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FIRST-COUNTABILITY, SEQUENTIALITY AND TIGHTNESS OF THE UPPER KURATOWSKI CONVERGENCE

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ABSTRACT. The first-countability of the upper Kuratowski convergence is characterized in terms of the underlying convergence. If we start with a topology, first-countability is also equivalent to countable tightness of the upper Kuratowski topology, equivalently upper Kuratowski convergence. This result is applied to consonance and its analogous for the real-valued continuous dual is applied to a generalized (real-valued) consonance result.

1. Introduction. Relations between topological properties of a space X and the corresponding properties of function spaces on X, and in particular those of various hyperspace topologies on its closed sets, have been intensively investigated ([**26, 30, 6, 5, 16**] among a lot of others). In such studies a great collection of hyperspace structures emerged. Moreover, several non topological structures appear naturally ([**32, 35, 7**]). The upper Kuratowski convergence plays a particular role in the lattice of all the convergence structures one can endow a hyperspace with. Indeed, this is the least convergence that makes the evaluation (by identifying closed sets with their indicator functions) jointly continuous. This least convergence, transposed on the lattice of open sets, is homeomorphic to the Scott convergence, classically used and studied since its introduction by Scott in [**42**] in the extensive literature on lattice theory and continuous lattices, e.g., [**28, 29, 27, 11**]. In the sublattice of hyperspace topological structures, studied for instance in [**13**] and [**14**], there is in general no such least structure available. However, the topological modification of the upper Kuratowski convergence, called upper Kuratowski topology (homeomorphically Scott topology), retains some of the nice properties of the upper Kuratowski convergence and turns out to be of particular interest. Hence, even if you focus your attention on topological hyperspace structures and, in

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particular, on the upper Kuratowski topology, the broader framework of convergences also provides information concerning topologies.

Let X be a topology and let $[X,\$]$ denote the associated upper Kuratowski convergence. Then $T[X,\$]$ denotes the associated upper Kuratowski topology. In [**3**], Alleche and Calbrix proved

Theorem 0.1 [**3**, Corollary 2.2]. *If X is a hereditarily Lindelöf topology, then $T[X,\$]$ is sequential.*

Recently Costantini, Holá and Vitolo proved the converse. Moreover,

Theorem 0.2 [**12**, Proposition 3.6]. *Let X be a Hausdorff topology. The following are equivalent*:

1. $T[X,\$]$ *is sequential*;
2. $T[X,\$]$ *is countably tight*;
3. X *is hereditarily Lindelöf.*

In this paper I characterize the first-countability of the upper Kuratowski convergence in the general context of convergences. The result seems to be new even for the upper Kuratowski convergence associated with a topology. In this latter particular case, it can be strengthened as follows (simplified version of Corollary 3.8).

Theorem 0.3. *The following are equivalent*:

1. X *is hereditarily Lindelöf*;
2. $[X,\$]$ *is first-countable*;
3. $[X,\$]$ *is sequential*;
4. $[X,\$]$ *is countably tight*;
5. $T[X,\$]$ *is sequential*;
6. $T[X,\$]$ *is countably tight.*

This refines and extends both Theorem 0.1 and Theorem 0.2. Notice that, contrary to Theorem 0.2, no separation is needed in Theorem 0.3.

These results appear as special instances of our main theorems.

In the last section, I provide applications of the results mentioned above to both consonance and **R**-consonance. Recall that a topology is *consonant* [**23**] if the cocompact and the upper Kuratowski topology coincide. Consonance has been intensively studied by several authors, e.g., Alleche and Calbrix [**3, 2**], Bouziad [**10**], Costantini [**13, 15**], Dolecki, Greco and Lechicki [**23**], Fremlin, Nogura and Shakhmatov [**40**], Saint-Raymond [**43**], Vitolo [**13**], Watson [**15**], among others. Analogously, I call **R**-*consonant* a topology for which the topological modification of the continuous convergence and the compact-open topology coincide on real-valued functions. Alleche and Calbrix proved [**3**, Corollary 2.5] that a hereditarily Lindelöf and strongly Fréchet topology is consonant if and only if the associated cocompact topology is sequential. I generalize this result to Theorem 0.4 and I moreover obtain a new analogue for **R**-consonance (simplified versions of Theorem 4.12 and Corollary 4.16). See Section 4 for precise definitions. A Hausdorff strongly Fréchet topology is quasi strongly k.

Theorem 0.4. *A quasi strongly k Hausdorff topology is hereditarily Lindelöf and consonant if and only if the associated cocompact topology is sequential.*

Theorem 0.5. *A Lindelöf Hausdorff* **R***-strongly k topology is* **R***-consonant if and only if the compact-open topology on real-valued functions is sequential.*

1. Convergences and upper Kuratowski convergence.

A *convergence* ξ on a set X is a relation between X and the filters on X, denoted by $x \in \lim_\xi \mathcal{F}$ whenever x and $\mathcal{F}$ are in relation, such that

$$\mathcal{F} \leq \mathcal{G} \Longrightarrow \lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{G},$$
$$x \in \lim_\xi (x),$$
$$\lim_\xi (\mathcal{F} \wedge \mathcal{G}) = \lim_\xi \mathcal{F} \cap \lim_\xi \mathcal{G},$$

for each fixed ultrafilter (x) and each pair of filters $\mathcal{F}$ and $\mathcal{G}$.

I denote by $|\xi|$ the underlying set of the convergence ξ. A convergence ξ is *finer* than a convergence ϑ ($\xi \geq \vartheta$) whenever $\lim_\xi \mathcal{F} \subset \lim_\vartheta \mathcal{F}$ for every filter $\mathcal{F}$. A map $f : |\xi| \to |\tau|$ is *continuous* if $f(\lim_\xi \mathcal{F}) \subset \lim_\tau f(\mathcal{F})$; this implies the definitions of initial and final convergences, hence of product, subspace and so on. If $f : |\xi| \to |\tau|$, then I will denote by f^- the inverse relation of f and by $f^-\tau$ the initial convergence with respect to f and τ.

We say that two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X *mesh*, in symbol $\mathcal{A}\#\mathcal{B}$, whenever $A \cap B \neq \varnothing$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A subset A of X is ξ-*closed* whenever $\lim_\xi \mathcal{F} \subset A$ for every filter $\mathcal{F}$ with $A\#\mathcal{F}$. The set of all ξ-closed sets gives rise to a topology, called *topological modification of* ξ and denoted $T\xi$. The map T is a bireflector called the *topologizer*. See [**1**] for a general definition of a reflector. In the category of convergences, a bireflector R can be characterized as follows. R assigns to each convergence ξ its reflection $R\xi \leq \xi$ so that $|R\xi| = |\xi|$ and R has the following properties: $\xi \geq \theta \Rightarrow R\xi \geq R\theta$; $R(R\xi) = R\xi$ and $R(f^-\xi) \geq f^-(R\xi)$ for every map f.

Let $\mathcal{F}$ be a filter on a convergence space X. The *adherence* of $\mathcal{F}$ is the union of the limits of all filters that are finer than $\mathcal{F}$:

$$\mathrm{adh}_\xi \mathcal{F} = \bigcup_{\mathcal{G} \geq \mathcal{F}} \lim{}_\xi \mathcal{G}. \tag{1.1}$$

In particular, the *adherence* $\mathrm{adh}_\xi A$ of a set A is the adherence of its principal filter, while the *closure* $\mathrm{cl}_\xi A$ of A is the (idempotent) adherence of A for $T\xi$. There are various ways to characterize the topologizer. For example,

$$\lim{}_{T\xi} \mathcal{F} = \bigcap_{C\#\mathcal{F}} \mathrm{cl}_\xi C. \tag{1.2}$$

For each point x, the neighborhood filter for $T\xi$ is denoted $\mathcal{N}_\xi(x)$.

A convergence ξ is a pretopology if $x \in \lim_\xi \mathcal{F}$ whenever $x \in \mathrm{adh}_\xi A$ for each $A\#\mathcal{F}$. The map P assigning to each convergence ξ the finest pretopology coarser than ξ, is a bireflector. It is known that [**20**],

$$\lim{}_{P\xi} \mathcal{F} = \bigcap_{A\#\mathcal{F}} \mathrm{adh}_\xi A. \tag{1.3}$$

For each point x, the infimum of all filters that ξ-converge to x is a $P\xi$-convergent filter called *vicinity filter* of x and denoted $\mathcal{V}_\xi(x)$. Consider the sequence of iterates of the adherence defined by $\mathrm{adh}_\xi^0 A = A$ and for each ordinal number $\alpha > 0$, by

$$\mathrm{adh}_\xi^\alpha A = \mathrm{adh}_\xi \Big(\bigcup_{\beta<\alpha} \mathrm{adh}_\xi^\beta A \Big).$$

The least ordinal $td(\xi)$ for which $\mathrm{adh}_\xi^{td(\xi)} = \mathrm{adh}_\xi^{td(\xi)+1}$ is called the *topological defect* of ξ and $\mathrm{adh}_\xi^{td(\xi)} = \mathrm{cl}_\xi$. The pretopology defined by the adherence operator (of sets) adh_ξ^α is denoted by $(P\xi)^\alpha$. See [**21**] for details.

