GLOBAL ATTRACTIVITY IN
A GENOTYPE SELECTION MODEL

XIAN HUA TANG AND SUI SUN CHENG

ABSTRACT. We obtain a sufficient condition for global attractivity to occur in the delay difference equation

$$x_{n+1} = x_n \exp(\beta_n(1 - x_{n-\tau})/(1 + x_{n-\tau})).$$

This leads to the fact that when $0 < \beta \le 3/(\tau + 1)$, the positive equilibrium $1/2$ of the genotype selection model

$$y_{n+1} = \frac{y_n e^{\beta(1-2y_{n-\tau})}}{1 - y_n + y_n e^{\beta(1-2y_{n-\tau})}}$$

is a global attractor for all solutions originated from positive initial conditions. Our result matches the computational result $0 < \beta \le 4\cos(\tau\pi/(2\tau + 1))$ suggested in [3].

1. Introduction. In [1, pp. 513–563], May proposed a genotype selection model of the form

$$y_{n+1} = \frac{y_n e^{\beta(1-2y_n)}}{1 - y_n + y_n e^{\beta(1-2y_n)}}, \quad n = 0, 1, 2, \ldots,$$

and investigated the local stability of the equilibrium solution $\{y_n\} = \{1/2\}$. Later in [2], Grove et al. showed that the equilibrium $1/2$ is globally asymptotically stable if $0 < \beta \le 4$, and unstable if $\beta > 4$. In the same paper [2], a positive integer delay τ is introduced into the above model to form

$$(1.1) \qquad y_{n+1} = \frac{y_n e^{\beta(1-2y_{n-\tau})}}{1 - y_n + y_n e^{\beta(1-2y_{n-\tau})}}, \quad n = 0, 1, 2, \ldots,$$

and it is shown that the equilibrium $1/2$ is locally asymptotically stable if $0 < \beta < 4\cos(\tau\pi/(2\tau + 1))$, and unstable if $\beta > 4\cos(\tau\pi/(2\tau + 1))$.

Furthermore, when $\tau \geq 1$, this equilibrium is a global attractor for all solutions $\{y_n\}_{n=-\tau}^{\infty}$ which satisfies the initial conditions

$$(1.2) \qquad\qquad 0 < y_{-\tau}, y_{-\tau+1}, \ldots, y_0 < 1,$$

if $0 < \beta \leq (2/\tau)$. On the basis of computer simulations, the authors of [2] also observe that the condition $0 < \beta \leq (2/\tau)$ is far from perfect, and therefore it is desirable to establish a better result which matches experimental conditions.

In this paper, we will show that when $\tau > 2$, the much improved condition $0 < \beta \leq (3/(\tau+1))$ can be established. To achieve our goal, it suffices to find a condition which implies that the solution $\{1\}$ is the global attractor of all solutions $\{x_n\}_{n=-\tau}^{\infty}$ of the nonautonomous delay difference equation

$$(1.3) \quad x_{n+1} = x_n \exp\left(\frac{\beta_n(1 - x_{n-\tau})}{1 + x_{n-\tau}}\right), \quad \beta_n > 0, \quad n = 0, 1, 2, \ldots,$$

originated from positive initial conditions $x_{-\tau}, \ldots, x_0 \in (0, \infty)$. Indeed, this can be seen by making the substitution [2]

$$x_n = \frac{y_n}{1 - y_n},$$

which transforms (1.1) into

$$(1.4) \qquad x_{n+1} = x_n \exp\left(\frac{\beta(1 - x_{n-\tau})}{1 + x_{n-\tau}}\right), \quad n = 0, 1, 2, \ldots,$$

and (1.2) into $x_{-\tau}, \ldots, x_0 \in (0, \infty)$.

2. Auxiliary inequalities. Our investigations lead to the following coupled pair of inequalities

$$(2.1) \qquad\qquad\qquad \ln u \leq 2\left(\frac{1 - v}{1 + v}\right),$$

$$(2.2) \qquad\qquad\qquad \ln v \geq 2\left(\frac{1 - u}{1 + u}\right).$$

Clearly, $(u, v) = (1, 1)$ is a solution of (2.1)–(2.2). We will need the fact that there are no other solutions in the region

$$\Phi = \{(u, v)| \, 0 < v \leq 1 \leq u\} \, .$$

Lemma 1. $(1, 1)$ *is the only solution of* (2.1)–(2.2) *in* Φ.

