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THE BIRTH OF HOMOLOGICAL ALGEBRA

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1. Introduction. There appeared, in the same issue of *Expositiones Mathematicae* in 1985, two papers, one by Jean Dieudonné, the other by Urs Stammbach [**2**], [**13**]. They were excellent papers; the first described the early history of algebraic topology, with special reference to the work of Poincaré, Brouwer and Alexander; the second showed how the integral homology groups of a group carried important information about that group. The papers had, of course, been written quite independently; moreover, as each author had written in his mother tongue (French, German, respectively), it was unlikely that many would be able to derive benefit from reading both. It occurred to me then to write a short note linking these two papers, and this I did [**7**]. I also gave a reference to the definitive paper by Saunders MacLane [**12**] tracing the history of homological algebra from its inception; but I felt then, and have continued to feel even more strongly since, that there were very remarkable features of the inception itself which deserved notice. Thus, when invited to speak at the Second Honolulu Conference on Abelian Groups and Modules, I decided that here was a splendid opportunity to study and review that remarkable period in the development of mathematics and the emergence of a new mathematical discipline. Obviously, I am very grateful to the conference organizers for providing me with this stimulating opportunity.

2. The contribution of Hurewicz. In a series of four notes which appeared in the *Proceedings of the Royal Dutch Academy of Sciences* in 1935/36 [**11**], Witold Hurewicz made a key contribution to the development of homotopy theory by showing that the higher homotopy groups played a vital role in the use of obstruction-theoretic methods to study problems of the extension of maps and the existence of homotopy relations between maps (see [**4**]). Many, indeed, credit Hurewicz with actually inventing the higher homotopy groups, but

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this is not strictly accurate. The Czech mathematician Eduard Čech had announced, at the International Congress in Zürich some years earlier [**1**], the discovery of these groups. However, he had not found any significant use for these groups; and he appears to have been persuaded by some of the skeptics who attended his Zürich talk that the higher homotopy groups were unlikely to be important since they were commutative; and some of the powerful topologists of that period believed that any information provided by commutative groups would be revealed by studying the homology groups – today we know much better.

However, there was in particular one result in the Hurewicz notes which, while it employed obstruction arguments in its proof, related more specifically to the fundamental group $\pi_1 X$ and not so much to the higher homotopy groups of X. We work with pointed spaces, pointed maps and pointed homotopies. It was then clear that, given any group π, one could always find a connected complex, which we write $K(\pi, 1)$, whose fundamental group is π and whose higher homotopy groups vanish. Indeed we now call such a complex an *Eilenberg-MacLane space.* Then Hurewicz showed that, for any connected complex X, there is a natural one-to-one correspondence

$$[X, K(\pi, 1)] \cong \mathrm{Hom}\,(\pi_1 X, \pi), \tag{2.1}$$

where $[X, Y]$ is the set of (pointed) homotopy classes of maps from X to Y. Moreover, the bijection (2.1) is simply induced by associating with $f : X \to K(\pi, 1)$ the homomorphism $f_* : \pi_1 X \to \pi$ of their fundamental groups; today we recognize (2.1) as a very important example of Kan's notion of *adjoint functors*, one of the most fundamental concepts of category theory.

It is easy to prove as a consequence of (2.1) that the homotopy type of a $K(\pi, 1)$ is entirely determined by π; even more precisely, given two groups π and $\bar{\pi}$, a homomorphism $\varphi : \pi \to \bar{\pi}$ and arbitrarily chosen model Eilenberg-MacLane spaces $K(\pi, 1)$ and $K(\bar{\pi}, 1)$, there is a unique homotopy class of maps $f : K(\pi, 1) \to K(\bar{\pi}, 1)$ inducing the homomorphism φ on the fundamental groups. Since the homology groups of a space X are invariants of the homotopy type of X, it further follows that the homology groups of $K(\pi, 1)$ are simply invariants of the group π.

3. Heinz Hopf enters the picture. At this stage, another of the great topologists of this epoch, Heinz Hopf, enters the story. Hopf argued as follows. If the homology groups of $K(\pi, 1)$ depend only on π, there must be a purely *algebraic* way of defining them as functions of π without first constructing a topological space $K(\pi, 1)$ – usually an infinite-dimensional complex – and then computing its homology groups H_n as the quotients Z_n/B_n of the group of n-cycles by the group of n-boundaries.

Hopf's work on this problem was further complicated by the outbreak of World War II. Fortunately, Hopf was working at that time in Switzerland (he had succeeded to the chair in the Mathematics Department at the E.T.H. previously occupied by Hermann Weyl), so he was able to continue his research and teaching; but the war made it very difficult, if not impossible, for mathematicians to communicate with each other so that, in particular, Hopf and his student Beno Eckmann were quite unaware of what was going on in the US – and conversely. However, Hopf published in 1941 his first contribution to the solution of the problem he had set himself – he showed how one could express $H_2K(\pi, 1)$ as a function of π [**9**]. Indeed, Hopf went considerably further, observing that for any connected space X with fundamental group π, the group he had constructed out of π yielded the quotient of H_2X by the subgroup of spherical cycles.