A convergence ξ is said to be *regular* if $\lim_\xi \mathcal{F} \subset \lim_\xi \mathrm{adh}_\xi^\sharp \mathcal{F}$ for every filter $\mathcal{F}$ where $\mathrm{adh}_\xi^\sharp \mathcal{F}$ is the filter generated by $\{\mathrm{adh}_\xi F : F \in \mathcal{F}\}$. Analogously, for any operator o, I denote by $o^\sharp \mathcal{A}$ the family $\{oA : A \in \mathcal{A}\}$ and I usually identify filter-bases with the filters they generate.

Let \$ denote the Sierpiński topology, that is, the topology on $\{0,1\}$ in which 0 is isolated while 1 is not. Continuous maps from a convergence ξ to \$ are precisely the indicator functions of ξ-closed sets (taking value 0 on A and 1 on A^c). Therefore, $T\xi = \bigvee_{f \in C(\xi,\$)} f^- \$$, and the continuous convergence $[\xi, \$]$ can be considered as a convergence on the set $\mathcal{C}(\xi)$ of ξ-closed sets. It turns out that $[\xi, \$]$ is homeomorphic to the upper Kuratowski convergence. Recall that (e.g., [**23**]) the upper Kuratowski convergence is characterized by

$$A \in \lim_{[\xi,\$]} \mathcal{G} \Longleftrightarrow \mathrm{adh}_\xi |\mathcal{G}| \subset A, \tag{1.4}$$

where the *reduced filter* $|\mathcal{G}|$ of $\mathcal{G}$ is generated by $\{\bigcup_{C \in G} C : G \in \mathcal{G}\}$. Notice that $|\mathcal{G}|$ may be degenerated, i.e., $|\mathcal{G}| = 2^{|\xi|}$, if $\mathcal{G}$ is the fixed ultrafilter generated by $\{\varnothing\}$. In this case there is no (non-degenerate) filter that meshes $|\mathcal{G}|$, so that extending the definition (1.1) of the adherence to families, $\mathrm{adh}\, |\mathcal{G}| = \varnothing$. In other words, the ultrafilter generated by $\{\varnothing\}$ converges to every ξ-closed set in $[\xi, \$]$. The *cocompact topology* $C_k(\xi, \$)$ on $\mathcal{C}(\xi)$ admits as a subbase the family

$$\{F \in \mathcal{C}(\xi) : F \cap K = \varnothing\}, \tag{1.5}$$

where K ranges over ξ-compact sets.

If $\mathcal{H}$ is a filter on $|\xi|$, I define on $|[\xi, \$]|$ the *polar-erected filter* $\uparrow^L \mathcal{H}$ generated by $\{C = \mathrm{cl}_\xi C \subset H : H \in \mathcal{H}\}$. The following is an immediate consequence of the definitions

$$\mathcal{H} \leq |\mathcal{G}| \Longleftrightarrow \uparrow^L \mathcal{H} \leq \mathcal{G}. \tag{1.6}$$

Consider the adherence of points ad defined by $\mathrm{ad}_\xi A = \bigcup_{a \in A} \lim_\xi(a)$, so that $\mathrm{ad}_{T\xi} A = \bigcup_{a \in A} \mathrm{cl}_\xi a$ and the associated interior operator

$$\mathrm{in}_\xi A = (\mathrm{ad}_\xi A^c)^c. \tag{1.7}$$

The following lemma follows immediately from the definitions.

Lemma 1.1. *A filter $\mathcal{G}$ fulfills $\mathcal{G} = ad^\sharp_{T\xi}\mathcal{G}$ if and only if $\mathcal{G} = |\uparrow^L \mathcal{G}|$.*

2. Classes of filters and compact-like properties.

2.1 Classes of filters. In this paper, φ denotes the class of all filters and $\varphi(X)$ the set of all filters on X. If $\mathfrak{J}$ is a class of filters, the elements of $\mathfrak{J}$ are called $\mathfrak{J}$-*filters*. If the class of filters $\mathfrak{J}(\cdot)$ is such that $\xi \geq \theta$ implies $\mathfrak{J}(\xi) \supset \mathfrak{J}(\theta)$, $\mathfrak{J}(\mathrm{Base}_\mathfrak{J}\xi) = \mathfrak{J}(\xi)$ and $f\mathcal{H} \in \mathfrak{J}(\tau)$ whenever $f : \xi \to \tau$ and $\mathcal{H} \in \mathfrak{J}(\xi)$, then the map $\mathrm{Base}_\mathfrak{J}$ defined by

$$\lim_{\mathrm{Base}_\mathfrak{J}\xi} \mathcal{F} = \bigcup_{\mathfrak{J} \ni \mathcal{H} \leq \mathcal{F}} \lim_\xi \mathcal{H} \tag{2.1}$$

is a bicoreflector [**18**]. See [**1**] for a general definition. In convergences, bicoreflectors may be characterized as follows. A bicoreflector C assigns to each convergence ξ its coreflection $C\xi \geq \xi$ so that $|C\xi| = |\xi|$ and has the following properties: $\xi \geq \theta \Rightarrow C\xi \geq C\theta$; $C(C\xi) = C\xi$ and $C(f^-\xi) \geq f^-(C\xi)$ for every map f. A convergence such that $\mathrm{Base}_\mathfrak{J}\xi = \xi$ is called $\mathfrak{J}$-*based*.

Let λ and $\varkappa$ be two, possibly finite, cardinal numbers. A filter is λ-*based* if it admits a filter-base of cardinality less than or equal to λ. Notice that a topology is of character λ if and only if each neighborhood filter is λ-based. More generally, the least cardinal number λ such that a convergence ξ is based in λ-based filters is the *character* $\chi(\xi)$ of the convergence. In particular, a convergence of countable character is traditionally called *first-countable* and the associated coreflector $\mathrm{Base}_{\mathfrak{J}}$ is denoted by B_ω. A pretopology (and in particular) a topology, is called finitely generated if each point has a smallest vicinity [**33**]. Equivalently, each vicinity filter is a 1-based filter. Hence, I call *finitely generated* each convergence that is based in principal filters. The associated coreflector is denoted by B_1 and is called the *finitely generated modifier* (in previous papers such as [**24, 37, 36**], among others, B_ω and B_1 were denoted by First and Fin, respectively). The class of λ-based filters is denoted by φ_λ and, in particular, φ_1 stands for the class of principal filters.

A filter $\mathcal{F}$ is $(\varkappa, \lambda)$-*tight* if, for every A of cardinality less than or equal to $\varkappa$ such that $A\#\mathcal{F}$, there exists $B \subset A$ of cardinality less than or equal to λ such that $B\#\mathcal{F}$. A filter is λ-*tight* if it is $(\varkappa, \lambda)$-*tight* for every $\varkappa$. Notice that a topology is λ-tight, i.e.,

$$x \in \operatorname{cl} A \Longrightarrow \exists_{B\subset A}, x \in \operatorname{cl} B \text{ and } \operatorname{card} B \leq \lambda,$$

if and only if every neighborhood filter is λ-tight. The least cardinal number λ for which a convergence ξ is based in λ-tight filters is called the *tightness* $t(\xi)$ of the convergence. The class of $(\varkappa, \lambda)$-tight filters, respectively of λ-tight filters, is denoted $\varphi_{\#(\varkappa,\lambda)}$, respectively $\varphi_{\#\lambda}$.

A filter $\mathcal{F}$ is λ-*deep* if $\bigcap \mathcal{A} \in \mathcal{F}$ for every family $\mathcal{A} \subset \mathcal{F}$ of cardinality less than or equal to λ. The class of λ-deep filters is denoted by $\varphi_{\wedge\lambda}$. Notice that a topology is a P-topology, i.e., each G_δ set is open if and only if each neighborhood filter is ω-deep.

In order not to overburden notations, I use the convention that

$$B_\square = \mathrm{Base}_{\varphi\square}. \tag{2.2}$$

Hence the countably tight modifier is $B_{\#\omega}$.

A class $\mathfrak{J}$ of filters is said to be *projectable* if $|\mathcal{G}|$ is a $\mathfrak{J}$-filter on $|\xi|$ provided that $\mathcal{G}$ is a $\mathfrak{J}$-filter on $|[\xi, \$]|$. A class $\mathfrak{J}$ of filters is *polar-stable* if $\uparrow^L \mathcal{H}$ is a $\mathfrak{J}$-filter on $[\xi, \$]$ whenever $\mathcal{H}$ is a $\mathfrak{J}$-filter on $|\xi|$. The class of λ-tight filters is projectable, but not polar-stable, and the class of filters generated by sequences is neither projectable nor polar-stable.

