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B-SCROLLS WITH NON-DIAGONALIZABLE SHAPE OPERATORS

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Dedicated to Professor D.E. Blair on the occasion of his sixtieth birthday

ABSTRACT. We study some Lorentzian surfaces in the three-dimensional Lorentzian space forms whose shape operators are not diagonalizable at least at one point. It is related to the so-called notion of 2-type surfaces. A local classification theorem in this respect is obtained.

1. Introduction. Let us denote by $\overline{M}_1^3(c)$ the standard model of a Lorentzian space form with constant curvature $c = 0, \pm 1$, that is, the Lorentz-Minkowski space L^3, the de Sitter space-time S_1^3 in E_1^4 and the anti de Sitter space-time H_1^3 in E_2^4, respectively. For $(n, \mu) = (3,1), (4,1)$ or (4,2), let E_μ^n be the corresponding pseudo-Euclidean space where $\overline{M}_1^3(c)$ is lying.

Suppose that $x : M_1^2 \to \overline{M}_1^3(c) \subset E_\mu^n$ is an isometric immersion of a two-dimensional connected Lorentzian surface into the three-dimensional Lorentzian space form. Denote by Δ the Laplacian operator of the Lorentzian surface M_1^2. The immersion x is said to be of finite type if each component of the position vector field of M_1^2 in E_μ^n, also denoted by x, can be written as a finite sum of eigenfunctions of the Laplacian operator Δ, that is, if

$$x = x_0 + x_1 + x_2 + \cdots + x_k, \tag{1.1}$$

where x_0 is a constant vector, $x_1, \dots, x_k$ are nonconstant maps satisfying $\Delta x_i = \lambda_i x_i$, $i = 1, \dots, k$. If, in particular, all eigenvalues

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$\lambda_1, \dots, \lambda_k$ are mutually different, then the immersion x (or the surface M_1^2) is said to be of k-type and the decomposition (1.1) is called the spectral decomposition of the immersion x. If one of $\lambda_1, \dots, \lambda_k$ is zero, then the immersion is said to be of null k-type. Note that we have the following formula:

$$\Delta x = -2H, \tag{1.2}$$

where H is the mean curvature vector field of M_1^2 in E_μ^n. It is well known that M_1^2 is of 1-type if and only if it is either minimal or non-flat totally umbilical in $\overline{M}_1^3(c)$ [**6**].

If M_1^2 is a null 2-type surface, then the position vector x takes the following decomposition:

$$x = x_1 + x_2, \qquad \Delta x_1 = 0, \qquad \Delta x_2 = \lambda x_2 \tag{1.3}$$

for some nonconstant maps x_1, x_2 and a constant $\lambda \neq 0$. From (1.2) and (1.3) we have

$$\Delta H = \lambda H, \quad \lambda \neq 0, \tag{1.4}$$

that is, the mean curvature vector field is an eigenvector function of Δ. Conversely, we have the following [**9**].;

Lemma 1.1. *There is a constant $\lambda \neq 0$ such that* (1.4) *holds if and only if M_1^2 is either of* 1*-type or of null* 2*-type.*

Now suppose that a surface M_1^2 in $\overline{M}_1^3(c)$ is of 2-type. Then (1.1) and (1.2) imply that

$$\Delta H = \lambda H + \mu(x - x_0), \tag{1.5}$$

where x_0 is a constant vector and λ and μ are two real constants.

Conversely, we have the following [**4, 9**].

Lemma 1.2. *Suppose that there exist constants λ and μ such that* (1.5) *holds. Then M_1^2 is of* 2*-type if and only if the polynomial $t^2 - \lambda t + 2\mu$ has two distinct real roots.*

In a series of papers ([**3, 5–8, 10–13, 16–18**]), the technique of finite type immersions has been used to characterize certain interesting families of Riemannian or Lorentzian surfaces. If the ambient space is the three-dimensional Riemannian space form E^3, S^3 or H^3, then the following theorems are well known [**6, 7, 16**].

