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UNIVALENT FUNCTIONS AND FREQUENCY ANALYSIS

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Dedicated to Professor William B. Jones on the occasion of his 70th birthday

ABSTRACT. The purpose of this article is to introduce an idea to an alternative method for solving the frequency analysis problem by using univalent functions. The method represents a bridge between univalent functions and frequency analysis. More specifically we want to present a way of using star-like functions in solving the frequency analysis problem.

1. Introduction. The frequency analysis problem is the problem of determining the unknown frequencies ω_j and amplitudes α_j in a trigonometric signal $x_N(m)$. The signal values from N observations are known.

An established method for solving the problem (with roots back to Wiener and Levinson) may be roughly described in the following way: From the signal values a certain absolutely continuous measure on the unit circle is constructed. This gives rise to an inner product, and in turn to moments and monic orthogonal polynomials (Szegö-polynomials) on the unit circle. Asymptotic values of some zeros of the polynomials then lead to the frequencies.

The methods (variations obtained by different choices of measures or modification of moments) are dealt with in [**2**]–[**4**], [**6**]–[**9**]. Throughout this article these methods all together will be referred to as the "Szegö polynomial method."

The purpose of this article is to introduce an alternative method to solve the problem by using a bridge between univalent functions and frequency analysis. More specifically we want to present a way of using star-like functions in solving the frequency analysis problem.

We begin with some definitions.

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Univalent functions. A function f analytic in the unit disk $\mathbf{D} = \{z : |z| < 1\}$ is said to be *univalent* in $\mathbf{D}$ if it does not take the same value twice, i.e.,

$$z_1 \neq z_2 \Longrightarrow f(z_1) \neq f(z_2), \quad z_1, z_2 \in \mathbf{D}.$$

The theory of univalent functions is largely concerned with the family S of functions f analytic and univalent in $\mathbf{D}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, thus having the form

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots, \quad z \in \mathbf{D}.$$

For our purposes, we will concentrate on the subfamily S^* of S consisting of star-like univalent functions. If $f \in S^*$ then $f(\mathbf{D})$ is star-like with respect to the origin O. This means that the line segment joining O to every other point $w \in f(\mathbf{D})$ is in $f(\mathbf{D})$.

The following result due to Nevanlinna [**5**] gives a useful analytic description of star-like functions f:

$$f \in S^* \Longleftrightarrow \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 0. \tag{1}$$

Any function, analytic in the unit disk and mapping $\mathbf{D}$ into the right halfplane, is called a Carathéodory function. The class of such functions φ where in addition $\varphi(0) = 1$, is denoted P. According to the Herglotz formula (see, for instance, [**1**, p. 22]) every $\varphi \in P$ can be represented as a Poisson-Stieltjes integral

$$\varphi(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t), \tag{2}$$

where $d\mu(t) \geq 0$ and $\int d\mu(t) = 1$.

Remarks.

- The function $(zf'(z)/f(z)) = F(z)$ when $f \in S^*$ is a Carathéodory function.
- Conversely, for each Carathéodory function F where $F(0) = 1$, there is a function $f \in S^*$ given by $F(z) = (zf'(z)/f(z))$.

Frequency analysis. A noiseless signal is received, assumed to be on the form

$$G(t) = \sum_{j=-I}^{I} A_j e^{2\pi i f_j t},$$

where t is the time and $f_j = -f_{-j}$, $j = 1, 2, \dots, I$, $f_0 = 0$, $A_j = A_{-j} \in \mathbf{R}$. Here, f_j are the frequencies and $|A_j|$ the amplitudes. The frequency analysis problem is to determine the unknown frequencies from signal values observed at times $m\Delta t$. Let $\omega_j = 2\pi f_j \Delta t$, where Δt is chosen such that we have reason to believe that $\omega_j < \pi$. With the assumption $A_0 = 0$ the signal may then be written in the form

$$x(m) = 2\sum_{j=1}^{I} A_j \cos m\omega_j. \tag{3}$$

The terms are arranged so that the normalized frequencies satisfy $0 < \omega_1 < \omega_2 < \cdots < \omega_I < \pi$.