Proof. Set

$$g(x) = \exp\left(\frac{2(1 - x)}{1 + x}\right), \quad x > -1.$$

Let (u, v) in Φ be a solution of (2.1)–(2.2), then

$$g(u) \leq v \leq 1 \leq u \leq g(v) \leq g(g(u)),$$

where the last inequality follows from the fact that g is strictly decreasing on $(-1, \infty)$. We assert that $g(g(u)) < u$ for $u > 1$. Indeed, this follows from the fact that the function

$$f(x) = x - g(g(x)), \quad x \geq 1,$$

has the derivative

$$f'(x) = \frac{(1 + x)^2 (1 + g(x))^2 - 16g(x)g(g(x))}{(1 + x)^2 (1 + g(x))^2},$$

and the numerator $h(x)$ in the righthand side of the above expression satisfies $h(1) = 0$ and

$$h'(x) = 2\left(1 + g(x)\right)\left(\left(1 + x\right)\left(1 + g(x)\right) - 4g(x)\right)$$
$$+ \frac{64}{(1 + x)^2} \, g(x)g(g(x)) \frac{(1 - g(x))^2}{(1 + g(x))^2}$$
$$> 0$$

for $x > 1$. Thus we have $u \leq g(v) \leq g(g(u)) < u$, which is a contradiction. By symmetry considerations, v cannot be greater than 1 either. The proof is complete. $\square$

We will need three other inequalities which can be proved by looking at the related first and second derivatives.

Lemma 2. (i) *For any* $v \in [0,1)$, $\ln\left(2e^{-v(1-\frac{v}{2})} - 1\right) \geq -2v$.

(ii) *For any* $u \in [0,\infty)$, $\ln\left(2e^{u(1+\frac{u}{2})} - 1\right) \geq 2u$.

(iii) *For any* $v \in (0,1)$,

$$(2.3) \quad \ln \frac{1 + \left(2e^{-v(1-\frac{v}{2})} - 1\right)e^{-vx}}{1 + e^{-vx}} \leq -v\left(1 - \frac{v}{2}\right) + \frac{v^2}{2}x, \quad x \geq 0.$$

The first statement can be seen by showing the function $f(v) = 2e^{-v(1-\frac{v}{2})} - e^{-2v}$ is nondecreasing on $[0,1)$ and $f(0) = 1$, while the second statement follows from showing $g(u) = 2e^{u(1+\frac{u}{2})} - e^{2u}$ is nondecreasing on $[0,\infty)$ and $g(0) = 1$. To see that the third statement holds, we let $h(x)$ be the lefthand side of (2.3) and then show that $h''(x) \leq 0$ for $x \geq 0$. Then

$$h(x) = h(0) + h'(0)x + \frac{h''(\xi)}{2}x^2 \leq h(0) + h'(0)x$$

$$= -v\left(1 - \frac{v}{2}\right) + \frac{vx}{2}\left(e^{v(1-\frac{v}{2})} - 1\right)$$

$$\leq -v\left(1 - \frac{v}{2}\right) + \frac{v^2}{2}x,$$

where the last inequality follows from $e^{x(1-\frac{x}{2})} \leq 1 + x$ for $x \geq 0$.

3. Global attractivity. Our main objective is to show the following result.

Theorem 1. *Suppose* $\{\beta_n\}_{n=0}^{\infty}$ *is a positive sequence which satisfies*

$$(3.1) \qquad\qquad \sum_{i=n-\tau}^{n} \beta_i \leq 3$$

for all large n, and

$$(3.2) \qquad\qquad \sum_{i=0}^{\infty} \beta_i = \infty.$$

Then every solution $\{x_n\}_{n=-\tau}^{\infty}$ *of* (1.3) *under the initial conditions* $x_{-\tau}, \dots, x_0 > 0$ *will tend to* 1.