Theorem 1.3. *Let M^2 be a surface in the 3-dimensional Euclidean space E^3. Then M^2 is of null 2-type if and only if M^2 is an open portion of a circular cylinder.*

Theorem 1.4. *A surface M^2 in the unit 3-sphere S^3, standardly embedded in E^4, is of 2-type if and only if M^2 is an open portion of the product surface of two plane circles of different radii.*

Theorem 1.5. *A surface M^2 in the hyperbolic space H^3, standardly embedded in E^4_1, is of 2-type if and only if M^2 is an open portion of the product surface $H^1(\sqrt{1+r^2}\,) \times S^1(r)$.*

If the ambient space is the 3-dimensional Lorentzian space form $\overline{M}^3_1(c)$, it is well known that the shape operator of a Lorentzian surface need not be diagonalizable; because of this fact there are substantial differences between the Lorentzian and Riemannian cases. Actually, there exists a wide family of examples of surfaces in Lorentzian space forms without Riemannian counterparts; the B-scrolls [**14**] and the complex circles [**19**] are some of these examples.

Ferrández and Lucas showed the following [**11**]:

Theorem 1.6. *Let M^2_1 be a null 2-type Lorentzian surface in L^3. Then the following hold*: i) *if the shape operator is diagonalizable on M^2_1, then M^2_1 is an open piece of a Lorentzian cylinder*, ii) *if the shape operator is not diagonalizable at a point p of M^2_1, then an open set of M^2_1 around p is a B-scroll.*

For Lorentzian surfaces in the non-flat Lorentzian space forms, extending above, Alías, Ferrández and Lucas gave nice classification theorems with certain conditions on the shape operator [**3**].

Theorem 1.7. *Let M_1^2 be a 2-type Lorentzian surface in S_1^3, standardly embedded in E_1^4. Then the following hold*: i) *if the shape operator is diagonalizable on M_1^2, then M_1^2 is an open piece of a Lorentzian cylinder $S_1^1(r) \times S^1(\sqrt{1-r^2})$*, ii) *if the shape operator is not diagonalizable at a point p of M_1^2, then an open set of M_1^2 around p is a B-scroll over a null curve.*

Theorem 1.8. *Let M_1^2 be a 2-type Lorentzian surface in H_1^3, standardly embedded in E_2^4. Then the following hold*: i) *if the shape operator is diagonalizable on M_1^2, then M_1^2 is an open piece of a Lorentzian cylinder, $H_1^1(r) \times S^1(\sqrt{r^2-1})$, or $S_1^1(r) \times H^1(\sqrt{1+r^2})$*, ii) *if the shape operator is not diagonalizable at a point p of M_1^2, then an open set of M_1^2 around p is a non-flat B-scroll over a null curve.*

Interestingly, there exist 2-type Lorentzian surfaces which contain points of both kinds. For example, a B-scroll is a case of such kind. Let D denote the set of all points of M_1^2 at which the shape operator S is diagonalizable, and $U = M_1^2 \setminus D$, the set of all points of M_1^2 at which S is not diagonalizable. It can be shown that U is an open subset of M_1^2, hence D is a closed subset of M_1^2. Thus it is natural to ask the following question:

"What can we say about the neighborhood of a 2-type Lorentzian surface around a point in the boundary of the set U?"

The purpose of this paper is to give an answer to this question. More precisely, we shall establish the following:

Theorem A. *If a null 2-type Lorentzian surface in L^3 admits a point where the shape operator is not diagonalizable, then it is locally a B-scroll.*

Theorem B. *If a 2-type Lorentzian surface in S_1^3, H_1^3 admits a point where the shape operator is not diagonalizable, then it is locally a B-scroll.*

2. B-scrolls and complex circles. In this section we will describe some examples of surfaces of non-flat space form $\overline{M}_1^3(c)$ which will be useful in order to give the classification results.