A class that is both projectable and polar-stable is called \$-*compatible* [**24**]. For example, the classes of λ-based filters and of λ-deep filters are \$-compatible.

The coreflector on convergences based in filters generated by sequences is denoted by Seq. A topology ξ is sequential, i.e., such that sequentially closed and closed sets coincide, if and only if

$$\xi \geq T\mathrm{Seq}\,\xi, \text{ equivalently } \xi \geq TB_{\omega}\xi. \tag{2.3}$$

Because (2.3) is meaningful for all convergences, we call a convergence *sequential* if it fulfills (2.3), see [**18**].

Obviously, each λ-based filter is λ-tight, so that a first-countable convergence is countably tight. However, a sequential convergence need not be countably tight (see Example 2.2 below) although $T\xi = TB_{\omega}\xi$ is countably tight as a sequential topology. More generally, it follows from Proposition 2.1 below that the tightness of $T\xi$ is less than or equal to the tightness of ξ for every convergence ξ.

Proposition 2.1. *For every ordinal* α*,*

$$(PB_{\#\lambda}\xi)^{\alpha} \geq B_{\#\lambda}(P\xi)^{\alpha}; \tag{2.4}$$

$$(PB_{\#(\varkappa,\lambda)}\xi)^{\alpha} \geq B_{\#(\varkappa,\lambda)}(P\xi)^{\alpha}. \tag{2.5}$$

Proof. The proof of (2.5) follows the lines of the proof of (2.4). I proceed by induction on α. $B_{\#\lambda}(P\xi)$ is the coarsest λ-tight convergence finer than $P\xi$ and $PB_{\#\lambda}\xi$ is λ-tight. Indeed, if $A \in \mathcal{V}_{B_{\#\lambda\xi}}(x)^{\#}$, then a filter $\mathcal{F}$ exists such that $x \in \lim_{B_{\#\lambda\xi}} \mathcal{F}$ and $A \in \mathcal{F}^{\#}$. Thus there exists $B \in \mathcal{F}^{\#}$ such that $B \subset A$ and $\operatorname{card} B \leq \lambda$. Consequently, $PB_{\#\lambda}\xi \geq B_{\#\lambda}P\xi$.

Assume that (2.4) holds for every $\alpha < \beta$. $B_{\#\lambda}(P\xi)^{\beta}$ is the coarsest λ-tight convergence finer than $(P\xi)^{\beta}$ and $(PB_{\#\lambda}\xi)^{\beta}$ is λ-tight. Indeed, $(P\xi)^{\beta} = P(\bigwedge_{\alpha<\beta}(P\xi)^{\alpha})$ and the infimum of a family of λ-tight convergences is clearly λ-tight. $\square$

As a corollary, the pretopological and topological modifications of a λ-tight convergence are λ-tight.

A convergence ξ is *Fréchet* if $\xi \geq PB_\omega \xi$. In other words, ξ is Fréchet if $\mathrm{adh}_\xi A \subset \mathrm{adh}_{\mathrm{Seq}\,\xi} A$ for every $A \subset |\xi|$. If moreover $\mathrm{adh}_\xi \mathcal{H} \subset \mathrm{adh}_{\mathrm{Seq}\,\xi} \mathcal{H}$ for every countably based filter, ξ is called *strongly Fréchet*.

Example 2.2 (A Fréchet noncountably tight convergence). Let ξ denote the quotient topology obtained from a disjoint sum of countably many copies $(I_p)_{p\in\mathbf{N}}$ of $[0,1]$ by identifying all points 0 to a single point x_0. It is well known that this topology is Fréchet but not strongly Fréchet. Indeed, the countably based filter $\mathcal{H}$ generated by $\{\bigcup_{p\geq n} I_p \setminus \{x_0\}\}_{n\in\mathbf{N}}$ verifies $x_0 \in \mathrm{adh}_\xi \mathcal{H} \setminus \mathrm{adh}_{B_\omega \xi} \mathcal{H}$. On the other hand, each element of $\mathcal{N}_\xi(x_0) \vee \mathcal{H}$ contains an interval, so that $\mathcal{N}_\xi(x_0) \vee \mathcal{H}$ is a uniform filter on the underlying set X of ξ, i.e., every element of the filter as the cardinality of X. Each uniform filter $\mathcal{F}$ admits a uniform ultrafilter. Otherwise, each ultrafilter of $\mathcal{F}$ would contain a set of cardinality strictly less than $\mathrm{card}\, X$ so that, by [**31**, Proposition 1.2.2], there would be a set of cardinality strictly less than $\mathrm{card}\, X$ in $\mathcal{F}$. Consequently, there exists a uniform ultrafilter $\mathcal{U}$ of $\mathcal{N}_\xi(x_0) \vee \mathcal{H}$. Let τ denote the convergence on X defined by $\lim_\tau \mathcal{F} = \lim_{B_\omega \xi} \mathcal{F}$ if $\mathcal{F} \neq \mathcal{U}$ and $\lim_\tau \mathcal{U} = \lim_\xi \mathcal{U} = \{x_0\}$. There is no countably based filter coarser than $\mathcal{U}$ that converges to x_0 in τ because otherwise x_0 would belong to $\mathrm{adh}_{B_\omega \xi} \mathcal{H}$. Thus $B_\omega \xi = B_\omega \tau$ so that $\tau \geq \xi \geq PB_\omega \xi = PB_\omega \tau$. The convergence τ is Fréchet but not countably tight. Indeed, $A \# \mathcal{U}$ if and only if $A \in \mathcal{U}$ and $\mathcal{U}$ is uniform, so that no countable subset of X belongs to $\mathcal{U}$.

2.2 Covers and compact-like properties. In this section, I use only one convergence, so that I often omit the symbol. A family $\mathcal{A}$ is $\mathfrak{J}$-*compactoid* in $\mathcal{B}$ if $\mathrm{adh} \mathcal{H} \# \mathcal{B}$ whenever $\mathcal{H}$ is a $\mathfrak{J}$-filter such that $\mathcal{H} \# \mathcal{A}$. A family is $\mathfrak{J}$-*compact* if it is $\mathfrak{J}$-compactoid in itself. A $\mathfrak{J}$-compact family for the class $\mathfrak{J} = \varphi$ of all filters is called compact. A set K is $\mathfrak{J}$-compact if $\{K\}$ is a $\mathfrak{J}$-compact family. A convergence is $\mathfrak{J}$-compact if the underlying set is $\mathfrak{J}$-compact. If $\mathfrak{J}$ is respectively the class of all, of ω-based, and ω-deep filters, then $\mathfrak{J}$-compactness extends to convergences the classical topological notions of compactness, countable compactness and Lindelöfness, respectively.

A convergence is *locally compact* if every convergent filter contains a compact set. The class of locally compact convergences is bicoreflective in the category of convergences and the associated coreflector is denoted by K. A convergence is *locally hereditarily compact* if it is based in filters that admit a base composed of compact sets. This property is also coreflective and the associated coreflector is denoted by K_{her}.

A family $\mathcal{S}$ of subsets of $|\xi|$ is a ξ-*cover* of $A \subset |\xi|$ if every filter that ξ-converges to a point in A contains an element of $\mathcal{S}$. For a topology it amounts to:

$$A \subset \bigcup_{S \in \mathcal{S}} \operatorname{int} S.$$

Hence, the notion is different from the set-theoretic notion of a cover but coincides in case of covers by open sets, as used in every definition of topological compact-like properties. See [**17**] and [**19**] for details on covers in convergences. In this latter paper, Dolecki shows how to translate in terms of filters every proposition using (open) covers. He shows, in particular, [**19**, Theorem 2.1]:

Proposition 2.3. *A family $\mathcal{S}$ is a ξ-cover of $A \subset |\xi|$ if and only if*

$$\operatorname{adh}_\xi \mathcal{S}_c \cap A = \varnothing.$$

In this proposition, $\mathcal{S}_c = \{S^c : S \in \mathcal{S}\}$ denotes the family of complements of elements of $\mathcal{S}$ and

$$\operatorname{adh} \mathcal{S}_c = \bigcup_{\mathcal{F} \# \mathcal{S}_c} \lim \mathcal{F}$$

extends to families (not necessarily filters) the notion of adherence. If $\mathcal{S}$ is an ideal, i.e., $\mathcal{S}$ is stable by finite union and subsets, then $\mathcal{S}_c$ is a (possibly degenerate) filter. Let $\mathfrak{J}$ be a class of filters. An ideal $\mathcal{S}$ is a $\mathfrak{J}$-cover of $A \subset |\xi|$ if it is a cover of A and if $\mathcal{S}_c$ is a $\mathfrak{J}$-filter. A cover $\mathcal{S}$ is *point-regular* if $\operatorname{in}^{\natural}_{T\xi} \mathcal{S}$ is again a cover.