In the method to be used we shall need a special normalization which, however, can be assumed without restriction of generality. Define $\alpha_j = A_j/K$ where $K > 0$ is such that

$$2\sum_{j=1}^{I} \alpha_j^2 = 1, \quad \text{i.e.,} \quad \alpha_j = \frac{A_j}{\sqrt{2\sum_{j=1}^{I} A_j^2}}.$$

The given signal (3) and the signal

$$\frac{x(m)}{K} =: s(m) = 2\sum_{j=1}^{I} \alpha_j \cos m\omega_j \tag{4}$$

then only differ by the factor $K = \sqrt{2\sum_{j=1}^{I} A_j^2}$.

Let, for any positive integer N, $\psi^{(N)}(\theta)$ be the positive measure on the unit circle $\partial\mathbf{D} = \{z : |z| = 1\}$ given by

$$\frac{d\psi^{(N)}(\theta)}{d\theta} = \frac{1}{2\pi}\left|\sum_{m=0}^{N-1} x(m) e^{-mi\theta}\right|^2. \tag{5}$$

In the next section we will use this ψ with $s(m)$ instead.

A crucial point in the theory of frequency analysis is the weak convergence of the measure (5) divided by N as N tends to infinity. Then the measure will converge in the weak $*$-topology towards a discrete measure with support at the points $\theta = \omega_j,\ j = 1, 2, \dots, I$. The weights are A_j^2 at these points.

2. The main result.

Theorem 1. *Given a trigonometric signal of the form*

$$s(m) = 2\sum_{j=1}^{I} \alpha_j \cos m\omega_j,$$

$$\alpha_j \in \mathbf{R}, \quad 0 < \omega_1 < \omega_2 < \cdots < \omega_I < \pi \quad \textit{and} \quad 2\sum_{j=1}^{I} \alpha_j^2 = 1.$$

Let

$$\frac{d\psi^{(N)}(\theta)}{d\theta} = \frac{1}{2\pi}\left|\sum_{m=0}^{N-1} s(m)e^{-\mathrm{i}m\theta}\right|^2 \tag{6}$$

be a positive measure on $\partial\mathbf{D}$ *and* N *be any positive integer. Finally, let*

$$F^{(N)}(z) = \frac{1}{N}\,\mu_0^{(N)} + 2\sum_{m=1}^{\infty} \frac{1}{N}\,\mu_m^{(N)} z^m$$

be the Carathéodory function with $\psi^{(N)}$ *as the representing measure, and let* $f^{(N)}(z)$ *be defined by*

$$\frac{zf^{(N)\prime}(z)}{f^{(N)}(z)} = F^{(N)}(z).$$

Then

$$\lim_{N\to\infty} f^{(N)}(z) =: f(z) = \frac{z}{\prod_{j=1}^{I}(1 - 2z\cos\omega_j + z^2)^{2\alpha_j^2}} \in S^*. \tag{7}$$

Before proving the theorem, we comment on the mapping properties of the function (7):

By letting $z = e^{i\theta}$, $\theta \in [0, 2\pi\rangle$, (7) may be written in the form

$$f(e^{i\theta}) = \frac{1}{2(\cos\theta - \cos\omega_1)^{2\alpha_1^2}(\cos\theta - \cos\omega_2)^{2\alpha_2^2} \cdots (\cos\theta - \cos\omega_I)^{2\alpha_I^2}}.$$

- $\theta \in [0, \omega_1\rangle$: $f(e^{i\theta}) \in \mathbf{R}^+$, with the smallest value for $\theta = \theta_0 = 0$ and $f(z) \to \infty$ as $\theta \to \omega_1$.
- $\theta \in [\omega_1, \omega_2\rangle$: $f(e^{i\theta})$ is on the slit with angle $2\pi\alpha_1^2$ with the positive real axis, starting at infinity for $\theta = \omega_1$. By Rolle's theorem there is at least one extremum $\theta = \theta_1$ in the interval. By differentiating we find that there is only one, giving a minimum for f: the slit endpoint of the interval. The argument is similar for all intervals $[\omega_i, \omega_{i+1}\rangle$, $i = 2, 3, \dots, I-1$.
- $\theta \in [\omega_I, \pi]$: $f(e^{i\theta}) \in \mathbf{R}^-$ since $\sum_{j=1}^{I} 2\pi\alpha_j^2 = \pi$. In this interval $f(e^{i\theta})$ starts at infinity for $\theta = \omega_I$, moving towards $f(-1)$ for $\theta = \theta_I = \pi$.