Let $\gamma(s)$ be a null curve in $\overline{M}_1^3(c) \subset E_\mu^4$, and let $\{A(s), B(s), C(s)\}$ be a Cartan frame of $\gamma(s)$, that is, $A(s), B(s), C(s)$ are tangent vector fields of $\overline{M}_1^3(c)$ along $\gamma(s)$ satisfying the following conditions:

$$\begin{aligned} &\langle A, A\rangle = \langle B, B\rangle = 0, \qquad \langle A, B\rangle = -1, \\ &\langle A, C\rangle = \langle B, C\rangle = 0, \qquad \langle C, C\rangle = 1, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \dot{\gamma}(s) &= A(s), \\ \dot{C}(s) &= -aA(s) - k(s)B(s), \end{aligned} \tag{2.2}$$

where a is a constant and $k(s)$ is a function of s.

Then the immersion $x(s,t) = \gamma(s) + tB(s)$ parametrizes a Lorentzian surface M_1^2 into $\overline{M}_1^3(c)$, which is called a B-scroll over a null curve γ [**14**]. In the last section we prove the existence and uniqueness of Cartan framed null curves in $\overline{M}_1^3(c)$ satisfying the appropriate differential equations. For a B-scroll over a null curve in the flat Lorentzian space form L^3, see [**11, 14**].

By a reparametrization, we may always assume that $\gamma(s)$ is a null geodesic of the B-scroll, which is equivalent to the condition $\langle \dot{A}(s), B(s)\rangle \equiv 0$. Note that the Laplacian Δ of the B-scroll M_1^2 is given by

$$\Delta h = 2h_{st} + 2Kth_t + Kt^2 h_{tt}, \quad h \in C^\infty(M_1^2),$$

where $K = c + a^2$ is the Gaussian curvature of the B-scroll M_1^2. Thus, from (1.2), we have

$$\begin{aligned} H(s,t) &= -KtB(s) + aC(s) - c\gamma(s), \\ \Delta H &= 2KH. \end{aligned}$$

For a non-flat B-scroll, we let

$$x_2 = -\frac{1}{K}\, H, \qquad x_1 = \frac{1}{K}\, \{aC(s) + a^2\gamma(s)\};$$

then we have

$$x = x_1 + x_2, \qquad \Delta x_1 = 0, \qquad \Delta x_2 = 2KH.$$

By a simple computation, we have

$$\frac{dx_1}{ds} = \frac{-ak(s)}{K}B(s).$$

Hence we obtain the following.

Proposition 2.1. *If $ak(s) \not\equiv 0$, then the non-flat B-scroll in $\overline{M}_1^3(c)$ is of null 2-type; otherwise, the non-flat B-scroll is of 1-type. However a flat B-scroll (hence $c = -1$ and $a^2 = 1$) is a biharmonic surface into H_1^3 and is of infinite type.*

If we choose a unit normal vector field $N = -atB(s) + C(s)$, then the shape operator S takes, in the usual frame $\{x_s, x_t\}$, the following form

$$\begin{pmatrix} a & 0 \\ k(s) & a \end{pmatrix}$$

and its minimal polynomial changes its degree.

Suppose that $k(s) \equiv 0$ in an open interval I of s. Then $x(s,t) + (1/a)N(s,t) = (1/a)C(s) + \gamma(s)$ is a nonzero constant vector y_0 with $\langle y_0, y_0 \rangle = c + a^{-2}$. Hence the B-scroll $x(s,t)$, restricted to $I \times R$, is just an open part of the following totally umbilic Lorentzian surface $L_1^2(y_0, c)$ determined by y_0:

$$L_1^2(y_0, c) = \{x \in E_\mu^4 \mid \langle x, x \rangle = \langle x, y_0 \rangle = c\}. \tag{2.3}$$