Hopf calculated $H_2K(\pi, 1)$, which we will write simply as $H_2\pi$, as follows.[1] Let

$$R \rightarrowtail F \twoheadrightarrow \pi$$

be a *free presentation* of π; that is, F is a free group mapping onto π with kernel R, so that $F/R = \pi$. Then

$$H_2\pi = (R \cap [F, F])/[F, R]. \tag{3.1}$$

Here $[F, F]$ is the commutator subgroup of F and $[F, R]$ is the subgroup of F generated by commutators $[x, r]$, $x \in F$, $r \in R$. It is of particular interest, and typical of the arguments in homological algebra, that the group on the right of (3.1) depends only on π and not on the choice of free presentation. We will see this feature again later.

Hopf provided a complete solution to this problem in a paper published in 1944[2] [**10**]. He proceeded as follows. Let

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbf{Z} \tag{3.2}$$

be a *free resolution* of the additive group of integers $\mathbf{Z}$, regarded as a trivial π-module. Thus, each C_n is a free π-module, the *augmentation* ε maps C_0 *onto* $\mathbf{Z}$ and, at each C_n, the kernel of the outward arrow is the image of the inward arrow. It is not at all difficult to construct such free resolutions. We remark that (3.2) is often abbreviated to $C_* \overset{\varepsilon}{\twoheadrightarrow} \mathbf{Z}$. Then we may tensor (3.2) with a π-module A, over π, to obtain a chain-complex of abelian groups

$$(3.3)\quad \cdots \to C_{n+1} \otimes_\pi A \overset{\partial\otimes 1}{\to} C_n \otimes_\pi A \overset{\partial\otimes 1}{\to} C_{n-1} \otimes_\pi A \to \cdots \to C_0 \otimes_\pi A,$$

where 1 stands for the identity map of A. Hopf made the key observation that the homology groups of this chain complex depend only on π and A, and not on the choice of free resolution (3.2); we will revert once more to this key result when considering the natural generalization of this process.

Hopf then pointed out that, if $X = K(\pi, 1)$ is a cell-complex, then the universal cover $\tilde{X}$ of X has a natural cell-complex structure and π acts freely on the set of cells of $\tilde{X}$ of a given dimension, permuting those that lie over a given cell of X. Thus the chain-complex $C(\tilde{X})$ is a free resolution of $\mathbf{Z}$ as π-module, since $\tilde{X}$ is contractible, with the augmentation ε taking the value 1 on each vertex of $\tilde{X}$. But we know that the homology groups of $C(\tilde{X}) \otimes_\pi A$ are the homology groups of X with local coefficients A; in particular, if A is a trivial π-module (that is, if π acts trivially on A), then $C(\tilde{X}) \otimes_\pi A = C(X) \otimes A$, so that the homology groups of (3.3) are just $H_n(K(\pi, 1); A)$ or, as we now prefer to write, $H_n(\pi; A)$. This last form of writing would, of course, be adopted even if the action of π is not trivial.

4. Others play their part. Hopf's student, Beno Eckmann, soon made a key contribution to the story [**3**]. First he noticed a particular natural and convenient free resolution (3.2). Second, he extended the argument to cohomology and thus introduced the cohomology *ring* of π (with coefficients in a commutative ring). The way to replace homology by cohomology is simply to replace (3.3) by

(4.1)

$$\cdots \leftarrow \operatorname{Hom}_\pi(C_{n+1}, B) \overset{\partial^*}{\leftarrow} \operatorname{Hom}_\pi(C_n, B) \overset{\partial^*}{\leftarrow} \operatorname{Hom}_\pi(C_{n-1}, B) \overset{\partial^*}{\leftarrow} \cdots \overset{\partial^*}{\leftarrow} \operatorname{Hom}_\pi(C_0, B)$$

with arrows ∂^* induced by the maps ∂ of (3.2), where B is a π-module. Then (4.1) is a cochain complex whose cohomology groups do not depend on the choice of free resolution (3.2); these groups are then written $H^n(\pi; B)$.

Meanwhile, and quite independently, Sammy Eilenberg and Saunders MacLane were also inventing the cohomology groups of a group, though their emphasis was by no means so strongly on the topological significance of these groups (see [**5**]). They were, however, very much concerned with the interpretation, in terms of known invariants, of the low-dimensional cohomology groups. They, in turn, invented two (equivalent) free resolutions (3.2) which played a key role in these interpretations; one of those resolutions, not unexpectedly, coincided with the Eckmann resolution. They were also – and uniquely – responsible for introducing analogs $K(\pi, n)$, $n \geq 1$, of the spaces[3] $K(\pi, 1)$ and they studied the homology groups of these spaces, a crucial and very difficult problem in modern algebraic topology. In this task the eminent French mathematician Henri Cartan was soon to become involved.