The function is symmetric about the real axis, thus its behavior in the interval $\langle\pi, 2\pi\rangle$ is given and $f(1)$, $f(-1)$ are slit endpoints.

Example 1. Let $I = 3$ with $\omega_1 = \pi/6$, $\omega_2 = \pi/2$, $\omega_3 = 3\pi/4$ and $\alpha_1 = \sqrt{3}/4$, $\alpha_2 = 1/2$, $\alpha_3 = 1/4$. Then

$$f(e^{i\theta}) = \frac{1}{2(\cos\theta - (\sqrt{3}/2))^{3/8}(\cos\theta)^{1/2}(\cos\theta + (\sqrt{2}/2))^{1/8}}. \tag{8}$$

This function maps $\mathbf{D}$ onto the complement of the slits given in Figure 1.

This is an exact illustration as we know the function $f(z) \in S^*$ and are able to map the whole unit disk $|z| < 1$. In the figure we count the number of angular openings in the upper halfplane to obtain the number I of frequencies. The amplitudes are given by the angles between the rays if we divide by 2π and take the square root of the result. Frequencies are obtained by one of the methods in Section 4.

In a practical case we cannot for obvious reasons, use $|z| < 1$. Thus, we map $|z| < R$ for some $R < 1$ or by the maximum principle $|z| = R$.

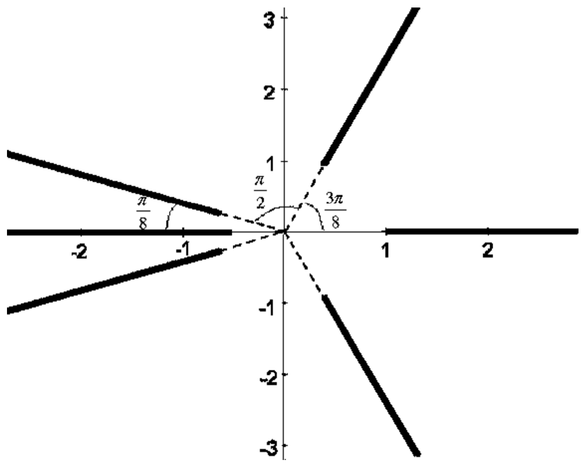

FIGURE 1.

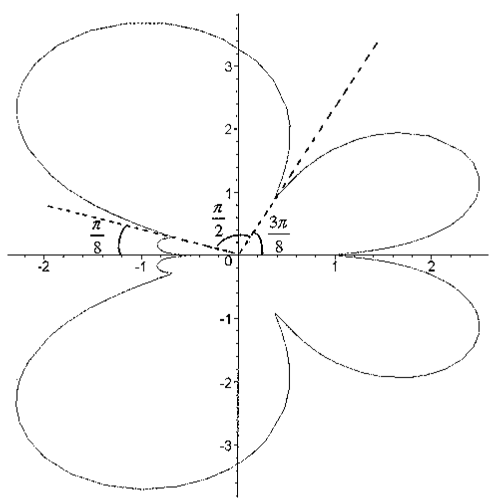

FIGURE 2.

Then the angular openings will be replaced by "bumps," and the number of bumps in the upper halfplane will give I. With $R = 0.98$, the function (8) produces the illustration in Figure 2. It is important that R is chosen sufficiently large to produce bumps.

We will return to this example later.

Proof. Based upon the signal (4) we construct a positive measure as given in (6). This measure gives rise to moments of the form

$$\frac{1}{N}\,\mu_m^{(N)} = \int_0^{2\pi} e^{-\mathrm{im}\theta}\, d\bigg(\frac{\psi^{(N)}(\theta)}{N}\bigg),$$

which may be rewritten in the form

$$\frac{1}{N}\,\mu_m^{(N)} = \frac{1}{N}\sum_{k=0}^{N-m-1} s(k)s(k+m), \quad m = 0, 1, \dots \tag{9}$$

for practical use (see, e.g., [**2**]). By the Riesz-Herglotz theorem (2) with our measure and moments at hand, the corresponding Carathéodory function takes the form

$$F^{(N)}(z) = \frac{1}{N}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\psi^{(N)}(\theta) = \frac{1}{N}\,\mu_0^{(N)} + 2\sum_{m=1}^{\infty} \frac{1}{N}\,\mu_m^{(N)} z^m.$$