Proof. Since every solution $\{x_n\}_{n=-\tau}^{\infty}$ of (1.3) that satisfies $x_{-\tau}, \dots,$ $x_0 > 0$ also satisfies $x_n > 0$ for $n \geq 1$, in order to prove the about result, we need to consider three kinds of solutions. The first kind consists of positive solutions that is eventually strictly bounded below by 1, and the second kind consists of positive solutions that is eventually strictly bounded above by 1. The third kind consists of positive solutions that oscillate about 1. It is easy to show that if (3.2) holds, then the first and the second kind of solutions will tend to 1. Indeed, let $\{x_n\}$ be a positive solution which is of the first kind. Then in view of (1.3), $\{x_n\}$ is decreasing and thus tends to a number $L \geq 1$. Furthermore,

$$\ln x_n - \ln x_{n+1} = \beta_n \frac{x_{n-\tau} - 1}{1 + x_{n-\tau}} \geq \beta_n \frac{L-1}{1+L}$$

for all large n. Summing the above inequality, we have

$$\ln x_n - \ln L \geq \frac{L-1}{1+L} \sum_{i=n}^{\infty} \beta_i,$$

which, together with (3.2), imply $L = 1$. The case where $\{x_n\}$ is a type two solution is similarly proved.

We will show that the same conclusion holds for the third kind of solutions. To see this, let $\{x_n\}_{n=-\tau}^{\infty}$ be a solution of (1.3) which oscillates about 1. We first show that $\{x_n\}$ is bounded above and strictly bounded below by 0. Pick a positive integer N so large that (3.1) holds for $n \geq N$. Let $n^* \geq N+\tau$ such that $x_{n^*+1} \geq \max\{x_{n^*}, 1\}$. Then in view of (1.3), $x_{n^*-\tau} \leq 1$, and thus

$$x_{n^*+1} = x_{n^*-\tau} \exp\left(\sum_{i=n^*-\tau}^{n^*} \beta_i \frac{1 - x_{i-\tau}}{1 + x_{i-\tau}} \right) \leq \exp\left(\sum_{i=n^*-\tau}^{n^*} \beta_i \right) \leq e^3.$$

It follows that $\limsup_{n\to\infty} x_n \leq e^3$. Next, let $n_* \geq N + 3\tau$ such that $x_{n_*+1} \leq \min\{1, x_{n_*}\}$. Then in view of (1.3), $x_{n_*-\tau} \geq 1$, and thus

$$x_{n_*+1} = x_{n_*-\tau} \exp\left(\sum_{i=n_*-\tau}^{n_*} \beta_i \frac{1 - x_{i-\tau}}{1 + x_{i-\tau}} \right) \geq \exp\left(\frac{3(1 - e^3)}{1 + e^3} \right).$$

Hence $\liminf_{n\to\infty} x_n \geq \exp\left(3(1 - e^3)/(1 + e^3)\right) > 0$.

Let $u = \limsup_{n\to\infty} x_n$ and $v = \liminf_{n\to\infty} x_n$. Since $\{x_n\}$ oscillates about 1, $0 < v \leq 1 \leq u < \infty$. Our proof will be complete if we can show that $u = v = 1$. Set

$$x(t) = x_n \left(\frac{x_{n+1}}{x_n}\right)^{t-n}, \quad \beta(t) = \beta_n, \quad n \leq t < n+1, \quad n = 0, 1, 2, \ldots.$$

Then $x(t)$ is continuous on $[0, \infty)$ and satisfies $x(n) = x_n$ for $n = 0, 1, 2, \ldots$. It is easy to see that $x(t)$ is monotonic on each interval $[n, n+1]$, which implies that

$$(3.3) \qquad\qquad u = \limsup_{t\to\infty} x(t), \quad v = \liminf_{t\to\infty} x(t).$$

Let $[\cdot]$ denote the greatest integer function. Then $x(t)$ satisfies the following equation with piecewise constant argument

$$(3.4) \qquad\qquad x'(t) = \beta(t)x(t)\frac{1 - x([t - \tau])}{1 + x([t - \tau])}, \quad t \geq 0,$$

where the derivative $x'(t)$ exists at each point $t \in [0, \infty)$, except possibly when $t \in \{0, 1, 2, \ldots\}$ at which the left-sided derivative exists.

