith's version [**13**] to the final version that appeared in [**20**]). Thus, it was almost a given that *if* Axiom 3 groups could be classified, then they would turn out in the end to be precisely the totally projective groups. And so it was.

An unexpected bonus was the straightforward proof [**26**] that simply presented p-groups satisfy Axiom 3.

7. Transitivity and other dividends from Axiom 3. There were a number of dividends that came with the Axiom 3 approach. The uniqueness theorem for Axiom 3 groups was established in the following strong form.

Suppose that G and G' are p-groups and that H and H', respectively, are nice subgroups for which G and G' have the same relative invariants.

Further, suppose that G/H and G'/H' satisfy Axiom 3. Then any height-preserving isomorphism from H onto H' can be extended to an isomorphism from G onto G'.

Assume now that H and H' are isomorphic finite subgroups of an Axiom 3 group G. Then the hypotheses of the preceding theorem are all satisfied (when $G' = G$) so any height-preserving isomorphism between H and H' can be extended to an automorphism of G. This means that the uniqueness theorem for Axiom 3 groups (as stated above) gives as an essentially free bonus the fact that Axiom 3 groups are transitive.

To show that transitivity is basically an Axiom 3 property, as opposed to a general p-group property (as once was suggested by Kaplansky), an example of a nontransitive p-group was needed. Such an example was supplied by Megibben [**53**]. The example of a nontransitive p-group was made even stronger when by using Axiom 3 methods I observed in [**25**] that p-groups G exist that are not even *potentially* transitive in the sense that G contains elements x and y with the same height sequence but $G/\langle x\rangle$ and $G/\langle y\rangle$ are not isomorphic, which obviously precludes an automorphism of G from mapping x onto y.

Aside from transitivity, the Axiom 3 approach provided additional new information about automorphisms and endomorphisms. For example, it was shown in [**24**] that if $p \neq 2$, any endomorphism of an Axiom 3 group is the sum of two automorphisms. In particular, the additive group of the endomorphism ring is generated by units.

In closing this section we mention the following remarkable feature that Axiom 3 groups enjoy, which has proved most useful in a variety of ways.

Let $\mathcal{P}$ be a group property (invariant, of course, under isomorphism). Suppose that the following hold:

1. The cyclic group of order p has property $\mathcal{P}$.

2. A direct sum of groups satisfying $\mathcal{P}$ again satisfies $\mathcal{P}$.

3. An elementary elongation of a group satisfying property $\mathcal{P}$ continues to satisfy $\mathcal{P}$.

Then $\mathcal{P}$ is an Axiom 3 property, that is, all Axiom 3 groups satisfy property $\mathcal{P}$.

8. Valuated groups and vector spaces. After the torrid pace of the 1960's in p-group development highlighted by the classification of Axiom 3 groups, emphasis began to shift in the 1970's to torsion-free groups and to other problems in p-groups. By the mid 1970's, as evidenced by the program for the Second New Mexico State Conference held in December, 1976, the studies of valuated vector spaces and that of valuated groups were becoming very popular. For a long time before, the socle of a p-group had been considered as a valuated vector space, but the formal study of valuated vector spaces began in earnest with [**11**].

The investigation of valuated groups was centered at New Mexico State and was led by Roger Hunter, Fred Richman and Elbert Walker (with papers on the subject written by various, if not all, nonempty subsets). It was my understanding that a special report was to be presented here on abelian groups at New Mexico State. Therefore, I would like to defer to one of the principals for additional remarks concerning the history of valuated groups. In the context of the development of p-groups, however, I need at least to mention that one of the main goals was to classify simply presented valuated p-groups. Using ideas and results of Laurel Rogers [**61**], Hunter, Richman and Walker achieved this result in [**43**].

Let me also point out here that the other side of benefits that came out of the study of valuated vector spaces were the concepts of compatibility and separability, both of which have since proven to be relevant to the mainstream structural theory of p-groups; see, for example, [**35**].

9. Beyond Ulm's theorem—new invariants. For whatever reason, it became fashionable in the 70's and 80's to refer to any theorem as *Ulm's Theorem* if the theorem concluded that two p-groups (including valuated ones) belonging to some designated class are isomorphic whenever they have the same invariants—whatever those invariants might be. However, my preference is to refer to such theorems as uniqueness theorems, especially if they involve invariants other than or in addition to the U-K invariants.

Since it is not possible using only the U-K invariants to classify any class of p-groups that properly contains the Axiom 3 groups and which is closed with respect to direct sums, additional new invariants have been sought that would suffice for larger classes of groups; for this purpose, it is understood that we should consider only classes that are closed with respect to direct sums to avoid trivialities.