Conversely, consider the Lorentzian surface $L_1^2(y_0, c) \subset \overline{M}_1^3(c)$ in (2.3). Then the surface is Lorentzian if and only if $\langle y_0, y_0 \rangle > c$. Hence we may assume that $\langle y_0, y_0 \rangle = c + a^{-2}$ for a positive real number a. Note that a unit normal vector field N of $L_1^2(y_0, c)$ in $\overline{M}_1^3(c)$ is given by $N = a(y_0 - x)$ and that the surface is of constant Gaussian curvature $K = c + a^2$. For any fixed point $p \in L_1^2(y_0, c)$, choose a pseudo-orthonormal basis $\{A_0, B_0\}$ of T_pL and put $C_0 = a(y_0 - p)$. If we let

$$\gamma(s) = p + A_0 s, A(s) = A_0,$$
$$B(s) = \frac{1}{2}KA_0 s^2 + (Kp - a^2 y_0)s + B_0, \tag{2.4}$$
$$C(s) = -aA_0 s + C_0 = a(\gamma_0 - \gamma(s),$$

then $\{A(s), B(s), C(s)\}$ is the unique Cartan frame of the null curve $\gamma(s)$ in $L_1^2(y_0, c)$ with $k(s) = k_1(s) \equiv 0$ satisfying $\gamma(0) = p$, $A(0) = A_0$, $B(0) = B_0$ and $C(0) = C_0$, see Appendix. The B-scroll $y(s,t) = \gamma(s) + tB(s)$ is a parametrization of the Lorentzian surface $L_1^2(y_0, c)$, which omits the null straight line $2y_0 - p + A_0 t$ of $L_1^2(y_0, c)$. Note that every null geodesic of $L_1^2(y_0, c)$ is a straight line. This property characterizes the totally umbilic submanifolds with indefinite metric of a pseudo-Euclidean space [**2**].

Now we fix a complex number $c + id = \kappa$ in C with $c^2 - d^2 = -1$. C^2 can be identified with R_2^4 by sending $(x_1 + ix_3, x_2 + ix_4)$ to (x_1, x_2, x_3, x_4). The metric on R_2^4 is given by $dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$. The mapping $x(z) = \kappa(\cos z, \sin z) \in C^2$, where $z = u_1 + iu_2 = (u_1, u_2)$, parametrizes a nonminimal flat Lorentzian surface into the anti de Sitter space H_1^3, which is called a complex circle of radius κ [**19**].

If we choose a unit normal vector field

$$N = (d + ic)(\cos z, \sin z),$$

then the shape operator S takes, in the usual frame $\{\partial x/\partial u_1, \partial x/\partial u_2\}$, the following form:

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

where $\alpha = -2cd/(c^2 + d^2)$, $\beta = 1/(c^2 + d^2)$. The mean curvature vector field H in R_2^4 is given by $H = \alpha N + x$. It is not difficult to show that a complex circle satisfies the condition (1.5) with $x_0 = 0$, $\lambda = -4/(c^2 + d^2)$ and $\mu = 2/(c^2 + d^2)^2$. However, since the polynomial $t^2 - \lambda t + 2\mu$ has vanishing discriminant, Lemma 1.2 shows that the complex circle is not of finite type.

3. Behavior of nondiagonalizable shape operator and *B*-scrolls as 2-type surfaces. In this section we prove the main theorems, Theorem A and B in Section 1. Let M_1^2 be a null 2-type (2-type) Lorentzian surface in the 3-dimensional Lorentz-Minkowski space L^3 (non-flat Lorentzian space form $\overline{M}_1^3(c)$, respectively). Then Alías, Ferrández and Lucas show that the shape operator S has constant trace, say $2a$, and constant square of length $|S|^2 = \mathrm{tr}\,(S^2)$ [**3, 11**]. Obviously, the constant a is nonzero. Let D denote the set of all points of M_1^2 at which the shape operator S is diagonalizable, and $U = M_1^2 \setminus D$, the set of all points of M_1^2 at which the shape operator S cannot be diagonalizable. It is well known that 1) if $c = 0$, each connected component of the interior of D is an open part of a Lorentzian cylinder or a de Sitter space-time $S_1^2[y_0, (1/|a|)]$, $y_0 \in L^3$, 2) if $c \neq 0$, each connected component of the interior of D is an open part of either a totally umbilic surface $L_1^2(y_0, c)$ of $\overline{M}_1^3(c)$ with $\langle y_0, y_0\rangle = c + a^{-2}$ or one of the standard product surfaces in Theorems 1.7 and 1.8 [**1**], [**2**].