In view of (3.1) and (3.3), for any $\varepsilon \in (0, v)$, there exists a positive integer $n_0 = n_0(\varepsilon)$ such that

$$(3.5) \qquad v_1 \equiv v - \varepsilon < x([t - \tau]) < u + \varepsilon \equiv u_1, \quad t \geq n_0,$$

and

$$(3.6) \qquad\qquad \int_{[t-\tau]}^{t} \beta(s)ds \leq 3, \quad t \geq n_0 - \tau.$$

Using (3.4) and (3.5), we have

$$(3.7) \qquad\qquad \frac{x'(t)}{x(t)} \leq \beta(t)\frac{1 - v_1}{1 + v_1}, \quad t \geq n_0,$$

and

$$(3.8) \qquad\qquad \frac{x'(t)}{x(t)} \geq -\beta(t)\frac{u_1 - 1}{u_1 + 1}, \quad t \geq n_0.$$

Let $\{n_i\}$ be an increasing sequence of integers such that $n_i > n_0 + \tau$, $x_{n_i} \geq \max\{x_{n_i-1}, x_{n_i+1}, 1\}$, $n_i \to \infty$ and $x_{n_i} = x(n_i) \to u$ as $i \to \infty$. It is easy to show that $x_{n_i-\tau} \geq 1$ and $x_{n_i-\tau-1} \leq 1$, and so there exists $\xi_i \in [n_i - \tau - 1, n_i - \tau]$ such that $x(\xi_i) = 1$. For $\xi_i \leq t \leq n_i$, by integrating (3.7) from $[t - \tau]$ to ξ_i, we obtain

$$x([t - \tau]) \geq \exp\left(-\frac{1 - v_1}{1 + v_1} \int_{[t-\tau]}^{\xi_i} \beta(s)\, ds\right), \quad \xi_i \leq t \leq n_i.$$

Substituting this into (3.4), we have

$$\frac{x'(t)}{x(t)} \leq \beta(t) \frac{1 - \exp\left(-\frac{1-v_1}{1+v_1} \int_{[t-\tau]}^{\xi_i} \beta(s)\, ds\right)}{1 + \exp\left(-\frac{1-v_1}{1+v_1} \int_{[t-\tau]}^{\xi_i} \beta(s)\, ds\right)}, \quad \xi_i \leq t \leq n_i.$$

Denote $(1 - v_1)/(1 + v_1)$ by v_2. Then $0 < v_2 < 1$. Thus

$$(3.9) \qquad \frac{x'(t)}{x(t)} \leq \min\left\{\beta(t)v_2, \frac{\beta(t)\left[1 - \exp\left(-v_2 \int_{[t-\tau]}^{\xi_i} \beta(s)\, ds\right)\right]}{1 + \exp\left(-v_2 \int_{[t-\tau]}^{\xi_i} \beta(s)\, ds\right)}\right\},$$
$$\xi_i \leq t \leq n_i.$$

Since $0 < v_2 < 1$, it follows from Lemma 2(i) that

$$\ln\left(2e^{-v_2(1-\frac{v_2}{2})} - 1\right) \geq -2v_2,$$

and so the number

$$\Gamma \equiv -\frac{1}{v_2} \ln\left(2e^{-v_2(1-\frac{v_2}{2})} - 1\right)$$

satisfies $0 < \Gamma \leq 2$. There are now two possibilities:

$$\int_{\xi_i}^{n_i} \beta(s)\, ds \leq \Gamma,$$

or

$$\Gamma < \int_{\xi_i}^{n_i} \beta(s)\, ds \leq 3.$$

In the former case, in view of (3.6) and (3.9),

$$
\begin{aligned}
\ln x(n_i) &\leq \int_{\xi_i}^{n_i} \frac{\beta(t)\left[1 - \exp\left(-v_2 \int_{[t-\tau]}^{\xi_i} \beta(s)\,ds\right)\right]}{1 + \exp\left(-v_2 \int_{[t-\tau]}^{\xi_i} \beta(s)\,ds\right)}\,dt \\
&= \int_{\xi_i}^{n_i} \frac{\beta(t)\left\{1 - \exp\left[-v_2 \left(\int_{[t-\tau]}^{t} \beta(s)\,ds - \int_{\xi_i}^{t} \beta(s)\,ds\right)\right]\right\}}{1 + \exp\left[-v_2 \left(\int_{[t-\tau]}^{t} \beta(s)\,ds - \int_{\xi_i}^{t} \beta(s)\,ds\right)\right]}\,dt \\
&\leq \int_{\xi_i}^{n_i} \frac{\beta(t)\left\{1 - \exp\left[-v_2 \left(3 - \int_{\xi_i}^{t} \beta(s)\,ds\right)\right]\right\}}{1 + \exp\left[-v_2 \left(3 - \int_{\xi_i}^{t} \beta(s)\,ds\right)\right]}\,dt \\
&= \int_{\xi_i}^{n_i} \beta(s)\,ds - \frac{2}{v_2} \ln \frac{1 + \exp\left[-v_2\left(3 - \int_{\xi_i}^{n_i} \beta(s)\,ds\right)\right]}{1 + \exp\left(-3v_2\right)}.
\end{aligned}
$$