When $N \to \infty$, the weak convergence of the measure then gives the limit moments $\mu_m := \lim_{N\to\infty} \frac{1}{N}\mu_m^{(N)} = \sum_{j=-I}^{I} e^{-im\omega_j}\alpha_j^2 = 2\sum_{j=1}^{I} \alpha_j^2 \cos m\omega_j$, in particular $\mu_0 = 2\sum_{j=1}^{I} \alpha_j^2 = 1$, and a limit Carathéodory function

$$\begin{aligned} F(z) := \lim_{N\to\infty} F^{(N)}(z) &= \sum_{j=-I}^{I} \frac{e^{i\omega_j}+z}{e^{i\omega_j}-z}\cdot \alpha_j^2 \\ &= 1 + 2\sum_{j=1}^{I} \frac{2\cos\omega_j z - 2z^2}{1-2\cos\omega_j z + z^2}\cdot \alpha_j^2 \\ &= \mu_0 + 2\sum_{m=1}^{\infty} \mu_m z^m, \end{aligned}$$

where in particular μ_0 is equal to 1. The corresponding star-like function $f \in S^*$ is, as in (1), given by the relation

$$\frac{zf'(z)}{f(z)} = F(z).$$

A simple calculation gives the function in (7). □

Comment. For an arbitrary signal we form the "moment" sequence by using the autocorrelation formulas, and divide by $\sum_{k=1}^{N} x(k)^2$ for normalization, i.e., the "moments" are

$$\widetilde{\mu}_m^{(N)} = \frac{\sum_{k=1}^{N} x(k)x(k+m)}{\sum_{k=1}^{N} x(k)^2}.$$

Important then is that we later return to the original amplitudes of the signal as follows:

Assume that the α_j-values are found, except possibly for signs (see next section). Let K be given by equation (4), i.e.,

$$K \cdot 2 \sum_{j=1}^{I} \alpha_j = x(0).$$

Then, the coefficients in the signal are

$$A_j = K \cdot \alpha_j = \frac{x(0)}{2\sum_{j=1}^{I} \alpha_j} \cdot \alpha_j.$$

3. Indication of an application I. How to find I and α_j. If we, somehow, can come from observations of the signal to $f(z)$ in S^*, it is easy to determine I and α_j^2: I is the number of angular openings in the upper halfplane, $2\alpha_j^2$ is the angle between the rays divided by π.

This is the ideal situation. In a practical case we have to do differently. Rather than letting $N \to \infty$, we solve the differential equation

$$\frac{f^{(N)\prime}(z)}{f^{(N)}(z)} - \frac{1}{z} = \frac{F^{(N)}(z) - 1}{z}$$

with the initial conditions $f^{(N)}(0) = 0$ and $f^{(N)\prime}(0) = 1$. The solution is the function

$$f^{(N)}(z) = z \exp\left(\sum_{m=1}^{\infty} \frac{2\widetilde{\mu}_m^{(N)}}{m} z^m\right). \tag{10}$$

In this expression we are forced to use a large value of N rather than letting N tend to infinity. Moreover, in the two sums involved—the sum *in* the exponent of $f^{(N)}$ and the sum *of* the exponent expression—we replace infinite sums by finite sums. The method to be presented is not completely developed, it is more a sketch of an idea. A careful discussion of errors caused by these substitutions is beyond the purpose of this paper.

A proper choice of the R value in $|z| = R$ is important.

Example 2. Unfortunately we have no "real signal" coming in to our computer. We have chosen the same signal as in Example 1.

$$s(m) = 2\sum_{j=1}^{I} \alpha_j \cos m\omega_j = \frac{\sqrt{3}}{2}\cos\left(\frac{\pi}{6}m\right) + \cos\left(\frac{\pi}{2}m\right) + \frac{1}{2}\cos\left(\frac{3\pi}{4}m\right)$$

for which $I = 3$, but pretend not to know it as soon as the moments are calculated. The computation is done by using MAPLE.

We first compute the autocorrelation coefficients in (9) as we choose to replace N by 10^6. In the program, these coefficients are called Am. By letting m be equal to 0 in Am we have the normalization coefficient to divide upon as commented in Section 2. For our signal, $A0$ is nearly 1 since the signal is of the "right form" already. Next, we use $Am/A0$ as a substitute for the moment limits μ_m to produce the function (10) with $\sum_1^\infty$ replaced by $\sum_1^{100}$. This function gives, with $R = 0.98$, the following figure.