Perhaps the most remarkable fact about developments in those early wartime days is that the German geometer Hans Freudenthal was also working along very similar lines to Hopf but, of course, neither knew of the other's work (see [**6**]). Indeed, Freudenthal's situation was particularly dire, as a German Jewish refugee in Holland, often in hiding from the occupying Nazi forces and always utterly isolated. One must pay tribute to his remarkable courage, determination and insight. Of course, Freudenthal continued to live and work in Holland after the war and is today generally thought of as a Dutch mathematician.

5. Two standard resolutions. We now pass to a purely mathematical description of developments from those days to the coming of age of homological algebra as a mature autonomous discipline. First we describe two equivalent standard resolutions of $\mathbf{Z}$ as trivial π-module; in fact, the homogeneous bar construction was due independently to Eckmann and to Eilenberg-MacLane, and the non-homogeneous version was made explicit by Eilenberg-MacLane.

The *homogeneous bar construction* is obtained by setting $C_n(\pi)$ equal to the free abelian group with basis the set of ordered $(n+1)$-tuples $\langle a_0, a_1, \dots, a_n \rangle$, $a_i \in \pi$, which becomes a free π-module on the subset of ordered $(n+1)$-tuples $\langle 1, a_1, \dots, a_n\rangle$ by defining

$$a\langle a_0, a_1, \cdots, a_n\rangle = \langle aa_0, aa_1, \cdots, aa_n\rangle, \quad a \in \pi. \tag{5.1}$$

The boundary homomorphism is the usual 'simplicial' boundary

$$\partial\langle a_0, a_1, \cdots, a_n\rangle = \sum_{i=0}^{n}(-1)^i\langle a_0, a_1, \cdots, \hat{a}_i, \cdots, a_n\rangle, \quad n \geq 1 \tag{5.2}$$

($\hat{a}_i$ means that a_i is omitted) and the *augmentation*, $\varepsilon : C_0(\pi) \xrightarrow{\varepsilon} \mathbf{Z}$ is given by

$$\varepsilon\langle\, a_0 \,\rangle = 1. \tag{5.3}$$

The standard argument shows that $\partial\partial = 0$ on $C_n(\pi)$, $n \geq 2$ and $\varepsilon\partial = 0$ on $C_1(\pi)$.

To see that $C_.(\pi) \xrightarrow{\varepsilon} \mathbf{Z}$ is indeed a resolution, we again follow the standard 'simplicial' procedure; that is, we introduce a *contracting homotopy*

$$s : C_n(\pi) \longrightarrow C_{n+1}(\pi),$$

which is an abelian group homomorphism given by

$$s\langle a_0, a_1, \cdots, a_n\rangle = \langle 1, a_0, a_1, \cdots, a_n\rangle. \tag{5.4}$$

Notice that, of course, s is *not* a module-map. However, we do have

$$\begin{aligned} \partial s + s\partial &= 1 \quad \text{on } C_n(\pi),\ n \geq 1;\\ \partial s + s\varepsilon &= 1 \quad \text{on } C_0(\pi), \end{aligned} \tag{5.5}$$

proving the exactness of $C_.(\pi) \xrightarrow{\varepsilon} \mathbf{Z}$.

We may modify this resolution by factoring out the designated *degenerate* simplexes. We choose to define a simplex $\langle a_0, a_1, \dots, a_n\rangle$ as degenerate if two successive vertices a_i, a_{i+1}, $0 \leq i \leq n-1$, coincide. Notice that, if $C_n^d(\pi)$ is the submodule generated by the degenerate n-simplexes, then a basis for this submodule consists of the degenerate n-simplexes in our chosen basis for $C_n(\pi)$; moreover, $\partial C_n^d(\pi) \subseteq C_{n-1}^d(\pi)$ and $sC_n^d(\pi) \subseteq C_{n+1}^d(\pi)$. Thus we may obtain the *normalized* homogeneous bar construction by factoring out the degenerate simplexes.

Further, if we wish to do cohomology using the normalized homogeneous construction, then an n-cochain f with values in M is a function of $(n+1)$ variables $f(a_0, a_1, \dots, a_n) \in M$, $a_i \in \pi$, which vanishes if any two successive variables a_i, a_{i+1} coincide.

The *non-homogeneous* bar construction yields a resolution $B_{.}(\pi) \overset{\varepsilon}{\twoheadrightarrow} \mathbf{Z}$ isomorphic to $C_{.}(\pi) \overset{\varepsilon}{\twoheadrightarrow} \mathbf{Z}$. Explicitly, $B_n(\pi)$ has π-module basis $|b_1, b_2, \dots, b_n|$, $b_i \in \pi$, and the boundary $\partial : B_n(\pi) \to B_{n-1}(\pi)$, $n \geq 1$, is given by

$$\begin{aligned} \partial|b_1, b_2, \dots, b_n| = b_1&|b_2, \dots, b_n| \\ &+ \sum_{i=0}^{n-1} (-1)^i |b_1, \dots, b_{i-1}, b_i b_{i+1}, b_{i+2}, \dots, b_n| \\ &+ (-1)^n |b_1, b_2, \dots, b_{n-1}|. \end{aligned} \tag{5.6}$$