Since the function $x - (2/v_2) \ln\left(1 + e^{-v_2(3-x)}\right)$ is increasing on $[0,3]$, we see from Lemma 2 that

$$
\begin{aligned}
\ln x(n_i) &\leq \Gamma - \frac{2}{v_2} \ln \left(\frac{1 + e^{-v_2(3-\Gamma)}}{1 + e^{-3v_2}}\right) \\
&= \Gamma + \frac{2}{v_2} \ln \left(\frac{1 + \left[2e^{-v_2(1-\frac{v_2}{2})} - 1\right]e^{-v_2(3-\Gamma)}}{1 + e^{-v_2(3-\Gamma)}}\right) \\
&\leq \Gamma + \frac{2}{v_2}\left[-v_2\left(1 - \frac{v_2}{2}\right) + \frac{v_2^2(3-\Gamma)}{2}\right] \\
&= -2(1 - 2v_2) - \frac{1 - v_2}{v_2} \ln\left(2e^{-v_2(1-\frac{v_2}{2})} - 1\right) \\
&\leq 2v_2.
\end{aligned}
$$

In the latter case, choose $p_i \in (\xi_i, n_i)$ such that

$$
\Gamma = \int_{p_i}^{n_i} \beta(s)\,ds.
$$

Then by (3.6), (3.9) and Lemma 2, we have

$$
\begin{aligned}
\ln x(n_i) &\le v_2 \int_{\xi_i}^{p_i} \beta(s)\, ds \\
&\quad + \int_{p_i}^{n_i} \frac{\beta(t)\left[1 - \exp\left(-v_2 \int_{[t-\tau]}^{\xi_i} \beta(s)\, ds\right)\right]}{1 + \exp\left(-v_2 \int_{[t-\tau]}^{\xi_i} \beta(s)\, ds\right)}\, dt \\
&\le v_2 \int_{\xi_i}^{p_i} \beta(s)\, ds \\
&\quad + \int_{p_i}^{n_i} \frac{\beta(t)\left\{1 - \exp\left[-v_2\left(3 - \int_{\xi_i}^{t} \beta(s)\, ds\right)\right]\right\}}{1 + \exp\left[-v_2\left(3 - \int_{\xi_i}^{t} \beta(s)\, ds\right)\right]}\, dt \\
&= v_2 \int_{\xi_i}^{p_i} \beta(s)\, ds + \int_{p_i}^{n_i} \beta(s)\, ds \\
&\quad - \frac{2}{v_2} \ln \frac{1 + \exp\left[-v_2\left(3 - \int_{\xi_i}^{n_i} \beta(s)\, ds\right)\right]}{1 + \exp\left[-v_2\left(3 - \int_{\xi_i}^{p_i} \beta(s)\, ds\right)\right]} \\
&= v_2 \int_{\xi_i}^{n_i} \beta(s)\, ds + (1 - v_2)\Gamma \\
&\quad - \frac{2}{v_2} \ln \frac{1 + \exp\left[-v_2\left(3 - \int_{\xi_i}^{n_i} \beta(s)\, ds\right)\right]}{1 + \exp\left[-v_2\left(3 + \Gamma - \int_{\xi_i}^{n_i} \beta(s)\, ds\right)\right]} \\
&\le 3v_2 + (1 - v_2)\Gamma - \frac{2}{v_2} \ln \frac{2}{1 + e^{-\Gamma v_2}} \\
&= -2(1 - 2v_2) - \frac{1 - v_2}{v_2} \ln\left(2e^{-v_2\left(1 - \frac{v_2}{2}\right)} - 1\right) \\
&\le 2v_2,
\end{aligned}
$$

where the third inequality follows from the fact that the function

$$
q(x) = v_2 x - \frac{2}{v_2} \ln \frac{1 + e^{-v_2(3-x)}}{1 + e^{-v_2(3+\Gamma-x)}}
$$