Figure 3 strongly indicates that $I = 3$ (number of bumps in the upper halfplane). By measuring the angles $r_j \approx 2\alpha_j^2\pi$ we find that $r_1 = 67.8^\circ$, $r_2 = 90.4^\circ$ and $r_3 = 21.8^\circ$. This strongly indicates that the exact values must be $3\pi/8$, $\pi/2$ and $\pi/8$. This coincides with the values we know. We will return to the frequencies later.

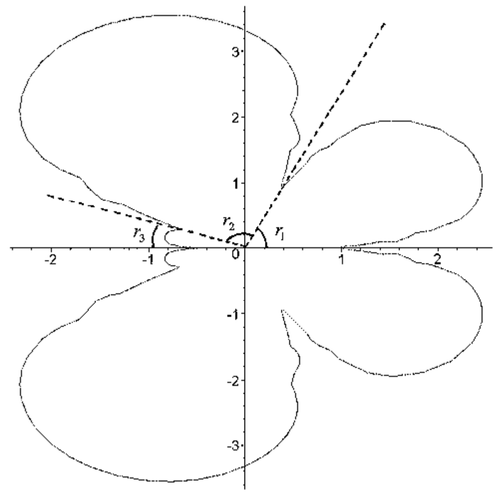

FIGURE 3.

Comments.

• *Signs of the amplitudes.* By measuring the angles on the figures obtained, we find $|\alpha_j|$. To get the correct signs on these amplitudes, we have to use all combinations of signs; 2^I possibilities, sum each of these combinations and compare the results with $s(1)$. This usually tells us the one and only correct combination of signs. But if more than one combination gives the wanted result, then we have to do the same compared with $s(2)$ and so on. Soon only one combination, with the correct signs, is left. We remark that this method leading to the signs is assumed to be done *after* we know the frequencies of the signal, Section 4. For practical reasons it is mentioned here.

• *Remarks on errors.* As mentioned earlier, our estimates produce errors, in particular of the following three types.

1. In $(1/N)\mu_m^{(N)}$ we use a large N instead of $N \to \infty$. Estimated error here is known to be $O(1/N)$.

2. In the exponent of $f^{(N)}(z)$ we use a high order Taylor polynomial rather than an infinite series. A rough upper bound for this estimated

error is (since $|c_m| \le 1$ for all Carathéodory functions $c_0 + 2\sum c_m z^m$)

$$\left| \sum_{m=q+1}^{\infty} \frac{2c_m}{m} z^m \right| \le \frac{2}{q+1} \cdot \frac{R^{q+1}}{1-R}.$$

3. For the function $f^{(N)}(z)$ itself, we use a Taylor polynomial of order 100 instead of an infinite series. To illustrate the accuracy of the approximated function (10) with $N = 10^6$ and upper limit 100, we include the first 5 terms of its Taylor expansion;

$$\begin{aligned} f^{(100)}(z) = z &+ 0.47273832z^2 - 0.20076003z^3 \\ &- 0.07119710z^4 + 0.13759672z^5 + O(z^6). \end{aligned}$$

Compared to the Taylor expansion of the exact function (7) with α_jś and ω_j's as in Example 1

$$\begin{aligned} f(z) = z &+ 0.47274235z^2 - 0.20075733z^3 - 0.07119792z^4 + 0.13759622z^5 \\ &+ O(z^6), \end{aligned}$$

we see that the expressions are comparable. Thus, (10) with the restrictions gives an estimated result which is good enough for our purposes. As long as we are able to use its mapping result to find the number of frequencies and sizes of amplitudes, we are satisfied with the figure not being exact. Of course, if the amplitudes are small or two or more frequencies are close, the bumps will be difficult to separate. But it is possible to zoom in the graph to make estimated counts.