Let $\mathfrak{J}$ and $\mathfrak{D}$ be two classes of filters. I call a subset A of X (*quasi*) *ideal-cover-$\mathfrak{D}/\mathfrak{J}$-compact* if, for every (point-regular) ideal $\mathfrak{D}$-cover $\mathcal{S}$ of A, there is a subfamily of $\mathcal{S}$ (of $\operatorname{in}^{\natural}_{T\xi} \mathcal{S}$) which is a $\mathfrak{J}$-cover of A.

In particular, if ξ is T_1 (points are closed) or if $\xi = T\xi$, then every cover is point-regular and quasi ideal-cover-$\mathfrak{D}/\mathfrak{J}$-compactness amounts to ideal cover-$\mathfrak{D}/\mathfrak{J}$-compactness. It is easy to check that ideal-cover-$\mathfrak{D}/\varphi_1$-compactness implies $\mathfrak{D}$-compactness. The converse does not hold for general convergences.

Notice that $\varphi_\varkappa/\varphi_\lambda$-compactness is traditionally called $(\varkappa, \lambda)$-*compactness*. By extension, I abridge $\varphi_\square/\varphi_\blacksquare$-compactness by $(\square, \blacksquare)$-compactness with the convention that $(\infty, \square)$-compact means $\varphi/\varphi_\square$-compact. Analogously, I abridge $\varphi_\square$-compact by $\square$-compact and ∞-compact means φ-compact. A topology is compact if and only if it is ∞-compact, if and only if it is (quasi) ideal-cover-$(\infty, 1)$-compact. Analogously, a topology is countably compact if and only if it is ω-compact, if and only if it is (quasi) ideal-cover-$(\omega, 1)$-compact.

Lemma 2.4. *The following classes of topologies coincide*:

1. *Lindelöf*;
2. $\wedge\omega$-*compact*;
3. (*quasi*) *ideal-cover-*(∞, ω)-*compact*;
4. (*quasi*) *ideal-cover-*$(\wedge\omega, \omega)$-*compact.*

The least cardinal number λ for which a convergence ξ is (∞, λ)-compact is the *Lindelöf degree* $l(\xi)$ of the convergence. The least cardinal number λ for which every open subset of a convergence ξ is (quasi) (∞, λ)-compact is the *hereditary* (*quasi*) *Lindelöf degree* $(hl^\circ(\xi))$ of the convergence.

These and more general notions of cover and filter compactness are studied in detail in [**17**].

3. Sequentiality, countable character of the upper Kuratowski convergence and their generalizations. The motivating question of this section is to characterize first-countability and sequentiality of the upper Kuratowski convergence. The convergence-theoretic approach in terms of classes of filters allows to obtain more.

Theorem 3.1. *Let* $\mathfrak{D}$ *and* $\mathfrak{J}$ *be two* \$-*compatible classes of filters.*

Then

$$Base_{\mathfrak{D}}([\xi,\$]) \geq Base_{\mathfrak{J}}([\xi,\$])$$

if and only if every ξ*-open set is quasi ideal-cover-*$\mathfrak{D}/\mathfrak{J}$*-compact.*

Proof. Assume $\mathrm{Base}_{\mathfrak{D}}([\xi,\$]) \geq \mathrm{Base}_{\mathfrak{J}}([\xi,\$])$ and consider an ideal point regular ξ-$\mathfrak{D}$-cover $\mathcal{S}$ of a ξ-open set U. By definition, the $\mathfrak{D}$-filter $\mathcal{G} = \mathrm{ad}^{\natural}_{T\xi}(\mathcal{S}_c)$ fulfills $\mathrm{adh}_\xi \mathcal{G} \cap U = \varnothing$. In view of Lemma 1.1, $|\uparrow^L \mathcal{G}| = \mathrm{ad}^{\natural}_{T\xi}\mathcal{G} = \mathcal{G}$, so that $U^c \in \lim_{[\xi,\$]} \uparrow^L \mathcal{G}$. Moreover, $\uparrow^L \mathcal{G}$ is a $\mathfrak{D}$-filter by \$-compatibility of $\mathfrak{D}$. Hence, $U^c \in \lim_{\mathrm{Base}_{\mathfrak{J}}([\xi,\$])}(\uparrow^L \mathcal{G})$. Thus there exists a $\mathfrak{J}$-filter $\mathcal{H}$ coarser than $\uparrow^L \mathcal{G}$ such that $U^c \in \lim_{[\xi,\$]} \mathcal{H}$. In other words, $\mathrm{adh}_\xi |\mathcal{H}| \cap U = \varnothing$. Since $\mathfrak{J}$ is a projectable class, $|\mathcal{H}|_c$ is a ξ-$\mathfrak{J}$-cover of U. As $|\mathcal{H}| \leq |\uparrow^L \mathcal{G}| = \mathcal{G}$, $|\mathcal{H}|_c$ is a subcover of $\mathcal{G}_c = \mathcal{S}$.

Conversely, assume that every ξ-open set is quasi ideal-cover-$\mathfrak{D}/\mathfrak{J}$-compact, and consider a ξ-closed set A and a $\mathfrak{D}$-filter $\mathcal{G}$ such that $A \in \lim_{[\xi,\$]} \mathcal{G}$, equivalently, $\mathrm{adh}_\xi |\mathcal{G}| \cap A^c = \varnothing$. Since $\mathfrak{D}$ is projectable, $|\mathcal{G}| = \mathrm{ad}^{\natural}_{T\xi}|\mathcal{G}|$ is a $\mathfrak{D}$-filter. It follows that $|\mathcal{G}|_c = \mathrm{in}^{\natural}_{T\xi}(|\mathcal{G}|_c)$. Hence, $|\mathcal{G}|_c$ is a point-regular ξ-$\mathfrak{D}$-ideal cover of the ξ-open set A^c. By assumption, there exists a subfamily $\mathcal{R}$ of $|\mathcal{G}|_c$ such that $\mathcal{R}_c$ is a $\mathfrak{J}$-filter that fulfills $\mathrm{adh}_\xi(\mathcal{R}_c) \cap A^c = \varnothing$. The filter $\uparrow^L \mathcal{R}_c$ converges for $[\xi,\$]$ to A, is a $\mathfrak{J}$-filter because $\mathfrak{J}$ is polar-stable, and $\mathcal{G} \geq \uparrow^L \mathcal{R}_c$ by (1.6). □

Theorem 3.1 applies with $\mathfrak{D} = \varphi$ and $\mathfrak{J} = \varphi_\lambda$ to the effect that

$$hl^\circ(\xi) = \chi([\xi,\$]). \tag{3.1}$$

In particular, if $\mathfrak{J} = \varphi_\omega$.

Corollary 3.2. *The upper Kuratowski convergence* $[\xi,\$]$ (*equivalently the Scott convergence on the lattice of* ξ*-open sets*) *is first-countable if and only if all* ξ*-open sets are quasi ideal-cover-Lindelöf.*

Analogously the upper Kuratowski convergence is finitely generated if and only if each set is quasi ideal-cover-compact. Notice that this last condition amounts to discreteness for convergences whose topological modification is Hausdorff. All ξ-open sets are quasi countably ideal-cover-compact if and only if the first-countable modification of the upper Kuratowski convergence $[\xi, \$]$ is finitely generated. Corollary 3.2 seems to be new even if ξ is a topology. In this case (3.1) becomes

$$hl(\xi) = \chi[(\xi, \$]),$$

and Corollary 3.2 rephrases as follows.

Corollary 3.3. *The upper Kuratowski convergence* $[\xi, \$]$ (*equivalently the Scott convergence on the lattice of* ξ*-open sets*) *associated with a topology* ξ *is first-countable if and only if* ξ *is hereditarily Lindelöf.*

Alleche and Calbrix showed [**3**, Corollary 2.2] that the upper Kuratowski topology associated with a hereditarily Lindelöf topology is sequential. If the upper Kuratowski convergence is first-countable, its topological modification is obviously sequential. Hence Corollary 3.3 refines the above result of Alleche and Calbrix.

If ξ is a topology, then

$$\mathrm{adh}_\xi \mathcal{F} = \mathrm{adh}_\xi (\mathrm{cl}_\xi^\sharp \mathcal{F}), \tag{3.2}$$

for every filter $\mathcal{F}$. Moreover, in the case of topologies, quasi ideal-cover-$\mathfrak{D}/\mathfrak{J}$-compactness and cover-$\mathfrak{D}/\mathfrak{J}$-compactness coincide. As the usual notions (countable compactness, Lindelöfness, compactness) of cover-compactness, ideal-cover-compactness and filter-compactness coincide for topologies, I omit the prefixes like "cover" or "ideal-cover" in the propositions concerning topologies.

If $\mathcal{A}$ is a family of subsets of $|\xi|$, B belongs to $\mathcal{O}_\xi(\mathcal{A})$ if B is ξ-open and if there exists $A \in \mathcal{A}$ such that $A \subset B$.