Moreover, the augmentation $\varepsilon : B_0(\pi) \twoheadrightarrow \mathbf{Z}$ given by

$$\varepsilon|\;| = 1. \tag{5.7}$$

The isomorphism between $C_{.}(\pi)$ and $B_{.}(\pi)$ is given by

$$\begin{cases} |b_1, b_2, \dots, b_n| \leftrightarrow \langle 1, b_1, b_1 b_2, \dots, b_1 b_2 \dots b_n \rangle \\ \langle a_0, a_1, \dots, a_n \rangle \leftrightarrow a_0 | a_0^{-1} a_1, a_1^{-1} a_2, \dots, a_{n-1}^{-1} a_n|. \end{cases} \tag{5.8}$$

We leave the reader to verify that these are, indeed, mutually inverse chain-maps. Notice that, using the given isomorphism to transfer the notion of degeneracy from $C_{.}(\pi)$, we must define $B_n^d(\pi)$ to be the submodule of $B_n(\pi)$ generated by the n-tuples $|b_1, b_2, \dots, b_n|$ with some $b_i = 1$. Thus we pass to the *normalized* non-homogeneous bar construction by factoring out $B_{.}^d(\pi)$; and, if we wish to do cohomology using the normalized non-homogeneous construction, then an n-cochain f with values in the π-module M is a function of n variables $f(b_1, b_2, \dots, b_n) \in M$, $b_i \in \pi$, which vanishes if any $b_i = 1$. Moreover, a cochain is a π-homomorphism, so that[4]

$$f(b_{.}(b_1, b_2, \cdots, b_n)) = b_{.} f(b_1, b_2, \cdots, b_n).$$

6. Interpreting the low-dimensional cohomology groups. We will use the normalized non-homogeneous bar construction to interpret

$H^0(\pi;A)$, $H^1(\pi;A)$ and $H^2(\pi;A)$, where A is an arbitrary π-module. We leave the reader to verify that $H^0(\pi;A)$ is just the subgroup A^π of A consisting of the elements of A fixed under π. As to $H^1(\pi,A)$, a 1-cocycle f is a function $f:\pi \longrightarrow A$ such that $f(1)=0$ and (since f is a cocycle) $f(x_1,x_2)=f(x_1)+x_1.f(x_2)$; thus, the group $Z^1(\pi;A)$ of 1-cycles is just the group $\operatorname{Der}(\pi,A)$ of derivations from π to A. It is then easy to see that the group $B^1(\pi;A)$ of 1-coboundaries is the subgroup $\operatorname{Ider}(\pi,A)$ of *inner derivations* from π to A. Here an inner derivation ∂_a, $a\in A$, is given by $\partial_a(x)=(x-1).a$. Thus

$$H^1(\pi;A)=\operatorname{Der}(\pi;A)/\operatorname{Ider}(\pi,A). \tag{6.1}$$

We will now state and prove the main result of this section. We consider, for a fixed π-module A, the short exact sequence[5]

$$A \rightarrowtail E \overset{\kappa}{\twoheadrightarrow} \pi \tag{6.2}$$

compatible with the action of π on A. Thus (6.2) induces an action of π on A in this way; if $\kappa y=x$, $y\in E$, $x\in\pi$, then yay^{-1} depends only on x and $a\in A$, and we set

$$x.a=yay^{-1}. \tag{6.3}$$

Then we say that (6.2) is consistent with the given action of π on A if (6.3) coincides with that action. We now declare two exact sequences $A\rightarrowtail E \overset{\kappa}{\twoheadrightarrow} \pi$ and $A\rightarrowtail E' \overset{\kappa'}{\twoheadrightarrow} \pi$, each consistent with the action, to be equivalent if there exists a homomorphism $\omega: E\to E'$ such that the diagram[6]

$$\begin{array}{ccccc} A & \rightarrowtail & E & \overset{\kappa}{\twoheadrightarrow} & \pi \\ \Big\Vert & & \Big\downarrow{\scriptstyle\omega} & & \Big\Vert \\ A & \rightarrowtail & E' & \overset{\kappa'}{\twoheadrightarrow} & \pi \end{array} \tag{6.4}$$

commutes; notice that this forces ω to be an isomorphism, so we do indeed have here an equivalence relation. We write $[E]$ for the equivalence class of E and call $[E]$ an *extension* of π by A. We further write $E(\pi;A)$ for the set of such extensions, consistent with the given action of π on A.

Theorem 6.1. *There is a natural bijection* $E(\pi; A) \cong H^2(\pi : A)$*.*

Proof. We will postpone until later a discussion of the significance of the attribute of naturality and will first concentrate on establishing the bijection.