is increasing in $[0, 3]$:

$$q'(x) = v_2 - 2\left[\frac{1}{1 + e^{v_2(3-x)}} - \frac{1}{1 + e^{v_2(3+\Gamma-x)}}\right], \quad 0 \le x \le 3,$$

$$q''(x) = -2v_2\left[\frac{e^{v_2(3-x)}}{\left(1 + e^{v_2(3-x)}\right)^2} - \frac{e^{v_2(3+\Gamma-x)}}{\left(1 + e^{v_2(3+\Gamma-x)}\right)^2}\right] < 0, \quad 0 \le x \le 3,$$

and

$$q'(x) \ge q'(3) = v_2 + 1 - e^{v_2(1-\frac{v_2}{2})} > 0, \quad 0 \le x < 3.$$

Both cases imply $\ln x(n_i) \le 2v_2$ for $n = 1, 2, \ldots$, so that in the limit, we have

$$\ln u \le 2\left(\frac{1 - v_1}{1 + v_1}\right) = 2\left(\frac{1 - v + \varepsilon}{1 + v - \varepsilon}\right)$$

for every small positive ε. Thus,

$$\ln u \le 2\left(\frac{1 - v}{1 + v}\right).$$

By almost symmetric arguments, we may also show that

(3.10)
$$\ln v \ge 2\left(\frac{1 - u}{1 + u}\right).$$

Indeed, let $\{m_i\}$ be an increasing sequence of integers such that $m_i \ge n_0 + k$, $x_{m_i} \le \min\{x_{m_i-1}, x_{m_i+1}, 1\}$, $m_i \to \infty$ and $x_{m_i} = x(m_i) \to v$ as $i \to \infty$. It is easy to show that $x_{m_i-k} \le 1$ and $x_{m_i-k-1} \ge 1$, and hence there is $\eta_i \in [m_i - k - 1, m_i - k]$ such that $x(\eta_i) = 1$. For $\eta_i \le t \le m_i$, integrating (3.8) from $[t - k]$ to η_i, we have

$$x([t - k]) \le \exp\left(\frac{u_1 - 1}{1 + u_1}\int_{[t-k]}^{\eta_i}\beta(s)\,ds\right), \quad \eta_i \le t \le m_i.$$

Substituting this into (3.4), we have

$$\frac{x'(t)}{x(t)} \ge \beta(t)\frac{1 - \exp\left(\frac{u_1-1}{1+u_1}\int_{[t-k]}^{\eta_i}\beta(s)\,ds\right)}{1 + \exp\left(\frac{u_1-1}{1+u_1}\int_{[t-k]}^{\eta_i}\beta(s)\,ds\right)}, \quad \eta_i \le t \le m_i.$$

Set $u_2 = (u_1 - 1)/(1 + u_1)$. Then $u_2 > 0$. And we have

$$(3.11) \qquad -\frac{x'(t)}{x(t)} \le \min\left\{\beta(t)u_2, \frac{\beta(t)\left[\exp\left(u_2 \int_{[t-k]}^{\eta_i} \beta(s)\,ds\right) - 1\right]}{1 + \exp\left(u_2 \int_{[t-k]}^{\eta_i} \beta(s)\,ds\right)}\right\},$$

$$\eta_i \le t \le m_i.$$

Since $0 < u_2 < 1$, we thus see that the number

$$\Theta = \frac{1}{u_2}\ln\left(2e^{u_2(1+\frac{u_2}{2})} - 1\right)$$

satisfies $0 < \Theta < 3$. There are two cases to consider:

$$\int_{\eta_i}^{m_i} \beta(s)\,ds \le 3 - \Theta,$$

or

$$3 - \Theta < \int_{\eta_i}^{m_i} \beta(s)\,ds \le 3.$$

In the former case, in view of (3.11) and Lemma 2,

$$-\ln x(m_i) \le u_2 \int_{\eta_i}^{m_i} \beta(s)ds \le 3u_2 - \ln\left(2e^{u_2(1+\frac{u_2}{2})} - 1\right) \le u_2.$$