A new invariant for a class of groups called N-groups was discovered in [**32**]. This class of groups is only slightly more general than d.s.c.'s. The class is closed with respect to both direct sums and direct summands, but not elongations. An isotype subgroup H of a d.s.c. is an N-group if (1) $p^\alpha(G/H) = (p^\alpha G + H)/H$ for each countable α and (2) $p^{\Omega+1}(G/H) = 0$. The extra invariant needed to classify N-groups is $\dim(\overline{H}[p]/H[p])$, where $\overline{H}$ denotes the completion of H in the p^Ω-topology.

If H is any isotype subgroup of an Axiom 3 group and μ is a limit ordinal not cofinal with ω, then the invariants

$$E_\mu(H) = \bigcap_{\alpha<\mu} (p^\alpha G + H)/(p^\mu G + H)$$

are, in fact, new invariants of H.

Now suppose that H is an isotype subgroup of a reduced Axiom 3 group G of length λ, where λ is a limit ordinal not cofinal with ω. Further, suppose that $p^\alpha(G/H) = (p^\alpha G + H)/H$ for all $\alpha < \lambda$. We call H a λ-elementary S-group if $G/H = Z(p^\infty)$. If G/H satisfies the much weaker condition of being a (not necessarily reduced) Axiom 3 group, H is said to be a λ-elementary A-group. An S-group or A-group, respectively is simply a direct sum of λ-elementary S groups or λ-elementary A-groups for various (not necessarily distinct) ordinals λ.

Warfield [**63**] classified S-groups using (the equivalent of) the new invariants mentioned above together with the classical invariants, and in the same way the larger class of A-groups were classified in [**33**].

Today we continue to look for new classes of p-groups and new invariants that will enable us to classify these groups. A suitable framework and guide for such investigations are given in [**40**]. Incidentally, some authors have erroneously used the terminology "classification" as a synonym for "uniqueness theorem," while the former requires *both* a uniqueness theorem and an existence theorem. But, having said that, it seems to be part of the culture to expect that a companion existence theorem can always be proven once a uniqueness theorem is established.

(I was kindly reminded by the referee that no companion existence theorem exists for the uniqueness theorem for simply presented valuated groups.)

10. Set theory sets in. To quote Paul Eklof and Alan Mekler from their book [**7**], "The modern era in set-theoretic methods in algebra can be said to have begun on July 11, 1973 when Saharon Shelah borrowed László Fuchs' *Infinite abelian groups* from the Hebrew University." In a paper [**49**] presented to the Colorado Springs Conference in 1995, Patrick Keef speaks of the *nearly pervasive influence* of set theory in the study of abelian groups in the preceding twenty years. (By contrast, however, a quick look at the programs for the last two conferences might suggest that set theory has at least temporarily vacated center stage.)

Shelah's renowned solution of the Whitehead problem (namely that it has no solution in ZFC) initiated a variety of new methods of using set theory in abelian groups. A favorite scheme was to show that some open problem is independent of ZFC by using contrasting set-theoretic hypotheses (such as $\diamondsuit$ and MA + $\sim$ CH) to obtain contradictory results. An example of this for p-groups is the independence of Crawley's problem [**54**].

It is probably fair to say that the work of Shelah and others in this area did not have the same impact on p-groups as on torsion-free groups. After all, there was no famous open problem in p-groups comparable to the Whitehead problem that lent itself to a set-theoretic resolution. Nevertheless, important results were obtained for p-groups using the new set-theoretic methods and special hypotheses. Some of these results had to do with groups being determined by their socle; see, for example, [**3**] and [**5**]. A number of interesting results were obtained about filtrations [**4**], [**7**]. Important structural results about separability and weak separability using set-theoretic methods were established in [**55**]. A most interesting new discovery appears in [**48**], where Keef proves that certain properties of abelian p-groups are equivalent to the Kurepa hypothesis.

11. Equivalent subgroups. The theory of abelian groups in general, and p-groups in particular, has been greatly enhanced in recent years by means of what has become known as equivalence theorems.

We say that two subgroups A and B of G are equivalent (as subgroups) if there is an automorphism of G that maps A onto B. This concept and terminology were used for p-groups at least as early as 1970 when it was shown that any two high subgroups of a countable p-group are equivalent [**27**]. This result was later generalized in [**31**] to include p^λ-high subgroups of Axiom 3 groups.