Suppose that p is a point in the set U. Choose a pseudo-orthonormal frame $\{X, Y\}$ on a normal neighborhood V_1 around p. Since the shape operator S is self-adjoint with trace $2a$, it satisfies

$$S(X) = aX + kY, \qquad S(Y) = jX + aY, \tag{3.1}$$

for some functions j and k on V_1. Hence, we have $|S|^2 = \mathrm{tr}\,(S^2) = 2(a^2 + jk)$, which implies that on V_1 the product function jk is a constant d. Thus the characteristics polynomial of S becomes

$$P_s(t) = (t-a)^2 - d. \tag{3.2}$$

If d is positive, then the shape operator S has two distinct real eigenvalues. In particular, it is diagonalizable at p, which is a contradiction. Suppose d is a negative constant $-b^2$. Then on V_1 the shape operator S has two complex eigenvalues $a \pm ib$. First we consider the case $c = 0$. Since the minimal polynomial $P_s(t)$ of S is constant on V_1, V_1 is an isoparametric surface in Magid's sense, and hence the shape operator cannot have a complex eigenvalue [**20**], Theorem 4.10. Now consider the case $c \neq 0$. We can choose an orthonormal frame $\{e_1, e_2\}$ on V such that the shape operator S satisfies

$$S(e_1) = ae_1 + be_2, \qquad S(e_2) = -be_1 + ae_2.$$

From the equations of Codazzi we see that the connection form ω_2^1 vanishes, and hence the neighborhood V is a flat Lorentzian surface in

H_1^3 with parallel second fundamental form in R_2^4. Therefore, V_1 is, up to congruences, an open part of a complex circle [**19**], which is not of finite type as we have already seen. These contradictions show that the product function $jk(=d)$ must vanish everywhere on V_1. From the hypothesis that p belongs to the set U, we may assume that $k(p) \neq 0$. Hence on a neighborhood V_2 of p, k is nonzero, and hence the function j must vanish there. Thus, on V_2, the shape operator S satisfies

$$S(X) = aX + kY, \qquad S(Y) = aY. \tag{3.3}$$

From (3.3) on V_2, we have

$$\overline{\nabla}_Y Y = \alpha Y, \tag{3.4}$$

where $\alpha = -\langle \nabla_Y Y, X\rangle$ and $\overline{\nabla}$ denotes the flat connection on the ambient pseudo-Euclidean space E_μ^n. Using the Codazzi equation, it can be shown from (3.3) and (3.4) that on V_2,

$$Y(k) = -2\alpha k. \tag{3.5}$$

Let $\gamma(s)$ be an integral curve of X starting from p. For each s, let $x(s,t)$ denote an integral curve of Y starting from $\gamma(s)$. Then $x(s,t)$ is a coordinate system of a neighborhood V around p. From (3.4), we see that

$$Y(x(s,t)) = f_s(t) Y(\gamma(s)), \tag{3.6}$$

where $f_s(t)$ is the positive function with $f_s(0)=1$, $f_s'(t)=\alpha(x(s,t))f_s(t)$. If we replace $\{X, Y\}$ by the pseudo-orthonormal frame $\{A, B\}$ defined by

$$A(x(s,t)) = f_s(t) X(x(s,t)), \qquad B(x(s,t)) = Y(\gamma(s)), \tag{3.7}$$

then on V, we have

$$S(A) = aA + hB, \qquad S(B) = aB, \tag{3.8}$$

where the nonzero function h is given by $h = kf_s^2$. Furthermore, (3.5), (3.6) and (3.7) imply that