Let $A \rightarrowtail E \overset{\kappa}{\twoheadrightarrow} \pi$ be consistent with the action. Choose a function $u : \pi \to E$ such that $\kappa u = \mathrm{Id}$; we will insist that $u(1) = 1$. Let $x_1, x_2 \in \pi$. Then $\kappa(u(x_1)u(x_2)) = x_1x_2 = \kappa u(x_1x_2)$, so that

$$u(x_1)u(x_2) = f(x_1, x_2)u(x_1x_2) \tag{6.5}$$

where $f(x_1x_2) \in A$. Notice that $f(1, x) = f(x, 1) = 0$, so that f is a 2-cochain in the normalized non-homogeneous bar construction. Traditionally, f has been called a *factor set* for (6.2).

We now exploit associativity. Thus,

$$\begin{aligned}(u(x_1)u(x_2))u(x_3) &= f(x_1, x_2)u(x_1x_2)u(x_3)\\ &= f(x_1, x_2)f(x_1x_2, x_3)u(x_1x_2x_3),\end{aligned}$$

while

$$\begin{aligned}u(x_1)(u(x_2)u(x_3)) &= u(x_1)f(x_2, x_3)u(x_2x_3)\\ &= x_1.f(x_2, x_3)f(x_1, x_2x_3)u(x_1x_2x_3).\end{aligned}$$

Thus

$$\delta f(x_1, x_2, x_3) = x_1.f(x_2, x_3) - f(x_1x_2, x_3) + f(x_1, x_2x_3) - f(x_1, x_2) = 0,$$

and f is a 2-cocycle, $f \in Z^2(\pi; A)$.

Obviously, f depends on the choice of u. If we choose $\bar{u}$ instead of u, then for $x \in \pi$, $\bar{u}(x) = g(x)u(x)$, $g(x) \in A$, so that g is a 1-cochain; notice that $g(1) = 0$, so g is normalized. Further, the choice of $\bar{u}$ gives rise to the 2-cocycle $\bar{f}$, where $\bar{u}(x_1)\bar{u}(x_2) = \bar{f}(x_1, x_2)\bar{u}(x_1x_2)$, so that

$$g(x_1)u(x_1)g(x_2)u(x_2) = \bar{f}(x_1, x_2)g(x_1x_2)u(x_1x_2),$$

or

$$g(x_1)(x_1.g(x_2))f(x_1, x_2)u(x_1x_2) = \bar{f}(x_1, x_2)g(x_1x_2)u(x_1x_2),$$

yielding $\bar{f}(x_1,x_2)-f(x_1,x_2)=x_1.g(x_2)-g(x_1x_2)+g(x_1)=\delta g(x_1,x_2)$. This shows that the 2-cycles f and $\bar{f}$ are cohomologous, so that the cohomology class of f is uniquely determined by the short exact sequence (6.2). Now let $A \rightarrowtail E' \xrightarrow{\kappa'} \pi$ be equivalent to the sequence (6.2), as in (6.4); and choose $u'=\omega u:\pi\to E'$. Then $\kappa' u'=\kappa'\omega u=\kappa u=\mathrm{Id}$ and $u'(1)=1$, as required. Moreover,

$$\begin{aligned} u'(x_1)u'(x_2) &= \omega u(x_1)\omega u(x_2)=\omega(u(x_1)u(x_2)) \\ &= \omega(f(x_1,x_2)u(x_1x_2)) = f(x_1,x_2)u'(x_1x_2), \end{aligned}$$

since the restriction of ω to A is the identity. Thus equivalent sequences yield the same cohomology class, so we have defined a function $\Phi: E(\pi;A)\longrightarrow H^2(\pi;A)$ by $\phi[E]=[f]$, the cohomology class of f.

We now define a function Ψ in the other direction, $\Psi: H^2(\pi;A)\longrightarrow E(\pi;A)$. Thus, given a cohomology class $[f]$, where f is a 2-cocycle, we define a short exact sequence.

$$A \rightarrowtail E_f \xrightarrow{\kappa_f} \pi \tag{6.6}$$

as follows. The underlying set of E_f is the cartesian product $A\times\pi$, and the multiplication is given by

$$(a_1,x_1)(a_2,x_2)=(a_1+x_1.a_2+f(x_1,x_2),x_1x_2). \tag{6.7}$$

Of course, A embeds in E_f by $a\longrightarrow(a,1)$, and $\kappa_f(a,x)=x$. We must verify that (6.7) defines a group structure on E_f and that (6.6) is consistent with the given action of π on A. As to the former, we see that (0,1) is a two-sided identity (since f is normalized) and that

$$(a,x)^{-1}=(-x^{-1}_{.}(a+f(x,x^{-1})),x^{-1}). \tag{6.8}$$

As to associativity, we leave it to the reader to verify that this is an immediate consequence of the fact that f is a *cocycle*.

To find the action of π on A determined by (6.6), we employ the section $x\longrightarrow(0,x)$. Then

$$\begin{aligned} (0,x)(a,1)(0,x)^{-1} &= (x.a,x)(-x^{-1}_{.}f(x,x^{-1}),x^{-1}), \quad \text{by (6.8)}, \\ &= (x.a,1), \end{aligned}$$

as required.