4. Indication of an application II: How to determine ω_j. We give a list of alternative methods:

i) *Combine with the Szegö polynomial method.* There are developed methods for how to solve the frequency analysis problem by using Szegö polynomials (see [**2**–**4**], [**6**–**9**]). In these methods, one usually does not know the number n_0 of frequencies throughout the argument. Thus, one has to pick an n assumed to be larger than n_0 and use this n in the discussion. This leads to the unknown frequencies, but in addition there are $n - n_0$ points as "leftovers," called uninteresting zeros. If we combine the methods, the S^*-method produces the exact number of frequencies n_0. Then, with this n_0, the Szegö polynomial method can derive the unknown frequencies without struggling with uninteresting zeros.

ii) *Solve equations* I. Solve the simultaneous set of equations

$$s(1) = 2\sum_{j=1}^{I} \alpha_j \cos\omega_j$$
$$s(2) = 2\sum_{j=1}^{I} \alpha_j \cos 2\omega_j \quad \text{etc.}$$

in which $s(1), s(2), \ldots, \alpha_1, \ldots, \alpha_I$ are known. With $\cos\omega_j = x_j$ we have $\cos m\omega_j = T_m(x_j)$ where T_m is a polynomial. The equations are thus made algebraic. Hereby we get the unknown frequencies.

iii) *Solve equations* II. Same procedure as in ii), but replace some of the equations with moment equations; see, for instance, (12) in Example 3.

Example 3. We use the same values of the α_j's and ω_j's as in Example 1 giving the signal values $s(1) = 0.3964466095$, $s(2) = -0.5669872980$, $s(3) = 0.3535533905$ and the moment value $\tilde{\mu}_1^{(N)} \approx \mu_1 = 0.2363711787$. In a practical case these values are measured. Then we forget the frequencies while the amplitudes are known from the earlier method.

If we let $\cos\omega_1 = x$, $\cos\omega_2 = y$ and $\cos\omega_3 = z$, thus $1 > x > y > z > -1$ by the assumptions in Theorem 1, the equations in (ii) take the form

(11)
$$2\cdot\frac{\sqrt{3}}{4}x + 2\cdot\frac{1}{2}y + 2\cdot\frac{1}{4}z = 0.3964466095$$
$$2\cdot\frac{\sqrt{3}}{4}(2x^2-1) + 2\cdot\frac{1}{2}(2y^2-1) + 2\cdot\frac{1}{4}(2z^2-1) = -0.5669872980$$
$$2\cdot\frac{\sqrt{3}}{4}(4x^3-3x) + 2\cdot\frac{1}{2}(4y^3-3y) + 2\cdot\frac{1}{4}(4z^3-3z) = 0.3535533905.$$

For the method in (iii) the first two equations in (11) are the same, while the third one is replaced with

$$2 \cdot \frac{1}{8}x + 2 \cdot \frac{1}{2}y + 2 \cdot \frac{1}{8}z = 0.2363711787. \tag{12}$$

We solve the systems of equations (11) or (12) by using MAPLE. Each calculation gives many solutions, but only one satisfies the order $1 > x > y > z > -1$:

For method (ii) this solution is

$$x = 0.8660254038,\ y = -0.4258130110 \cdot 10^{-10},\ z = -0.7071067812,$$

for method (iii) where the third equation is replaced with (12), this solution is

$$x = 0.8660254032,\ y = 0.9550022899 \cdot 10^{-9},\ z = -0.7071067823.$$

Both cases strongly indicate that $\omega_1 = \pi/6$, $\omega_2 = \pi/2$, $\omega_3 = 3\pi/4$.

Note. The methods described in this example do not distinguish between amplitudes of different signs. If we, for instance, replace the signs $+++$ assumed in Examples 1 and 2, by $+--$, the set of equations (11) will be

$$\begin{aligned} 2 \cdot \frac{\sqrt{3}}{4}x - 2 \cdot \frac{1}{2}y - 2 \cdot \frac{1}{4}z &= 1.103553390 \\ 2 \cdot \frac{\sqrt{3}}{4}(2x^2 - 1) - 2 \cdot \frac{1}{2}(2y^2 - 1) - 2 \cdot \frac{1}{4}(2z^2 - 1) &= 1.433012702 \\ 2 \cdot \frac{\sqrt{3}}{4}(4x^3 - 3x) - 2 \cdot \frac{1}{2}(4y^3 - 3y) - 2 \cdot \frac{1}{4}(4z^3 - 3z) &= 0.3535533905. \end{aligned}$$

which gives the solution

$$x = 0.8660254036,\ y = 0.4742328749 \cdot 10^{-9},\ z = -0.7071067808.$$

As we see, it leads to the same frequencies.

iv) *Use of slit endpoints.* By measuring the distances from the origin to the slit endpoints in $f^{(N)}(\mathbf{D})$, we are able, more theoretically, to