$$\overline{\nabla}_B B = 0, \tag{3.9}$$

which shows that the null geodesic in the direction of B is a straight line segment and B is parallel along the line segment. Hence a neighborhood V of p consists of a one-parameter family of null straight lines. By the same argument as in the proof of (3.5), it follows from (3.8) and (3.9) that

$$B(h) = 0. \tag{3.10}$$

From (3.8) we see that, for a fixed unit normal N, the function $\langle \overline{\nabla}_A A, N\rangle = -h$ vanishes nowhere on V, and hence the null geodesic in the direction of B is the unique null geodesic line segment through p. For each p in U we denote by $l(p)$ the maximal null geodesic line segment through p.

We are going to see what happens when we extend this segment of null line. The following lemma shows that the extended line never meets the set D; either it ends at a boundary point of M_1^2 or stays indefinitely in U.

Lemma 3.1. *Let $l(p)$ be the maximal null geodesic line segment through a point $p \in U$. Then $l(p) \subset U$.*

Proof. We parametrize $l(p)$ by $p + tB(p)$. Suppose that the line $l(p)$ contains a point $q \in D$. Then there exists a point $p_0 = p + t_0 B(p)$ of $l(p)$ such that $p_0 \in D$ and the points of $l(p)$ with $t \in [0, t_0)$ belong to U. By the above results, we may extend $\{A, B\}$ to an open set containing the half open line segment pp_0 so that (3.8) and (3.9) hold there. Then (3.10) shows that the function h is constant on the half open line segment pp_0. Since $p_0 \in D$, by continuity, h must vanish on that line segment pp_0, which is a contradiction. □

For a point p in the boundary $\mathrm{bd}\,(U)$ of the set U, we prove the following.

Lemma 3.2. *Let $p \in \mathrm{bd}\,(U) \subset M_1^2$ be a point of the boundary of the set U. Then through p there passes a unique open segment of null line $l(p) \subset M_1^2$. Furthermore, $l(p) \subset \mathrm{bd}\,(U)$, that is, the boundary of U is formed by segments of null lines.*

Proof. Let $p \in \mathrm{bd}\,(U)$. On a neighborhood V around p, let $\{X, Y\}$ be a pseudo-orthonormal frame on V. Then the shape operator S satisfies (3.1) for some functions j and k on V. Furthermore, we see as above that on $V \cap U$ the product function jk vanishes everywhere, but j and k do not vanish simultaneously. Since p is a limit point of U, it is possible to choose a sequence $\{p_n\}$ in $V \cap U$ which converges to p as $n \to \infty$.

Without loss of generality, we may assume that there exists such a sequence $\{p_n\}$ as above with $k(p_n) \neq 0$, $n = 1, 2, \dots$. Then in a neighborhood of p_n, the function j vanishes; hence, the shape operator S satisfies (3.3) there. Put $\phi : (-\delta_1, \delta_1) \times W \to V$ be the C^∞ unique trajectory of Y with $\phi(0, q) = q$ in a neighborhood W of p. Then $\phi(t, p_n)$ is nothing but a parametrization of the null straight line segment $l(p_n)$ through p_n. This shows that $(\overline{\nabla}_Y Y)(\phi(t, p_n))$ is parallel to $Y(\phi(t, p_n))$ for each $n = 1, 2, \dots,$ and $|t| < \delta_1$. By letting $n \to \infty$, we see that $(\overline{\nabla}_Y Y)(\phi(t, p))$ and $Y(\phi(t, p))$ are parallel for all t with $|t| < \delta_1$. Thus $\phi(t, p)$ is a parametrization of the null line segment through p in the direction of Y.