If we chose a cohomologous cocycle $\bar{f}$ with $\bar{f} - f = \delta g$, we may construct the commutative diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, Rightarrow, no head] & E_f \arrow[r, two heads, "\kappa_f"] \arrow[d, "\omega"] & \pi \arrow[d, Rightarrow, no head] \\ A \arrow[r, tail] & E_{\bar{f}} \arrow[r, two heads, "\kappa_{\bar{f}}"] & \pi \end{tikzcd} \tag{6.9}$$

where $\omega(a, x) = \overline{(a - g(x), x)}$. The diagram obviously commutes, and the fact that ω is a homomorphism follows from the relation $\bar{f} - f = \delta g$, that is, from the relation

$$\bar{f}(x_1, x_2) - f(x_1, x_2) = x_1.g(x_2) - g(x_1x_2) + g(x_1).$$

Thus $[f] \longrightarrow [E_f]$ yields a function $\Psi : H^2(\pi; A) \longrightarrow E(\pi; A)$. It remains only to show that Φ, Ψ are mutual inverses. First consider $\Psi\Phi[E] = \Psi[f] = [E_f]$. We construct a homomorphism $\omega : E \longrightarrow E_f$ so that the diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, Rightarrow, no head] & E \arrow[r, two heads, "\kappa"] \arrow[d, "\omega"] & \pi \arrow[d, Rightarrow, no head] \\ A \arrow[r, tail] & E_f \arrow[r, two heads, "\kappa_f"] & \pi \end{tikzcd} \tag{6.10}$$

commutes. Suppose f was obtained from the section $u : \pi \longrightarrow E$. If $y \in E$, let $\kappa y = x$ so that there is a unique $a \in A$ with $y = au(x)$. We set $\omega(y) = (a, x)$; then the diagram (6.10) obviously commutes. It remains to show that ω is a homomorphism. Now let $y_1 = a_1u(x_1)$, $y_2 = a_2u(x_2)$. Then

$$\begin{aligned} \omega(y_1y_2) &= \omega(a_1u(x_1)a_2u(x_2)) = \omega(a_1(x_1.a_2)u(x_1)u(x_2)) \\ &= \omega(a_1(x_1.a_2)f(x_1, x_2)u(x_1x_2) \\ &= (a_1 + x_1.a_2 + f(x_1, x_2), x_1x_2) \\ &= (a_1, x_1)(a_2, x_2) = \omega(y_1)\omega(y_2). \end{aligned}$$

Thus ω is a homomorphism, so that $\Psi\Phi[E] = [E]$ and $\Psi\Phi$ is the identity.

Finally we study $\Phi\Psi[f] = \Phi[E_f]$. We choose as section $u : \pi \longrightarrow E_f$ (as we did before) the function $u(x) = (0, x)$. Then

$$\begin{aligned} u(x_1)u(x_2) &= (0, x_1)(0, x_2) = (f(x_1, x_2), x_1, x_2) \\ &= (f(x_1, x_2), 1)(0, x_1x_2) = (f(x_1, x_2), 1)u(x_1x_2). \end{aligned}$$

This shows that the resulting 2-cocycle is f itself, so that $\Phi\Psi[f] = [f]$ and $\Phi\Psi$ is the identity. This completes the proof of Theorem 6.1 except for clarifying the notion of naturality which we do in Remark 2 below. □

Remark 1. Notice that, in this theorem, we have no choice in defining the equivalence relation on short exact sequences. There are, of course, other reasonable possibilities: we may declare two such sequences $A \rightarrowtail E \overset{\kappa}{\twoheadrightarrow} \pi$ and $A \rightarrowtail E' \overset{\kappa'}{\twoheadrightarrow} \pi$ to be equivalent if E, E' are isomorphic; or we may declare them equivalent if there are automorphisms α_A of A and α_π of π such that the diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\alpha_A"] & E \arrow[r, two heads, "\kappa"] \arrow[d, "\omega"] & \pi \arrow[d, "\alpha_\pi"] \\ A \arrow[r, tail] & E' \arrow[r, two heads, "\kappa'"] & \pi \end{tikzcd}$$

is commutative. However, it is only with *our* choice of equivalence relation that we get a bijection of $E(\pi; A)$ with $H^2(\pi; A)$, enabling us to calculate $E(\pi; A)$ and to transfer the abelian group structure from $H^2(\pi; A)$ to $E(\pi; A)$.

Remark 2. The abelian group $H^2(\pi; A)$ – indeed $H^n(\pi; A)$ for any n – is a *functor* of the two variables π and A, covariant in A, contravariant in π. As to the dependence on A, this means that a π-homomorphism θ from A to $\bar{A}$, where $A, \bar{A}$ are π-modules, induces in a natural way a homomorphism from $H^2(\pi; A)$ to $H^2(\pi; \bar{A})$. However, there is also a natural way in which θ allows us to pass from an element of $E(\pi; A)$ to an element of $E(\pi; \bar{A})$. These functions induced by θ are then compatible with Φ and Ψ in an obvious sense.