In the latter case, choose $q_i \in (\eta_i, m_i)$ such that

$$\int_{\eta_i}^{q_i} \beta(s)\,ds = 3 - \Theta.$$

Then by (3.11) and Lemma 2,

$$
\begin{aligned}
-\ln x(m_i) &\le u_2 \int_{\eta_i}^{q_i} \beta(s)\,ds \\
&\quad + \int_{q_i}^{m_i} \frac{\beta(t)\left[\exp\left(u_2 \int_{[t-k]}^{\eta_i} \beta(s)\,ds\right) - 1\right]}{1 + \exp\left(u_2 \int_{[t-k]}^{\eta_i} \beta(s)\,ds\right)}\,dt \\
&\le u_2 \int_{\eta_i}^{q_i} \beta(s)\,ds \\
&\quad + \int_{q_i}^{m_i} \frac{\beta(t)\left\{\left[\exp u_2\left(3 - \int_{\eta_i}^{t} \beta(s)\,ds\right)\right] - 1\right\}}{1 + \exp\left[u_2\left(3 - \int_{\eta_i}^{q_i} \beta(s)\,ds\right)\right]}\,dt \\
&= u_2 \int_{\eta_i}^{q_i} \beta(s)\,ds - \int_{q_i}^{m_i} \beta(s)\,ds \\
&\quad - \frac{2}{u_2} \ln \frac{1 + \exp\left[u_2\left(3 - \int_{\eta_i}^{m_i} \beta(s)\,ds\right)\right]}{1 + \exp\left[u_2\left(3 - \int_{\eta_i}^{q_i} \beta(s)\,ds\right)\right]} \\
&= (1+u_2)(3-\Theta) - \int_{\eta_i}^{m_i} \beta(s)\,ds + 2\left(1 + \frac{u_2}{2}\right) \\
&\quad - \frac{2}{u_2} \ln \frac{1 + \exp\left[u_2\left(3 - \int_{\eta_i}^{m_i} \beta(s)\,ds\right)\right]}{2} \\
&\le 2(1+2u_2) - \frac{1+u_2}{u_2} \ln\left(2e^{u_2\left(1+\frac{u_2}{2}\right)} - 1\right) \\
&\le 2u_2,
\end{aligned}
$$

where the last inequality follows from the fact that

$$
h(x) = -x - \frac{2}{u_2} \ln \frac{1 + e^{u_2(3-x)}}{2}
$$

is increasing in $[0,3]$. Both cases imply

$$
-\ln x(m_i) \le 2u_2, \quad i = 1, 2, \dots .
$$

Letting $i \to \infty$ and $\varepsilon \to 0$, we finally see that (3.10) holds.

Therefore, by Lemma 1, $u = v = 1$. Our proof is complete. $\square$

4. Remarks. When $\beta_n \equiv \beta > 0$ for $n = 0, 1, 2, \dots$, and $\tau \geq 2$, since the condition

$$0 < \beta \leq \frac{3}{\tau + 1}$$

implies (3.1) and (3.2), we see from Theorem 1 that every solution $\{x_n\}_{n=-\tau}^{\infty}$ of (1.4) which satisfies $x_{-\tau}, \dots, x_0 > 0$ will also tend to 1. This in turn implies that the equilibrium $1/2$ is a global attractor for all solutions $\{y_n\}_{n=-\tau}^{\infty}$ of (1.1) which satisfies $0 < y_{-\tau}, y_{-\tau+1}, \dots, y_0 < 1$.

We remark that computer simulations suggest that the best possible condition [**3**, p. 87] for $1/2$ to be a global attractor for positive solutions of (1.1) is

$$0 < \beta \leq 4 \cos \frac{\tau \pi}{2\tau + 1}.$$

Since

$$\lim_{\tau \to \infty} \frac{4 \cos \frac{\tau \pi}{2\tau+1}}{\frac{3}{\tau+1}} = \frac{\pi}{3} \approx 1.0472,$$

our condition appears to match the experimental results quite well for large τ.

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DEPARTMENT OF APPLIED MATHEMATICS, CENTRAL SOUTH UNIVERSITY OF TECHNOLOGY, CHANGSHA, HUNAN 410083, P. R. CHINA
E-mail address: xhtang@cs.hn.cn

DEPARTMENT OF MATHEMATICS, TSING HUA UNIVERSITY, HSINCHU 30043, TAIWAN, R.O. C.
E-mail address: sscheng@math.nthu.edu.tw