Suppose that there exists another sequence $\{q_n\}$ in $V \cap U$ with $j(q_n) \neq 0$, $n = 1, 2, \dots,$ which converges to p as $n \to \infty$. Then, as before, we see that the unique trajectory $\psi(t, q_n)$ of X, $|t| < \delta_2$, converges to a line segment $\psi(t, p)$ through p. For sufficiently large n, the line segment $\phi(t, p_n)$ through p_n should meet the line segment $\psi(t, p)$ at a point q in V. This is a contradiction, because Lemma 3.1 shows that $\phi(t, p_n)$ and $\psi(t, p)$ belong to the sets U and D, respectively. This contradiction shows that the integral curve $\phi(t, p)$ of Y is a parametrization of the unique null geodesic line segment through p, which we will denote by $l(p)$.

Next we assert that every point of $l(p)$ on M_1^2 is a boundary point of U. In fact, if $q \in l(p)$, there exists a sequence $q_n = \phi(t, p_n)$ in U with $p_n \to p$, and hence $q_n \to q$ as $n \to \infty$. Thus q belongs to the closure of U. Assume that q does not belong to $\mathrm{bd}\,(U)$. Then $q \in U$ and $l(p)$ is the unique null geodesic line segment through q and hence that $p \in U$, a contradiction. □

Note that each component of $\mathrm{int}\,(D)$ is as follows.

1) If $c = 0$, $\mathrm{int}\,(D)$ is one of an open part of a cylinder $S_1^1(r) \times R$,

$R_1^1 \times S^1(r)$, with $r = 1/(2|a|)$ or a de Sitter space-time $S_1^2(y_0, r)$ with $r = 1/|a|$, $y_0 \in L^3$.

2) If $c \neq 0$, $\text{int}\,(D)$ is an open part of either a totally umbilic Lorentzian surface $L_1^2(y_0, c)$ in (2.3) or one of the standard product surfaces in Theorems 1.7 and 1.8. Since the cylinders and the product surfaces contain no null geodesic line segment, we see that each component of $\text{int}\,(D)$ is an open part of a de Sitter space-time $S_1^2(y_0, r)$ (in case $c = 0$) or a Lorentzian surface $L_1^2(y_0, c)$ of $\overline{M}_1^3(c)$ with $\langle y_0, y_0 \rangle = c + a^{-2}$ (in case $c \neq 0$).

Now we give the proof of the main theorems. It suffices to show that the theorems hold in a neighborhood of a point $p \in \text{bd}\,(U)$. Let p be a point in the boundary of U and $\{X, Y\}$ a pseudo-orthonormal frame in a neighborhood of p. Then the shape operator S satisfies (3.1). Without loss of generality, we may assume that the line segment $l(p)$ is in the direction of Y. Then the proof of Lemma 3.2 shows that there exists a neighborhood V of p such that $\overline{\nabla}_Y Y$ is parallel to Y on $V \cap U$. Since every null geodesic of $\text{int}\,(D)$ is a straight line segment, we see that $\overline{\nabla}_Y Y$ is parallel to Y in $V \cap \text{int}\,(D)$ and hence, by continuity, in the whole neighborhood V. This implies that in V the function $j = -\langle S(Y), Y \rangle$ vanishes, and hence the shape operator S satisfies (3.3) for some function k on V. Obviously, we have $k^{-1}(0) = V \cap D$.

Let $\gamma(s)$ be the integral curve of X through p and $A(s), B(s), C(s)$ the restrictions of X, Y, N along $\gamma(s)$, respectively. Then (3.3) shows that the Cartan frame $\{A(s), B(s), C(s)\}$ satisfies (2.2). Hence we see that in a neighborhood of p, M_1^2 can be parametrized by a B-scroll $x(s, t) = \gamma(s) + tB(s)$. This completes the proofs of Theorems A and B.