Theorem 3.4 [**24**, Corollary 5.3]. *Let* ξ *be a topology, and let* $\mathfrak{J}$ *be a* $\$$*-compatible class of filters. Then a family* $\mathcal{B}$ *of* ξ*-closed sets is* $\mathrm{Base}_\mathfrak{J}[\xi, \$]$*-open if and only if* $\mathcal{B}_c = \mathcal{O}_\xi(\mathcal{B}_c)$ *is a* $\mathfrak{J}$*-compact family.*

Let $\mathfrak{D}$ and $\mathfrak{J}$ denote two classes of filters. The class of $\mathfrak{D}$-filters $\mathcal{F}$ for which $\bigcap \mathcal{H} \in \mathcal{F}$ for every $\mathfrak{J}$-filter $\mathcal{H} \leq \mathcal{F}$ is denoted by $\mathfrak{D}_{\wedge \mathfrak{J}}$. Hence, the notation $\varphi_{\wedge\lambda}$ for the class of λ-deep filters is a shorthand for $\varphi_{\wedge \varphi_\lambda}$.

Theorem 3.5. *Let ξ be a topology. If $\mathfrak{D}_{\wedge\mathfrak{J}}$ is polar-stable, if $\varphi_1 \subset \mathfrak{J}$ and if $\mathrm{cl}^\sharp\mathfrak{J} \subset \mathfrak{J}$, then*

$$\mathrm{Base}_{\mathfrak{D}_{\wedge\mathfrak{J}}}[\xi, \$] \geq TBase_{\mathfrak{J}}[\xi, \$],$$

implies that every ξ-open set is $\mathfrak{D}_{\wedge\mathfrak{J}}/\mathfrak{J}$-compact.

Proof. Let U be a ξ-open set and $\mathcal{S}$ an ideal open ξ-$\mathfrak{D}_{\wedge\mathfrak{J}}$-cover of U. Since $\mathrm{adh}_\xi\mathcal{S}_c \subset U^c$, the polar-erected filter $\uparrow^L \mathcal{S}_c$ is a $\mathfrak{D}_{\wedge\mathfrak{J}}$-filter that converges to U^c for $[\xi, \$]$. Thus,

$$U^c \in \lim_{T\mathrm{Base}_{\mathfrak{J}}[\xi,\$]} \uparrow^L \mathcal{S}_c. \tag{3.3}$$

Suppose that there exists no $\mathfrak{J}$-subfamily of $\mathcal{S}$ that covers U. In particular, $U \notin \mathcal{S}$. In other words,

$$U \bigcap \mathrm{adh}_\xi\mathcal{H} \neq \varnothing,$$

for every $\mathfrak{J}$-filter $\mathcal{H} \leq \mathcal{S}_c$. The family $\mathcal{A} = \mathcal{O}_\xi(\mathcal{S}_c^{\#}) \bigcup \mathcal{O}_\xi(U)$ is therefore ξ-$\mathfrak{J}$-compact. Indeed, if $\mathcal{G}$ is a $\mathfrak{J}$-filter that meshes $\mathcal{A}$, then $\mathcal{G}\#\mathcal{O}_\xi(\mathcal{S}_c)^{\#}$, equivalently $\mathrm{cl}_\xi^\sharp\mathcal{G}\#\mathcal{S}_c^{\#}$, that is, $\mathrm{cl}_\xi^\sharp\mathcal{G} \leq \mathcal{S}_c^{\#\#} = \mathcal{S}_c$, and thus $\mathrm{adh}_\xi\mathrm{cl}_\xi^\sharp\mathcal{G} \bigcap U \neq \varnothing$. Moreover, $\mathrm{adh}_\xi\mathcal{G} \bigcap U \neq \varnothing$ because

$$\mathrm{adh}_\xi\mathcal{G} = \bigcap_{G\in\mathcal{G}} \mathrm{cl}_\xi G = \mathrm{adh}_\xi\mathrm{cl}_\xi^\sharp\mathcal{G}, \tag{3.4}$$

as ξ is a topology. By (3.4), $\mathrm{adh}_\xi\mathcal{G} \in \mathcal{S}_c$ because $\mathrm{cl}_\xi^\sharp\mathcal{G} \in \mathfrak{J}$ and $\mathcal{S}_c \in \mathfrak{D}_{\wedge\mathfrak{J}}$. Thus, $\mathrm{adh}_\xi\mathcal{G}\#\mathcal{A}$. Hence, by Theorem 3.4, $\mathcal{A}_c$ is a $\mathrm{Base}_{\mathfrak{J}}[\xi, \$]$-open neighborhood of U^c. On the other hand, $\mathcal{S} \bigcap \mathcal{A} = \varnothing$ by definition of $\mathcal{A}$. Thus, for every $S \in \mathcal{S}$, $S^c = \mathrm{cl}_\xi S^c \notin \mathcal{A}_c$, so that $\uparrow^L S^c \not\subseteq \mathcal{A}_c$. Hence, $\mathcal{A}_c \notin \uparrow^L \mathcal{S}_c$, in contradiction with (3.3). Consequently, there exists a $\mathfrak{J}$-subcover of $\mathcal{S}$ that covers U. □

In particular, in case $\mathfrak{D}$ is the class of all filters and $\mathfrak{J}$ is the class of countably based filters, we obtain the equivalence between Lindelöfness of every ξ-open set, first-countability and sequentiality of the upper-Kuratowski convergence $[\xi, \$]$. Thus, hereditary Lindelöfness of ξ is equivalent to the sequentiality of $T[\xi, \$]$, which proves the converse of [**3**, Corollary 2.2].

Lemma 3.6. *Let U be an open set, and let $\mathfrak{A}$ be the set of all compact families $\mathcal{A} = \mathcal{O}(\mathcal{A})$ containing U. If $\mathcal{B}$ is a family of subsets such that $\mathcal{A} \bigcap \mathcal{B} \neq \varnothing$ for every $\mathcal{A} \in \mathfrak{A}$, then $U \subset \bigcup_{B\in\mathcal{B}} B$.*

Proof. If there exists $x_0 \in U \setminus \bigcup_{B\in\mathcal{B}} B$, then $\mathcal{O}_\xi(x_0)$ is a compact family that contains U but none of the elements of $\mathcal{B}$. □

Theorem 3.7. *Let ξ be a topology. The upper Kuratowski topology $T[\xi, \$]$ is $(\varkappa, \lambda)$-tight if and only if every ξ-open set is $(\varkappa, \lambda)$-compact.*

Proof. If $\mathcal{F}$ is a $\varkappa$-based filter such that $U \cap \mathrm{adh}_\xi \mathcal{F} = \varnothing$, then $U^c \in \lim_{[\xi,\$]}(\uparrow^L \mathrm{cl}_\xi^\sharp \mathcal{F})$. Thus, if $\{F_\alpha : \alpha \in I\}$ is a filter-base of $\mathcal{F}$ of cardinality less than or equal to $\varkappa$, then $\{\mathrm{cl}_\xi F_\alpha : \alpha \in I\}$ is a set of cardinality less than or equal to $\varkappa$ that meshes $\mathcal{N}_{[\xi,\$]}(U^c)$. Since $T[\xi, \$]$ is $(\varkappa, \lambda)$-tight, there exists $J \subset I$ such that $\mathrm{card}\, J \leq \lambda$ and $\{\mathrm{cl}_\xi F_\alpha : \alpha \in J\} \# \mathcal{N}_{[\xi,\$]}(U^c)$. In view of Lemma 3.6 and Theorem 3.4,

$$U \subset \bigcup_{\alpha\in J} (\mathrm{cl}_\xi F_\alpha)^c.$$

Conversely, assume that each ξ-open set is $(\varkappa, \lambda)$-compact, and let $\mathcal{A} \# \mathcal{N}_{[\xi,\$]}(A_0)$ be a family such that $\mathrm{card}\, \mathcal{A} \leq \varkappa$. By Theorem 3.4, for every compact family $\mathcal{G}$ of open sets that contains A_0^c, there exists $A_\mathcal{G} \in \mathcal{A}$ such that $(A_\mathcal{G})^c \in \mathcal{G}$. By Lemma 3.6, $A_0^c \subset \bigcup_{A\in\mathcal{A}} A^c$. Consequently, there exists a subfamily $\mathcal{B}$ of $\mathcal{A}$ (which can be assumed to be stable under finite union) such that $\mathrm{card}\, \mathcal{B} \leq \lambda$ and $A_0^c \subset \bigcup_{A\in\mathcal{B}} A^c$. Thus $\bigcup_{A\in\mathcal{B}} A^c$ belongs to every compact family $\mathcal{G}$ containing A_0^c. For every such $\mathcal{G}$, there exists $A \in \mathcal{B}$ such that $A^c \in \mathcal{G}$, by compactness of $\mathcal{G}$. In other words, $\mathcal{B} \# \mathcal{N}_{[\xi,\$]}(A_0)$. □

Corollary 3.8. *Let ξ be a topology. The following are equivalent*:

1. *ξ is hereditarily Lindelöf*;
2. $B_{\wedge\omega}[\xi,\$] \geq B_{\omega}[\xi,\$]$;
3. *$[\xi,\$]$ is first-countable*;
4. *$[\xi,\$]$ is countably tight*;
5. *$[\xi,\$]$ is sequential*;
6. *$T[\xi,\$]$ is sequential*;
7. *$T[\xi,\$]$ is countably tight.*

Proof. Equivalences between 1, 2 and 3 are a combination of Corollary 3.3 and Lemma 2.4. On the other hand, $3 \Rightarrow 4$ is obvious, $4 \Rightarrow 7$ follows from Proposition 2.1, and $7 \Rightarrow 1$ is a special instance of Theorem 3.7. Finally, $3 \Rightarrow 5 \Rightarrow 6 \Rightarrow 7$ follows directly from the definitions. □

This refines a recent result [**12**, Proposition 3.6] of Costantini, Holá and Vitolo that states the equivalence for a Hausdorff topology X, between 1, 6 and 7 of Corollary 3.8. Notice that, while the implication $6 \Rightarrow 7$ is true for every topology, the corresponding implication $5 \Rightarrow 4$ is a special property of the upper Kuratowski convergence, as shown in Example 2.2.