The dependence of $H^2(\pi; A)$ on π is a little more subtle. If γ is a homomorphism from $\bar{\pi}$ to π, and if A is a π-module, then γ allows A to be regarded as a $\bar{\pi}$-module; moreover, γ then induces a homomorphism γ^* from $H^2(\pi; A)$ to $H^2(\bar{\pi}; A)$. Notice that this effect is *contravariant* in that γ and γ^* go in opposite directions. Moreover, γ also induces a function γ^* from $E(\pi; A)$ to $E(\bar{\pi}, A)$; and these two induced functions γ^* are again compatible with Φ and Ψ.

Remark 3. The fact that we have provided interpretations of $H^0(\pi; A)$, $H^1(\pi; A)$ and $H^2(\pi; A)$ from classical algebra should not lead us to suppose that all the cohomology groups (and homology groups) of π admit such interpretations. Nor should we have hoped for this; for our homological methods enable us to systematize constructions which were not hitherto seen to stand in any special relation to each other and to devise an infinite sequence of abelian groups which all subscribe to the same rule of construction. That the first three groups in the sequence are of so fundamental a nature is, of course, an excellent reason for expecting the entire sequence to be important and to provide useful new group invariants.

7. The general procedure of homological algebra. Today, homological algebra is a vast subject, with applications to many other areas of mathematics; and the application to group theory described above would be regarded as just one example, though an important one, of the scope of the theory. By expressing ideas in homological form, we are not only led to significant generalizations of those ideas but also to systematic proof-procedures. In this section we will first describe the fundamental ideas of homological algebra and then give a few examples of the kinds of proofs which the theory provides.

We suppose, given two abelian categories $\mathfrak{A}$ and $\mathfrak{B}$, and the additive functor $T : \mathfrak{A} \to \mathfrak{B}$; the reader should keep in mind the example $\mathfrak{A} = \mathfrak{M}_\Lambda$, the category of (right) modules over the unitary ring Λ; $\mathfrak{B} = \mathfrak{Ab}$, the category of abelian groups; and $TA = A \otimes_\Lambda B$ for some fixed left Λ-module B. One takes an arbitrary object A of $\mathfrak{A}$ and constructs a projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{\partial} P_{n-1} \longrightarrow \cdots \xrightarrow{\partial} P_0 \xrightarrow{\varepsilon} A, \tag{7.1}$$

which, for brevity, we write $\underline{\mathrm{P}} \xrightarrow{\varepsilon} A$. Notice that we need not insist on a free resolution; a projective module P is one such that, given the diagram

$$\begin{array}{ccc} & & U \\ & & \downarrow\!\!\downarrow \alpha \\ P & \xrightarrow[\varphi]{} & V \end{array} ,$$

there exists $\psi : P \to U$ with $\alpha\psi = \varphi$. Every free object (where that notion makes sense in $\mathfrak{A}$) is projective, but the converse is, in general, false. Indeed, in $\mathfrak{M}_\Lambda$, the projective objects are precisely the direct summands in free objects.

One now applies the functor T to the resolution $\underline{\mathrm{P}}$. Then $T\underline{\mathrm{P}}$ is a chain complex in $\mathfrak{B}$, and the *nth derived functor* of T is given by

$$(D_nT)(A) = H_n(T\underline{\mathrm{P}}). \tag{7.2}$$

Of course, crucial to this definition is the fact that $H_n(T\underline{\mathrm{P}})$ does *not* depend on the choice of projective resolution $\underline{\mathrm{P}}$ of A; the argument establishing this is itself quite typical of homological-algebraic method (see, for example, [**8**, Chapter 4]). In the case cited with $TA = A \otimes_\Lambda B$, we write $\mathrm{Tor}_n^\Lambda(A,B)$ for $(D_nT)(A)$. Another crucial example is that in which $\mathfrak{A}, \mathfrak{B}$ are as before, but T is the contravariant functor given by $TA = \mathrm{Hom}_\Lambda(A,B)$ where B is a fixed right Λ-module. Then $(D_nT)(A)$ is written $\mathrm{Ext}_\Lambda^n(A,B)$. It turns out, moreover, that there is a remarkable balance between the roles played by A and B. As to Tor, one also obtains $\mathrm{Tor}_n^\Lambda(A,B)$ by taking a projective resolution of B (instead of A). As to Ext, one also obtains $\mathrm{Ext}_\Lambda^n(A,B)$ by taking an injective resolution of B (instead of a projective resolution of A). Here an injective resolution of B is an exact sequence

$$B \rightarrowtail I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow I_n \longrightarrow \cdots ,$$

where each I_n is an injective right Λ-module. We define I as *injective* if, given

$$\begin{array}{ccc} & & U \\ & & \uparrow \alpha \\ I & \xleftarrow[\varphi]{} & V \end{array}$$

there exists $\psi : U \to I$ with $\psi\alpha = \varphi$. Notice that, whereas 'injective' is obviously dual to 'projective,' there is no obvious dual to the notion 'free.' Notice, too, that the cohomology groups of π with coefficients in B are just $\mathrm{Ext}^n_{\mathbf{Z}\pi}(\mathbf{Z}, B)$; the homology groups with coefficients in B are likewise $\mathrm{Tor}_n^{\mathbf{Z}\pi}(\mathbf{Z}, B)$.