Let us consider a B-scroll $x(s, t) = \gamma(s) + tB(s)$ with $(s, t) \in R^2$. As we already have seen in Section 2, we may assume that $\langle \dot{A}(s), B(s) \rangle \equiv 0$, so that the metric tensor (g_{ij}) is given by

$$\begin{pmatrix} Kt^2 & -1 \\ -1 & 0 \end{pmatrix}.$$

Hence a non-flat B-scroll $x(s, t)$ is isometric to the parametrization $y(s, t)$ of the non-flat totally umbilic surface $L_1^2(y_0, c)$ defined by (2.4). Since $y(s, t)$ omits a null straight line of $L_1^2(y_0, c)$, it is not complete. Thus $x(s, t)$ is not complete either. On the other hand, a flat B-scroll is complete, because it is isometric to the Lorentz-Minkowski plane L^2.

Actually, our proofs show the following:

Theorem 3.3. *Let M_1^2 be a Lorentzian surface in the three-dimensional Lorentz-Minkowski space L^3. If the mean curvature and the Gaussian curvature are constant and the shape operator is not diagonalizable at a point, then M_1^2 is locally a B-scroll.*

Theorem 3.4. *Let M_1^2 be a Lorentzian surface in the three-dimensional non-flat Lorentzian space form $\overline{M}_1^3(c)$. If the mean and Gaussian curvatures are constant and the shape operator is not diagonalizable at a point, then M_1^2 is either a complex circle or locally a B-scroll.*

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Appendix

Let $\gamma(s)$ be a null curve in $\overline{M}_1^3(c) \subset E_\mu^4$ with a Cartan frame $\{A(s), B(s), C(s)\}$ with $\dot{\gamma}(s) = A(s)$. Obviously we have $(c, \mu) = (1,1), (-1,2)$. If we let $X(s)$ be the matrix $[A(s), B(s), C(s), \gamma(s)]$, with column vectors A, B, C and γ, then the 4×4 matrix $X(s)$ must satisfy the condition

$$X(s)^t E X(s) = T, \tag{4.1}$$

where $X(s)^t$ denotes the transpose of the matrix $X(s)$, $E = \operatorname{diag}(\nu_1, \nu_2, \nu_3, \nu_4)$, $\nu_i = -1$ for $1 \le i \le \mu$ and $\nu_i = 1$ for $\mu + 1 \le i \le 4$, and

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c \end{pmatrix}.$$

And (2.2) implies that

$$\begin{aligned} \dot{A}(s) &= k_1(s)A(s) - k(s)C(s), \\ \dot{B}(s) &= -\,k_1(s)B(s) - aC(s) + c\gamma(s), \\ \dot{C}(s) &= -\,aA(s) - k(s)B(s), \\ \dot{\gamma}(s) &= A(s), \end{aligned} \tag{4.2}$$

where $k(s)$ and $k_1(s)$ are continuous functions on the domain J of γ. On J, we put

$$M(s) = \begin{pmatrix} k_1(s) & 0 & -a & 1 \\ 0 & -k_1(s) & -k(s) & 0 \\ -k(s) & -a & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix},$$

then (4.2) is equivalent to the matrix equation:

$$\dot{X}(s) = X(s)M(s). \tag{4.3}$$

Since the matrix T and E satisfies $T^2 = E^2 = I$ where I denotes the identity matrix, it can be easily shown that a matrix X satisfies $X^tEX = T$ if and only if X satisfies $XTX^t = E$.

For any $X(0) = [A(0), B(0), C(0), \gamma(0)]$ satisfying $X(0)^tEX(0) = T$, there is a unique solution $X(s)$ of (4.3) with initial value $X(0)$. Furthermore, $X(s)$ is defined on the whole domain J of s. Since MT is skew symmetric, we see that $(d/ds)(X(s)TX(s)^t) = 0$. Thus $X(s)$ satisfies $X(s)TX(s)^t = E$ or equivalently, $X(s)^tEX(s) = T$. Therefore, the columns $\{A(s), B(s), C(s)\}$ of $X(s) = [A(s), B(s), C(s), \gamma(s)]$ is a desired Cartan frame along a null curve $\gamma(s)$ in $\overline{M}_1^3(c) \subset E_\mu^4$.

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