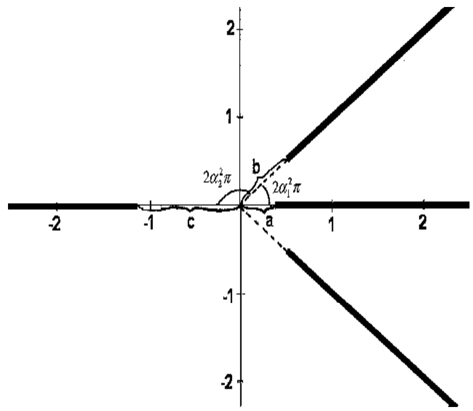

FIGURE 4.

determine the unknown frequencies. We include a simple example for which $I = 2$ and the exact function (7) is used, just to illustrate the method.

Example 4. In this example we have picked two amplitudes $\alpha_1 = \sqrt{2}/4$ and $\alpha_2 = \sqrt{6}/4$, and two frequencies $\omega_1 = \pi/3$ and $\omega_2 = 3\pi/4$, to produce the slit mapping region in Figure 4. Then we pretend not to know these quantities as soon as the region is drawn.

By the method presented in Section 3, we find that $I = 2$ and $\alpha_1 = \sqrt{2}/4$ and $\alpha_2 = \sqrt{6}/4$. The function (7) with $z = e^{i\theta}$ then takes the form

$$f(e^{i\theta}) = \frac{1}{2(\cos\theta - \cos\omega_1)^{1/4}(\cos\theta - \cos\omega_2)^{3/4}}.$$

Elementary calculus leads to the θ-value representing the endpoint on the $\pi/4$-angle slit. This is given for $\theta = \theta_1$ where $\cos\theta_1 = (3/4)\cos\omega_1 + (1/4)\cos\omega_2$. Let a, b, c be the distances from the origin to the endpoints of the slits, as shown in Figure 4. By the uniqueness of the Riemann mapping theorem the third endpoint is determined when the two others are given. In our example we determine $a \approx 0.41$, $b \approx 0.73$ and get the equations

$$\frac{1}{2(1-\cos\omega_1)^{1/4}(1-\cos\omega_2)^{3/4}} = 0.41,$$
$$\frac{1}{2\left(\frac{1}{4}\right)^{1/4}\left(\frac{3}{4}\right)^{3/4}(\cos\omega_1 - \cos\omega_2)} = 0.73.$$

We solve this system with MAPLE and find that the only possible solution is

$$\cos\omega_1 = 0.5025215490 \quad \text{and} \quad \cos\omega_2 = -0.6993725269,$$

i.e., $\omega_1 \approx \pi/3$ and $\omega_2 \approx 3\pi/4$.

This method depends upon careful measuring of the distances to the slit endpoints. If the figure is an estimate as in Example 2, the result can become rather inaccurate. Also for this method, calculations become more complicated for larger number of frequencies.

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REFERENCES

1. P. Duren, *Univalent functions*, Springer-Verlag, 1983.

2. W.B. Jones, O.Njåstad, W.J. Thron and H. Waadeland, *Szegö polynomials applied to frequency analysis*, J. Comput. Appl. Math. **46** (1993), 217–228.

3. W.B. Jones, O.Njåstad and H. Waadeland, *An alternative way of using Szegö polynomials in frequency analysis*, in *Continued fractions and orthogonal functions* (S.C. Cooper and W.J. Thron, eds.), Marcel Dekker, New York, 1994, 141–152.

4. W.B. Jones and V. Peterson, *Continued fractions and Szegö polynomials in frequency analysis and related topics*, Acta Appl. Math. **61** (2000).

5. R. Nevanlinna, *Über die konforme Abbildung von Sterngebieten*, Overview by Finska Vetenskaps Soc. Forh. **63** (A), no. 6, (1920-1921), 1–21.

6. O. Njåstad and H. Waadeland, *General Szegö theory in frequency analysis*, J. Math. Anal. Appl. **206** (1997), 280–307.

7. ———, *Asymptotic properties of zeros of orthogonal rational functions*, J. Comput. Appl. Math. **77** (1997), 255–275.

8. K. Pan and E.B. Saff, *Asymptotics for zeros of Szegö polynomials associated with trigonometric polynomial signals*, J. Approx. Theory **71** (1992), 239–251.

9. V. Petersen *On measures in frequency analysis*, J. Comp. Appl. Math. **105** (1999), 437–443.

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