The general theory now proceeds. In particular, there is a very beautiful result associating with a short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ of objects of $\mathfrak{A}$ and an additive functor T, a long exact sequence of derived functors of T. This, of course, is a generalization of results on long exact sequences familiar to algebraic topologists and plays a crucial role in many applications of homological algebra (see [**8**, Chapter 4]).

Finally let us give some applications of the methods of homological algebra to classical algebra. Our first applications are to the theory of Lie algebras. If $\mathfrak{g}$ is a Lie algebra over the field K, then it is known (the celebrated Birkhoff-Witt theorem) that $\mathfrak{g}$ embeds in its universal enveloping (associative) algebra $U_\mathfrak{g}$ as a Lie subalgebra. A $\mathfrak{g}$-module A is then an abelian group with an action of $\mathfrak{g}$ such that, for $x, y \in \mathfrak{g}$, $a \in A$,

$$[x, y].a = x.(y.a) - y.(x.a). \tag{7.3}$$

There is then a unique extension of the action of $\mathfrak{g}$ to $U_\mathfrak{g}$, and we define

$$H^n(\mathfrak{g}, A) = \mathrm{Ext}^n_{U_\mathfrak{g}}(K, A). \tag{7.4}$$

The British algebraic topologist, J.H.C. (Henry) Whitehead, proved two famous 'lemmas' about these groups, namely,

Theorem 7.1. *If $\mathfrak{g}$ is a finite-dimensional semi-simple Lie algebra over K and A is a finite-dimensional $\mathfrak{g}$-module, then $H^1(\mathfrak{g}; A) = 0$.*

Theorem 7.2. *If $\mathfrak{g}$ is a semi-simple Lie algebra over K and A is a finite-dimensional $\mathfrak{g}$-module, then $H^2(\mathfrak{g}; A) = 0$.*

In fact, Theorem 7.1 has as almost immediate consequence Weyl's theorem that every finite-dimensional $\mathfrak{g}$-module over a semi-simple Lie algebra $\mathfrak{g}$ is a direct sum of simple $\mathfrak{g}$-modules; and from Theorem 7.2 there follows, with similar facility, the theorem of Levi-Malcev that every finite-dimensional Lie algebra $\mathfrak{g}$ is the split extension of a semi-simple Lie algebra by the radical of $\mathfrak{g}$. Here the *radical* is the unique maximal solvable ideal of $\mathfrak{g}$.

For our last example we return to group theory. A famous theorem of Issai Schur asserts that if G is a group such that G/Z is finite (where Z is the center of G) then G', the commutator subgroup of G, is also finite. Now an easy theorem in the homology theory of groups shows that, if U is a group, then

$$H_nU, \quad n \geq 1, \quad \text{is finite if } U \text{ is finite.} \tag{7.5}$$

Indeed, if $|U| = m$, it is not hard to show that $mH_nU = 0$, $n \geq 1$, and the bar construction (Section 5) shows immediately that H_nU is finitely generated if U is finite.

A more subtle result relates to the homology groups of the groups occurring in a short exact sequence. This result takes on a particularly convenient form if the short exact sequence is a central extension. We apply it to $Z \rightarrowtail G \twoheadrightarrow G/Z$ to obtain an exact five-term sequence

$$H_2G \longrightarrow H_2G/Z \xrightarrow{\alpha} Z \xrightarrow{\beta} G_{ab} \twoheadrightarrow (G/Z)_{ab}\ , \tag{7.6}$$

where $G_{ab} = G/G'$. Now since G/Z is finite, $H_2(G/Z)$ is finite. Thus $\alpha H_2(G/Z)$ is finite, but

$$\operatorname{im}\alpha = \ker\beta = G' \cap Z,$$

so $G' \cap Z$ is finite. On the other hand, $G'/G' \cap Z$ imbeds in G/Z and so is finite. Since $G' \cap Z$ and $G'/G' \cap Z$ are finite, so is G'.

ENDNOTES

1. We will likewise write $H_n\pi$ for $H_nK(\pi, 1)$ when we give Hopf's solution to the problem in general.

2. Note that, at that time, homology groups were called 'Betti groups' after the topologist Betti.

3. $K(\pi, n)$ is a space whose only non-vanishing homotopy group is $\pi_nK(\pi, n)$, which equals π. Of course, if $n \geq 2$, then π must be commutative.

4. Notice that we have abandoned here the uprights in favor of more conventional symbols.

5. Remember that A is written additively. However, when regarded as a subgroup of E, its members will be combined multiplicatively.

6. "$\rightarrowtail$" is one-to-one, "$\twoheadrightarrow$" is surjective (onto).

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