On the other hand, equivalences between 1, 3, 4 and 7 obviously extend to the corresponding ordinal invariants.

Corollary 3.9. *Let ξ be a topology. Then*

$$\mathrm{hl}\,(\xi) = \chi([\xi,\$]) = t([\xi,\$]) = t(T[\xi,\$]).$$

4. Application to consonance. Recall that a topology ξ is called *consonant* [**23**] if the upper Kuratowski convergence $T([\xi,\$])$ and the cocompact topology $C_k(\xi,\$)$ associated with ξ (1.5) coincide. In view of Corollary 3.8, we immediately obtain the following.

Corollary 4.1. *If ξ is a consonant topology, the following are equivalent*:

1. ξ *is hereditarily Lindelöf*;
2. $C_k(\xi, \$)$ *is countably tight*;
3. $C_k(\xi, \$)$ *is sequential.*

This refines [**12**, Corollary 3.13] of Costantini, Holá and Vitolo, that states the equivalences between 1 and 3 for Čech-complete topologies. A topology is *quasi Čech-complete* if there exists a sequence $(\mathcal{B}_n)_n$ of open covers of the underlying space such that, whenever $\mathcal{F}$ is a family of closed sets with the finite intersection property such that $\mathcal{F} \bigcap \mathcal{B}_n \neq \varnothing$ for every n, then $\bigcap \mathcal{F} \neq \varnothing$. A Tychonoff quasi Čech-complete topology is Čech-complete. By [**23**, Theorem 4.1], regular quasi Čech-complete (in particular, Čech-complete) topologies are consonant. Analogously,

Corollary 4.2. *Let* ξ *be a consonant topology. Then*

$$hl(\xi) = t(C_k(\xi, \$)).$$

This last fact was apparently noticed [**12**, Comment on Proposition 2.11], only for regular quasi Čech-complete spaces.

It is known that the upper Kuratowski convergence is a topology if and only if ξ is core-compact [**28, 24**], i.e., for every point x and every neighborhood V of x, there is a neighborhood W of x which is relatively compact in V. For a Hausdorff topology, core-compactness amounts to local compactness. In this case, $[\xi, \$] = T([\xi, \$]) = [K\xi, \$] = [K_{\mathrm{her}}\xi, \$]$ is the cocompact topology. Assume that ξ is still a Hausdorff topology. In general, $[\xi, \$] \geq [K\xi, \$]$ because $K\xi \geq \xi$. Since $[K\xi, \$]$ is topological,

$$T([\xi, \$]) \geq [K\xi, \$]. \tag{4.1}$$

Of course, in general $C(\xi, \$) \subset C(K\xi, \$)$. The equality holds for example for k-convergences, i.e., convergences ξ for which $\xi \geq TK\xi$. Hence $[K\xi, \$]$ and $[K_{\mathrm{her}}\xi, \$]$ denote here the restrictions of these convergences to $C(\xi, \$)$. In order not to overburden the notations, the restrictions do not appear but are implicit in this context. Hence $[K\xi, \$]$ is the best candidate to obtain the upper Kuratowski topology $T[\xi, \$]$ as a continuous dual. The question of characterizing the topologies ξ for which $T([\xi,\$])$ can be obtained as a continuous dual is very natural. This is exactly the problem of consonance. Indeed, let $C_k(\xi,\sigma)$ denote the *compact-open topology* on the set of continuous functions from ξ to a topology σ, which admits as a subbase the family of sets

$$\langle K,V\rangle = \{f \in C(\xi,\sigma) : f(K) \subset V\}$$

where K ranges over ξ-compact sets and V ranges over σ-open sets. If σ is the Sierpiński topology \$, the compact-open topology is (homeomorphic to) the cocompact topology on $\mathcal{C}(\xi)$.

Proposition 4.3. *Let σ be a topology. Then*

$$[K\xi,\sigma] \geq C_k(\xi,\sigma) \geq [K_{\text{her}}\xi,\sigma].$$

Proof. If $f_0 \notin \lim_{[K_{\text{her}}\xi,\sigma]}\mathcal{F}$, there exists $x_0 \in \lim_{K_{\text{her}}\xi}\mathcal{G}$ such that $f_0(x_0) \notin \lim_\sigma ev(\mathcal{F}\times\mathcal{G})$. Hence, there exists $V_0 \in \mathcal{N}_\sigma(f_0(x_0))$ such that $ev(F\times G)\cap V_0^c \neq \varnothing$ for every $F \in \mathcal{F}$ and every $G \in \mathcal{G}$. Since f_0 is continuous, there exists a ξ-compact set $K_0 \in \mathcal{G}$ such that $f_0(K_0) \subset V_0$, so that $\langle K_0,V_0\rangle \in \mathcal{N}_{C_k(\xi,\sigma)}(f_0)$. But $\langle K_0,V_0\rangle \notin \mathcal{F}$ and thus $f_0 \notin \lim_{C_k(\xi,\sigma)}\mathcal{F}$.

If $f_0 \notin \lim_{C_k(\xi,\sigma)}\mathcal{F}$, there exists a ξ-compact set K_0 and a σ-open set V_0 such that $f_0(K_0) \subset V_0$ and $ev(F\times K_0)\cap V_0^c \neq \varnothing$ for every $F \in \mathcal{F}$. Thus there exists an ultrafilter $\mathcal{U}$ on K_0 (which is of course $K\xi$-convergent) such that $V_0 \notin ev(\mathcal{F}\times\mathcal{U})$. Otherwise, for every $\mathcal{U} \in \beta(K_0)$ there exists $U_\mathcal{U} \in \mathcal{U}$ and $F_\mathcal{U} \in \mathcal{F}$ such that $ev(F_\mathcal{U}\times U_\mathcal{U}) \subset V_0$. Therefore (see, for example, [**31**, Proposition 1.2.2]) there exists a finite subfamily $\{U_i : i \in 1,\dots,n\}$ of $\{U_\mathcal{U} : \mathcal{U} \in \beta(K_0)\}$ such that $K_0 \subset \bigcup_{i=1}^n U_i$. Consequently,

$$ev\left(\bigcap_{i=1}^n F_{\mathcal{U}_i} \times \bigcup_{i=1}^n U_i\right) \subset V_0,$$

and $ev(\bigcap_{i=1}^n F_{\mathcal{U}_i} \times K_0) \subset V_0$, a contradiction. Thus, $f_0 \notin \lim_{[K,\xi,\sigma]}\mathcal{F}$. □

In particular, if ξ is either a regular or a Hausdorff convergence, then $K\xi = K_{\text{her}}\xi$.

Corollary 4.4. *If a convergence ξ is either regular or Hausdorff, then (the restriction to $C(\xi,\sigma)$ of) $[K\xi,\sigma]$ is exactly the compact-open topology.*

Let σ denote a topology. I call *σ-consonant* a convergence ξ for which $C_k(\xi,\sigma) = T[\xi,\sigma]$. The problem of characterizing σ-consonant topologies was already formulated in [**4**, Section 5] in different terms. In view of Corollary 4.4, a Hausdorff convergence ξ is σ-consonant if and only if $[K\xi,\sigma] = T[\xi,\sigma]$. Several results on $-consonance, that we will review later, are based on the following general scheme.