Why are equivalence theorems important? They have proved to be very significant in a variety of applications. First and foremost, an equivalence theorem can sometimes be used to prove that a subgroup satisfying suitable hypotheses (such as being an isotype subgroup of an Axiom 3 group) is uniquely determined by invariants. Indeed, this method was used to classify S-groups and A-groups. Moreover, as we indicated in the previous section, equivalence theorems seem to offer the best hope for future classification theorems. As of now, the most definitive and no doubt the most pleasant equivalence theorem is a result that appeared in [**40**]: two isotype subgroups H and K of an Axiom 3 group G are equivalent if and only if (1) H and K have the same U-K invariants and (2) G/H and G/K are isomorphic as valuated groups equipped with the coset valuation.

Another type of application of an equivalence theorem is the proof, under suitable hypotheses, that subgroups inherit various structural properties. For example, this was the procedure used in [**40**] to prove that any isotype subgroup of an Axiom 3 group is transitive. We observe here also that an equivalence theorem has the potential of leading to simultaneous decompositions of a group G and a subgroup H; say, $G = \oplus_i G_i$ and $H = \oplus_i H_i$, where $H_i = H \cap G_i$. The idea here is to show that the pair (G, H) is equivalent to a pair $(G' \cong G, H' \cong H)$ which, by construction or otherwise, is *known* to admit such a decomposition. An example of this can be found in [**34**]. Perhaps we should remark that, by the preceding pair being equivalent, we mean simply that there is an isomorphism from G onto G' that maps H onto H'.

12. Criteria for axiom 3 and other structures. Before we turn to positive tests for Axiom 3 and related structures, let us mention first a couple of negative tests for Axiom 3. There are some p-groups (such as the torsion completion $\overline{B}$ of an unbounded but countable direct sum B of cyclic groups) that are so far removed from being an Axiom 3 group that they can be shown by a simple direct approach not to be Axiom 3 groups by a demonstration that they cannot possess an Axiom 3 system of nice subgroups. For instance, in the parenthetical example, it is obvious that the countable subgroup B of $\overline{B}$ is not contained in any countable nice subgroup of $\overline{B}$. A property that Axiom 3 groups have which is not an immediate consequence of the definition is that they are absolutely separable in the sense that whenever they appear as an isotype subgroup of any p-group G, they must be separable in G. Therefore, one way to show that a p-group is not an Axiom 3 group is to embed it as a nonseparable and isotype subgroup of some p-group.

Now we turn to the positive results. As is well known, among p-groups any of the following provides an alternative description of Axiom 3 groups: totally projective, simply presented, or balanced projective. However it is not often practical to use one of these alternate descriptions to show that a group satisfies Axiom 3.

In contrast to the negative test of separability mentioned above, as was shown in [**29**], if H is an isotype subgroup of an Axiom 3 group G, then H itself is Axiom 3 if (and only if) G satisfies Axiom 3 over H with respect to separable subgroups.

A criterion for an isotype subgroup H of an Axiom 3 group G to be an S-group was established in [**40**]. Likewise, we also gave a criterion for H to be an A-group.

There are a number of criteria for a p-group G to be an Axiom 3 group when there are restrictions imposed on the length or size of the group G. For example, if a p-group G of cardinality not exceeding $\aleph_1$ has a v-basis, then G must be an Axiom 3 group.

The reduced Axiom 3 groups of length not exceeding ω are, of course, precisely the direct sums of cyclic p-groups. Criteria for a group to be a direct sum of cyclic groups have been known for a long time. No doubt the best known is Kulikov's criterion: an abelian p-group without elements of infinite height is a direct sum of cyclic groups if it is the union of an ascending sequence of subgroups G_n each of which has finite height spectrum when heights are computed in G. When we say that a subgroup H of a p-group G has finite height spectrum (in G), we mean that the set of heights $\{|x|_G\}$ is finite. William Ullery and I recently generalized the preceding result so that it now applies to groups of arbitrary countable length [**42**].

The following is another useful criterion that applies to arbitrary p-groups. Suppose that the p-group G is the set-theoretical union of a countable number of isotype subgroups H_n. If H_n is an Axiom 3 group for each n, then G must be an Axiom 3 group [**28**], [**30**].

It is not known at the time of this writing whether or not $V(G)/G$ must be an Axiom 3 group for an arbitrary p-group G, where $V(G)$ denotes the normalized units in the group algebra $F[G]$ with F being a perfect field of characteristic p (or even for $F = \mathbf{Z}/p\mathbf{Z}$); see [**35**] for a more complete discussion of this problem. This just proves once again what we already knew. There is no perfect criterion that works all the time, so we need to continue to search for more and better ones.

13. Summary. In summary, abelian p-groups as well as abelian groups in general have experienced tremendous growth in the last forty years. My expectation is that the next forty years will bring forth the same kind of new growth and developments. The rich structure theory of p-groups makes the subject an appealing area in which to work.

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