Theorem 4.5. *Let E be a bicoreflector.*

1. *If ξ is σ-consonant and if $[\xi,\sigma] \geq TE[\xi,\sigma]$, then $C_k(\xi,\sigma) \geq TEC_k(\xi,\sigma)$.*

2. *If $E[\xi,\sigma] = EC_k(\xi,\sigma)$ and if $C_k(\xi,\sigma) \geq TEC_k(\xi,\sigma)$, then ξ is σ-consonant.*

Proof. If $[\xi,\sigma] \geq TE[\xi,\sigma]$, then $T[\xi,\sigma] \geq TET[\xi,\sigma]$. If ξ is moreover σ-consonant, then $C_k(\xi,\sigma) \geq TEC_k(\xi,\sigma)$. If $C_k(\xi,\sigma) \geq TEC_k(\xi,\sigma)$ and $E[\xi,\sigma] = EC_k(\xi,\sigma)$, then $C_k(\xi,\sigma) \geq TET[\xi,\sigma] \geq T[\xi,\sigma]$ so that ξ is σ-consonant. □

Corollary 4.6. *If ξ is such that*

$$E[\xi,\sigma] = EC_k(\xi,\sigma); \tag{4.2}$$

$$[\xi,\sigma] \geq TE[\xi,\sigma]; \tag{4.3}$$

then ξ is σ-consonant if and only if $C_k(\xi,\sigma) \geq TEC_k(\xi,\sigma)$.

One advantage of Corollary 4.6 is that it gives a criterion of σ-consonance that does not use any internal description of σ-consonance. All the applications of this scheme by other authors concern the case in which E is the sequentially based modifier Seq and σ is the Sierpiński topology \$. Thus, if no contrary mention is given, $E = \mathrm{Seq}$ and $\sigma = \$$ in the sequel.

A convergence is k' if, whenever $y \in \mathrm{adh}_\xi A$, then $y \in \mathrm{adh}_\xi(A \cap K)$ for some compact set K. If, moreover, $y \in \mathrm{adh}_\xi(\mathcal{H} \vee K)$ for some compact K whenever $y \in \mathrm{adh}_\xi \mathcal{H}$, where $\mathcal{H}$ is a countably based filter, then ξ is *strongly* k'. The first consonance result that uses these ideas is [**23**, Theorem 4.3] of Dolecki, Greco and Lechicki, that states that a Hausdorff k'-topology in which every open set is hemicompact is consonant. The condition that every open set is hemicompact ensures that $C_k(\xi, \$)$ is first-countable, hence sequential. The proof of [**23**, Theorem 4.3] essentially consists of verifying that the second condition of k'-ness ensures $\mathrm{Seq}\,[\xi, \$] = \mathrm{Seq}\, C_k(\xi, \$)$. Indeed, the result follows from the second point of Theorem 4.5.

Another related result is [**13**, Proposition 2.6] of Costantini and Vitolo, that states the equivalence among metrizable separable topologies between consonance and sequentiality of the associated cocompact topology. To prove this proposition, Costantini and Vitolo observed that $\mathrm{Seq}\,[\xi, \$] = \mathrm{Seq}\, C_k(\xi, \$)$, that is (4.2), if ξ is first-countable (in particular if ξ is metrizable) and that the upper Kuratowski topology $T[\xi, \$]$ associated with a metrizable topology ξ is sequential, that is (4.3), if and only if ξ is separable [**13**, Proposition 2.5]. Hence, the hypothesis in [**13**, Proposition 2.6] is designed to apply Corollary 4.6. The above result of Costantini and Vitolo has been refined by Alleche and Calbrix as follows.

Theorem 4.7 [**3**, Theorem 2.4]. 1. *The cocompact topology associated with a consonant hereditarily Lindelöf topology, is sequential.*

2. *If X is a strongly Fréchet topology and if the associated cocompact topology is sequential, then X is consonant.*

Corollary 4.8 [**3**, Corollary 2.5]. *A hereditarily Lindelöf and strongly Fréchet topology is consonant if and only if the associated cocompact topology is sequential.*

They first proved that the upper Kuratowski topology associated with a hereditarily Lindelöf topology is sequential [**3**, Corollary 2.2], so that (4.3) is verified provided that ξ is a hereditarily Lindelöf topology. Then they proved that the upper Kuratowski convergence and the cocompact topology associated with a strongly Fréchet topology coincide on sequences [**3**, Lemma 2.3]. In other words, (4.2) holds provided that ξ is a strongly Fréchet topology. Hence, their results follow from Theorem 4.5 and Corollary 4.6. It is interesting to note that each Hausdorff strongly Fréchet topology is a k'-topology, so that they could have refined their result for Hausdorff topologies thanks to the condition of Dolecki, Greco and Lechicki [**23**, Theorem 4.3] for (4.2). In view of Corollary 3.8, we have, moreover,

Proposition 4.9. *A k' Hausdorff topology is hereditarily Lindelöf and consonant if and only if the associated cocompact topology is sequential.*

Costantini, Holá and Vitolo recently obtained a characterization on a topology ξ for (4.2) to hold (for $E = \mathrm{Seq}$ and $\sigma = \$$). Their condition (which is weaker than k-ness, hence weaker than k'-ness, and weaker than being a P-space) reads as follows.

Condition 4.10. Given any countable collection $\mathcal{G}$ of ξ-closed sets and any point $x \in |\xi|$, if every neighborhood of x intersects infinitely many members of $\mathcal{G}$, then every neighborhood of x contains a compact set which intersects infinitely many members of $\mathcal{G}$.

In view of Corollary 3.8, the hypothesis of hereditary Lindelöfness to obtain (4.3) cannot be refined. Thus, the following result of Costantini, Holá and Vitolo is the best possible application of Theorem 4.5 and Corollary 4.6, when $E = \mathrm{Seq}$ and $\sigma = \$$.

Theorem 4.11 [**12**, Theorem 3.8]. *A topology that fulfills Condition* 4.10 *is hereditarily Lindelöf and consonant if and only if the associated cocompact topology is sequential.*

In view of Corollary 3.8, one may ask if $T[\xi, \$]$ must be first-countable when ξ is hereditarily Lindelöf. The answer is 'No'. Indeed, the cocountable topology on an uncountable set is hereditarily Lindelöf and fulfills Condition 4.10, so that $T[\xi, \$] = C_k(\xi, \$)$ is sequential. But the cocountable topology is not σ-compact, hence not hemicompact, so that $T[\xi,\$] = C_k(\xi,\$)$ is not first-countable.

I will now discuss another approach to results akin to Theorem 4.5. In the particular case of \$-consonance, I will obtain a weaker result than Theorem 4.11, but the approach allows to obtain a similar counterpart for **R**-consonance.

Following [**37**], if E is a coreflector, I consider the modifier

$$\mathrm{Epi}_E^\sigma \xi = i^-([E[\xi,\sigma],\sigma]),$$

which defines a reflector. E is omitted when E is the identity functor. The reflection $\mathrm{Epi}_E^\sigma \xi$ is the coarsest convergence among convergences θ on $|\xi|$ for which

$$E[\xi,\sigma] = E[\theta,\sigma]. \tag{4.4}$$

By Corollary 4.4, $C_k(\xi,\sigma) = [K\xi,\sigma]$ if ξ is a Hausdorff convergence, so that (4.2) is equivalent to

$$\xi \geq \mathrm{Epi}_E^\sigma K\xi, \tag{4.5}$$

because of (4.4).

In view of (2.3), either $E = \mathrm{Seq}$ or $E = B_\omega$ can be used to obtain sequentiality of $[\xi,\sigma]$ via (4.3). Of course, the condition for the coincidence of $[\xi,\sigma]$ and $[K\xi,\sigma]$ on every countably based filter is in general stronger than the condition for the coincidence on sequences. However, I use $E = B_\omega$ because I have explicit descriptions of reflectors Epi_E^σ in this case. In particular, $A = \mathrm{Epi}^\$$ is the reflector on Antoine convergences [**9, 24**] and $A_\omega = \mathrm{Epi}_{B_\omega}^\$$ is the reflector on countably Antoine convergences [**36, 24**]. Analogously, I denote by c_ω the reflector $\mathrm{Epi}_{B_\omega}^{\mathbf{R}}$, while $c = \mathrm{Epi}^{\mathbf{R}}$ is the reflector on c-embedded spaces in the sense of Binz [**8**], and $\Omega = \mathrm{Epi}_{B_1}^{\mathbf{R}}$ is the reflector on completely regular topologies. See [**38**] and [**37**] for details on reflectors Epi_E^σ.

Consider the case $E = B_\omega$ and $\sigma = \$$ in the above scheme. In case ξ is Hausdorff, [**36**, Theorem 2.3] applies to the effect that

$$\xi \geq A_\omega K\xi \iff \forall \mathcal{H} \in \varphi_\omega, \quad \mathrm{adh}_\xi \mathcal{H} \subset \mathrm{cl}_{K\xi}(\mathrm{adh}_{K\xi}\mathcal{H}). \tag{4.6}$$

I call *strongly k* the convergences ξ that fulfill (4.6), by analogy with strongly sequential convergences [**36**]. A T_1 convergence ξ is *strongly sequential* if $\mathrm{adh}_\xi \mathcal{H} \subset \mathrm{cl}_{\mathrm{Seq}\,\xi}(\mathrm{adh}_{\mathrm{Seq}\xi}\mathcal{H})$ for every countably based $\mathcal{H}$.

The relationship of strongly sequential spaces to sequential spaces is analogous to that of strongly Fréchet spaces with respect to general Fréchet spaces. See [**36**] for details. Here strongly k-spaces are related to k-spaces like strongly k'-spaces to k'-spaces. Since we only need (4.2) to hold in restriction to $\mathcal{C}(\xi)$, instead of $\mathcal{C}(K\xi)$, we only need ξ to be *quasi strongly* k, that is,

$$\mathrm{adh}_\xi \mathcal{H} \subset \mathrm{cl}_\xi(\mathrm{adh}_{K\xi}\mathcal{H}),$$

for every countably based filter $\mathcal{H}$. Observe that quasi strongly k-ness and strongly k-ness coincide among k-convergences. On the other hand, a Hausdorff strongly Fréchet topology is strongly k, hence quasi strongly k. In view of (4.5), (4.6), Theorem 4.5 and Corollary 3.8,

Theorem 4.12. *A quasi strongly k Hausdorff convergence is hereditarily Lindelöf and consonant if and only if the associated cocompact topology is sequential.*

I now turn to the case $\sigma = \mathbf{R}$ and $E = B_\omega$ in Theorem 4.5 and Corollary 4.6. Feldman proved (compare with Corollary 3.2)

Theorem 4.13 [**26**, Theorem 1]. *A convergence $\xi = c\xi$ is cover-Lindelöf if and only if $[\xi, \mathbf{R}]$ is first-countable.*

Thus a cover-Lindelöf convergence ξ fulfills (4.3). On the other hand, if ξ is Hausdorff, (4.2) amounts to $\xi \geq c_\omega K\xi$. I call such a convergence $\mathbf{R}$-*strongly k*. Notice that, as $A_\omega \geq c_\omega$, each strongly k and in particular each strongly k' space is $\mathbf{R}$-strongly k. A convergence ξ is $\mathbf{R}$-k if $\xi \geq \Omega K\xi$. Obviously, each k-convergence is $\mathbf{R}$-k. In view of [**39**, Theorem 5.3],

Proposition 4.14. *A convergence ξ is $\mathbf{R}$-strongly k if and only if it is $\mathbf{R}$-k and $\mathrm{adh}_\xi\mathcal{H} \neq \varnothing$ implies $adh_{K\xi}\mathcal{H} \neq \varnothing$ for every countably based filter $\mathcal{H}$ based in $\Omega K\xi$-open sets.*

Theorem 4.5 and Corollary 4.6 apply to the effect that

Theorem 4.15. 1. *If ξ is an $\mathbf{R}$-consonant cover-Lindelöf convergence, then the associated compact-open topology (on real-valued continuous functions) is sequential.*

2. *If ξ is an $\mathbf{R}$-strongly k Hausdorff convergence and if the associated compact-open topology is sequential, then ξ is $\mathbf{R}$-consonant.*

Corollary 4.16. *A cover-Lindelöf Hausdorff $\mathbf{R}$-strongly k convergence is $\mathbf{R}$-consonant if and only if the compact-open topology is sequential.*

In view of the following result of Dolecki, Greco and Lechicki, published without proof, I obtain Theorem 4.19 as a new condition for consonance, that involves the compact-open topology on real-valued functions rather than the cocompact topology on closed sets.

Theorem 4.17 [**22**, Theorem 4.4]. *A completely regular $\mathbf{R}$-consonant topology is consonant.*

Proof. Assume ξ is not consonant. There exists $\mathcal{B}$ which is $[\xi,\$]$-closed but not $C_k(\xi,\$)$-closed, so that there exists $B_0\in \mathrm{cl}_{C_k(\xi,\$)}\mathcal{B}\setminus\mathcal{B}$. Since ξ is completely regular, there exists $\{f_i : i\in I\}\subset C(\xi,\mathbf{R})$ such that $B_0=\bigcap_{i\in I} f_i^{-}(0)$. Notice that $\bigcup_{i\in I}(f_i^{-}(0))^c\#\mathcal{B}$ because $\bigcap_{i\in I} f_i^{-}(0)\notin\mathcal{B}$. Since $\mathcal{B}$ is $[\xi,\$]$-closed, there exists a finite subset F of I such that $\bigcup_{i\in F}(f_i^{-}(0))^c\#\mathcal{B}$, by [**23**, Corollary 3.2] (see also the dual statement of Theorem 3.4.). In other words, $\bigcap_{i\in F} f_i^{-}(0)\notin\mathcal{B}$ so that we can assume that there exists f_0 such that $B_0=f_0^{-}(0)$. Let

$$\tilde{\mathcal{B}}=\{f\in C(\xi,\mathbf{R}) : \exists_{B\in\mathcal{B}}, f(B)=0\}.$$

Observe that $\tilde{\mathcal{B}}$ is $[\xi,\mathbf{R}]$-closed, because of Lemma 4.18 and because $f\in\tilde{\mathcal{B}}$ if and only if $f^{-}(0)\in\mathcal{B}$. Since $B_0\in\mathrm{cl}_{C_k(\xi,\$)}\mathcal{B}$, the set $N_K=\{F\in\mathcal{C}(\xi) : F\cap K=\varnothing\}$ meshes $\mathcal{B}$ for every ξ-compact set K disjoint from B_0. For every $r>0$, the set $K_r=K\cap f_0^{-}([r,+\infty))$ is compact so that there exists $B_r\in\mathcal{B}$ such that $B_r\cap B_0=\varnothing$ and $B_r\cap K_r=\varnothing$. Since ξ is completely regular, there exists $g_r\in C(\xi,\mathbf{R})$

such that $g_r(B_r) = 0$ and $g_r(K_r) = 1$. Thus $f_r = f_0 \wedge (g_r \times f_0)$ belongs to $\tilde{\mathcal{B}}$ because $f_r^-(0) = B_0 \bigcup B_r \in \mathcal{B}$. Moreover, $\sup_K |f_r - f_0| < r$ so that $(f_{1/n})_{n \in \mathbf{N}}$ converges to f_0 in $C_k(\xi, \mathbf{R})$. Since $f_0 \notin \tilde{\mathcal{B}}$, the set $\tilde{\mathcal{B}}$ is not $C_k(\xi, \mathbf{R})$-closed. $\square$

Lemma 4.18. *If $\mathcal{F}^-(0)$ denotes the filter generated by $\{f^-(0) : f \in F\}_{F \in \mathcal{F}}$, then*

$$h \in \lim_{[\xi,\mathbf{R}]} \mathcal{F} \Longrightarrow h^-(0) \in \lim_{[\xi,\$]} \mathcal{F}^-(0).$$

Proof. Let $x \in \lim_\xi \mathcal{G}$. Since $h \in \lim_{[\xi,\mathbf{R}]} \mathcal{F}$, there exists $F_\varepsilon \in \mathcal{F}$ and $G_\varepsilon \in \mathcal{G}$ such that $ev(F_\varepsilon \times G_\varepsilon) \subset [h(x) - \varepsilon, h(x) + \varepsilon]$ for every $\varepsilon > 0$.

If $\mathcal{G} \# |\mathcal{F}^-(0)|$, there exists $f \in F_\varepsilon$ and $y \in G_\varepsilon$ such that $f(y) = 0$. Thus $0 \in \bigcap_{\varepsilon > 0} [h(x) - \varepsilon, h(x) + \varepsilon]$ so that $h(x) = 0$. $\square$

Theorem 4.19. *A Tychonoff $\mathbf{R}$-strongly k topology ξ is consonant provided that the associated compact-open topology $C_k(\xi, \mathbf{R})$ is sequential.*

McCoy and Ntantu give the following criterion of Fréchetness for $C_k(\xi, \mathbf{R})$. A family $\mathcal{A}$ of subsets of $|\xi|$ is a k-cover if, for every ξ-compact set K, there exists $A \in \mathcal{A}$ such that $K \subset A$. A *k-cover-sequence* is a sequence $\{A_n : n \in \mathbf{N}\}$ of subsets of $|\xi|$ with the property that, for every ξ-compact set K, there exists $m \in \mathbf{N}$ such that $K \subset A_n$ for every $n \geq m$.

Proposition 4.20 [**16**, Theorem 4.7.4]. *$C_k(\xi, \mathbf{R})$ is Fréchet if and only if every open k-cover of ξ contains a k-cover-sequence.*

Corollary 4.21. *A Tychonoff $\mathbf{R}$-strongly k topology in which every open k-cover contains a k-cover-sequence is consonant.*

On the other hand, Pytkeev proved in [**41**] that $C_k(\xi, \mathbf{R})$ is a k-space if and only if $C_k(\xi, \mathbf{R})$ is Fréchet, so that the condition of Proposition 4.20 is the best possible for the sequentiality of $C_k(\xi, \mathbf{R})$.

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